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A LIE SYMMETRY APPROACH FOR THE COMPUTATION OF THE VIBRATION MODES OF BEAMS

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Abstract. *Lie symmetries constitute an interesting tool to solve differential equations, consisting in reducing and simplifying these equations by only retaining some invariant solutions. In this paper, Lie symmetries are applied to solve the differential equation of Euler-Bernoulli beams with a view to computing their vibration modes. It is shown that the differential equation can be mapped on a subspace where smooth solutions can be obtained. This suggests nice features (i.e., good conditioning) for computing the high order modes of the beams.*

Keywords: *Lie symmetries, Euler-Bernoulli beam, vibration modes.*

1. INTRODUCTION

A Lie symmetry approach is proposed to compute the vibration modes of Euler-Bernoulli beams subject to arbitrary kinds of boundary conditions. Indeed, the computation of these modes via the direct resolution of the differential equation can be prone to ill-conditioning, especially at high frequencies where small wavelengths are dealt with. For clamped-clamped boundary conditions, the issue is mostly due to the occurrence of evanescent terms e^{kx} and e^{-kx}, with $e^{kx}/e^{-kx} \gg 1$, which highly vary in space and are therefore difficult to capture by means of numerical equations. Instead, the proposed approach consists in mapping the differential beam equation on some appropriate subspaces where smooth solutions — i.e., which slowly vary in space — can be obtained. This suggests nice features (i.e., good conditioning) for computing the high order modes of the beams.

The mathematical theory named Lie Symmetries, proposed by Sophus Lie, constitutes an efficient tool to determine groups of transformation for differential equations that retain invariant solutions, and to linearize and reduce these equations (Bluman and Anco, 2008; Oliveri, 2010). Hence, the Lie symmetry approach can be advantageously used to solve the beam differential equation and obtain a more natural way to solve it (analytically or numerically) by providing smooth solutions which can be accurately described even at high frequencies. This paper is a first attempt to apply the Lie symmetry theory to compute the vibration modes of structures, and further, the more general solutions of the wave propagation problems in periodic structures (Mencik, 2018; Pereira, 2009). The potential of the proposed approach is highlighted here for computing the vibration modes of beams subject to different types of boundary conditions.

2. LIE SYMMETRIES

The symmetry concept for differential equations is that of transformations to map these equations on a particular subspace which keeps the solutions and physical quantities unchanged (Bocko *et al.*, 2012). Consider the following ordinary differential equation (Olver, 1986; H. C. Costa Basquerotto *et al.*, 2018):

$$\mathcal{F}\left(x, u, u', ..., u^{(k)'}\right) = 0 \tag{1}$$

where x and u refer to independent and dependent variables, respectively, and where superscripts $'$ and $^{(k)'}$ denote the first and $k-$th order derivatives about x. Define the infinitesimal generator as follows:

$$\mathcal{X} = \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} \cdot \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial u} \end{Bmatrix} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}, \tag{2}$$

where ξ and η are infinitesimal functions. The Lie condition, which if true guarantees the presence of symmetries, is given by (Olver, 1986; Bluman and Anco, 2008):

$$\left(\mathcal{X}+\mathcal{U}^{(k)}\right)\mathcal{F}=0 \quad \Rightarrow \quad \xi\frac{\partial \mathcal{F}}{\partial x}+\eta\frac{\partial \mathcal{F}}{\partial u}+\beta^{(1)}\frac{\partial \mathcal{F}}{\partial u'}+...+\beta^{(k)}\frac{\partial \mathcal{F}}{\partial u^{(k)'}}=0. \tag{3}$$

Here, $\beta^{(k)}$ are the so-called prolongations, with $\beta^{(0)}=\eta$ and

$$\beta^{(k)}=\mathcal{D}_x\left(\beta^{(k-1)}\right)-u^{(k-1)'}\mathcal{D}_x\left(\xi\right) \qquad \text{for } k\geq 1, \tag{4}$$

where $\mathcal{D}_x$ is the total derivative:

$$\mathcal{D}_x=\frac{\partial}{\partial x}+u'\frac{\partial}{\partial u}+...+u^{(k)'}\frac{\partial}{\partial u^{(k-1)'}} \tag{5}$$

By determining the prolongations and applying the Lie condition, a system of equations can be obtained by grouping the terms with the same derivative order, which are therefore equal to zero. The determination of the infinitesimal functions ξ and η follows from the resolution of this system of equations. Several pairs of infinitesimal functions (ξ,η) can be obtained, each of them yielding a suitable group of transformations for the differential equation. For each set of infinitesimal functions, a domain mapping can be achieved by means of canonical coordinates r and $s(r)$ through the following equations (Silva Júnior, 2016):

$$\begin{cases} \xi\dfrac{\partial}{\partial r}s+\eta\dfrac{\partial}{\partial s}s=0 \\ \xi\dfrac{\partial}{\partial r}r+\eta\dfrac{\partial}{\partial s}r=1 \end{cases} \tag{6}$$

3. BEAM EQUATION

The transverse vibration of an Euler-Bernoulli beam vibrating in one of its natural modes is governed by the following fourth order ordinary differential equation (Grover, 1977; Timoshenko *et al.*, 1990):

$$\mathcal{F}=\frac{d^4}{dx^4}u(x)-k^4u(x)=0 \qquad x\in]0\,;\,\ell[, \tag{7}$$

where u is the transverse displacement, $k=\omega/c$ is the wave number, ω is the angular frequency, $c=c(\omega)$ is the wave speed and $x\in[0\,;\,\ell]$ is the position along the beam (ℓ being the length). The general solution of Eq. (7) is given by:

$$u(x)=\mathcal{C}_1e^{kx}+\mathcal{C}_2e^{-kx}+\mathcal{C}_3\sin(kx)+\mathcal{C}_4\cos(kx), \tag{8}$$

being $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ constants. By applying the Lie condition to Eq. (7) and solving the resulting system of equations, six pairs of infinitesimal functions (ξ,η) can be calculated. Among these, the third pair is given by $\xi=0$ and $\eta=e^{kx}$, which yield the following generator $\mathcal{X}_3$:

$$\mathcal{X}_3=e^{kx}\frac{\partial}{\partial u}. \tag{9}$$

The determination of the canonical coordinates r and $s=s(r)$ involves solving Eq. (6). This yields:

$$r=x \quad , \quad s=\frac{u}{e^{kx}}. \tag{10}$$

Introducing $u=se^{kr}$ and $x=r$ into the original differential equation (7) yields a differential equation expressed in the canonical domain, the mapped equation:

$$\frac{d^4}{dr^4}s(r)+4k\frac{d^3}{dr^3}s(r)+6k^2\frac{d^2}{dr^2}s(r)+4k^3\frac{d}{dr}s(r)=0, \tag{11}$$

whose closed-form solution is given by:

$$s(r)=\mathcal{C}_1+\mathcal{C}_2e^{-2kr}+\mathcal{C}_3e^{-kr}\sin(kr)+\mathcal{C}_4e^{-kr}\cos(kr). \tag{12}$$

The determination of the constants $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ involves considering the boundary conditions at $x=r=0$ and $x=r=\ell$. For instance, for a beam subject to clamped-clamped boundary conditions, one has:

$$u(0)=u(\ell)=0 \quad , \quad \left.\frac{d}{dx}u(x)\right|_{x=0}=\left.\frac{d}{dx}u(x)\right|_{x=\ell}=0, \tag{13}$$

By considering the canonical coordinates r and $s(r)$, this gives:

$$s(0) = s(\ell)e^{k\ell} = 0 \quad , \quad \frac{d}{dr}\left(s(r)e^{kr}\right)\bigg|_{r=0} = \frac{d}{dr}\left(s(r)e^{kr}\right)\bigg|_{r=\ell} = 0. \tag{14}$$

and developing the derivative terms:

$$\frac{d}{dr}\left(s(r)e^{kr}\right)\bigg|_{r=0} = \frac{d}{dr}s(r)\bigg|_{r=0} + k\cancelto{0}{s(0)} = 0,$$

$$\frac{d}{dr}\left(s(r)e^{kr}\right)\bigg|_{r=\ell} = e^{k\ell}\frac{d}{dr}s(r)\bigg|_{r=\ell} + ke^{k\ell}\cancelto{0}{s(\ell)} = 0.$$

Therefore, the mapped boundary conditions are:

$$s(0) = s(\ell) = 0 \quad , \quad \frac{d}{dr}s(r)\bigg|_{r=0} = \frac{d}{dr}s(r)\bigg|_{r=\ell} = 0. \tag{15}$$

In this case, one found $k_i = \left(i + \frac{1}{2}\right)\pi$, $(i = 1, 2, \dots)$. It is important to emphasize that the proposed domain transformation does not influence the values of k_i, being the same found in the literature. Two sets of mode shapes, $\{u_i(x)\}$ and $\{s_i(r)\}$, can be defined as follows:

$$u_i(x) = \cosh(k_i x) - \cos(k_i x) - \frac{\cosh(k_i\ell) - \cos(k_i\ell)}{\sinh(k_i\ell) - \sin(k_i\ell)}(\sinh(k_i x) - \sin(k_i x)), \tag{16}$$

$$s_i(r) = e^{-2k_i r} + \frac{1}{\sin(k_i\ell)}\big[- e^{-k_i r}(\sin(k_i\ell - k_i r) + \cos(k_i\ell - k_i r)) - e^{-k_i(\ell+r)}(\sin(k_i r) + \cos(k_i r)) + e^{-k_i\ell} - \cos(k_i\ell)\big]. \tag{17}$$

The procedure illustrated for the clamped-clamped beam is useful and can be developed for other situations. In Table 1 are shown the boundary conditions described in the original domain for four cases: clamped-clamped, simply-supported, free-ends and clamped-free beam. By considering the canonical transformation, the boundary conditions can be mapped in the domain represented by $s(r)$ and the constants $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ of Eq. (12) can be determined, as shown in Table 2 (where $\alpha_1 = 2\sin(k\ell)e^{-k\ell} + e^{-2k\ell} - 1$, $\alpha_2 = e^{k\ell} - e^{-k\ell} - 2\sin(k\ell)$, $\alpha_3 = e^{-k\ell} - \sin k\ell + \cos k\ell$ and $\mathcal{A}_i$ is the constant of mode amplitude).

Table 1. Boundary conditions analyzed.

	clamped-clamped	simply-supported	free-free	clamped-free
$x{=}0$	$u(0){=}\frac{d}{dx}u(x)\big\vert_{x=0}{=}0$	$u(0){=}\frac{d^2}{dx^2}u(x)\big\vert_{x=0}{=}0$	$\frac{d^2}{dx^2}u(x)\big\vert_{x=0}{=}\frac{d^3}{dx^3}u(x)\big\vert_{x=0}{=}0$	$u(0){=}\frac{d}{dx}u(x)\big\vert_{x=0}{=}0$
$x{=}\ell$	$u(\ell){=}\frac{d}{dx}u(x)\big\vert_{x=\ell}{=}0$	$u(\ell){=}\frac{d^2}{dx^2}u(x)\big\vert_{x=\ell}{=}0$	$\frac{d^2}{dx^2}u(x)\big\vert_{x=\ell}{=}\frac{d^3}{dx^3}u(x)\big\vert_{x=\ell}{=}0$	$\frac{d^2}{dx^2}u(x)\big\vert_{x=\ell}{=}\frac{d^3}{dx^3}u(x)\big\vert_{x=\ell}{=}0$

Table 2. Constants $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ occurring in the mapped equation.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
clamped-clamped	$\frac{\mathcal{A}_i}{\alpha_1}e^{-k\ell}(\cos(k\ell){-}\sin(k\ell){-}e^{-k\ell})$	$-\frac{\mathcal{A}_i}{\alpha_1}[e^{-k\ell}(\cos(k\ell){+}\sin(k\ell)){-}1]$	$\frac{\mathcal{A}_i}{\alpha_1}({-}2\cos(k\ell)e^{-k\ell}{+}e^{-2k\ell}{+}1)$	$\mathcal{A}_i$
simply-supported	0	0	$\mathcal{A}_i$	0
free-free	$\frac{\mathcal{A}_i}{\alpha_2}(\cos(k\ell){-}\sin(k\ell){-}e^{-k\ell})$	$\frac{\mathcal{A}_i}{\alpha_2}({-}\cos(k\ell){-}\sin(k\ell){+}e^{k\ell})$	$\frac{\mathcal{A}_i}{\alpha_2}(2\cos(k\ell){-}e^{k\ell}{-}e^{-k\ell})$	$\mathcal{A}_i$
clamped-free	$-\frac{\mathcal{A}_i}{\alpha_3}(\cos(k\ell){+}e^{-k\ell})$	$\frac{\mathcal{A}_i}{\alpha_3}\sin(k\ell)$	$\frac{\mathcal{A}_i}{\alpha_3}(\cos(k\ell){+}\sin(k\ell){+}e^{-k\ell})$	$\mathcal{A}_i$

4. NUMERICAL RESULTS

Numerical experiments are undertaken by considering a beam with a length of $\ell = 1$ m. Here, the remapped solutions $\bar{u}_i(x)$ are compared to the original solutions $u_i(x)$. The so-called remapped solution is computed by considering $s_i(r)$ and the inverse transformation $x = r$ and $\bar{u}_i(x) = s_i(r)e^{k_i r}$. Thus, by the closed-form solutions $u_i(x)$ of Eq. (8) and the mapped ones $s_i(r) = s_i(x)$ of Eq. (12), in addition to their respective constants for each beam situation of Table 2, the first and the tenth mode shapes are displayed (see Figs. 1 to 4).

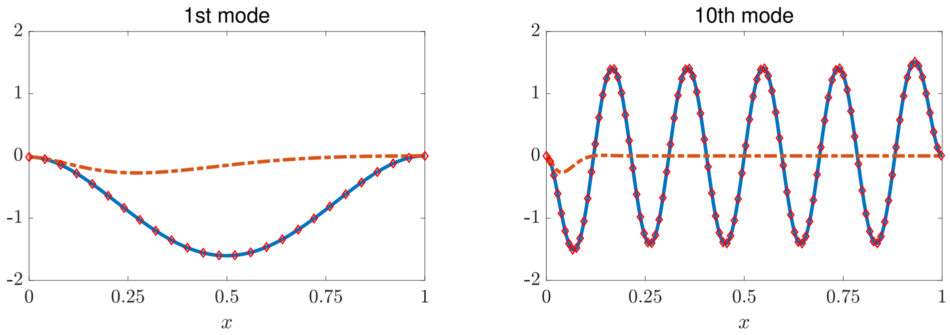

Figure 1. Mode shapes of the clamped-clamped beam.

$u_i(x)$ $s_i(x)$ $\bar{u}_i(x)$

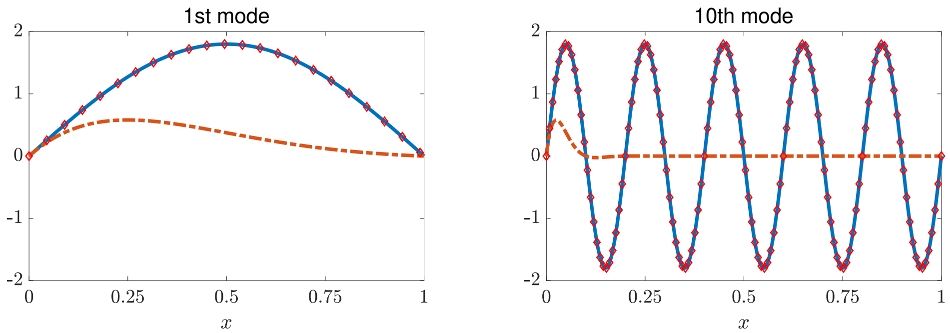

Figure 2. Mode shapes of the simply-supported beam.

$u_i(x)$ $s_i(x)$ $\bar{u}_i(x)$

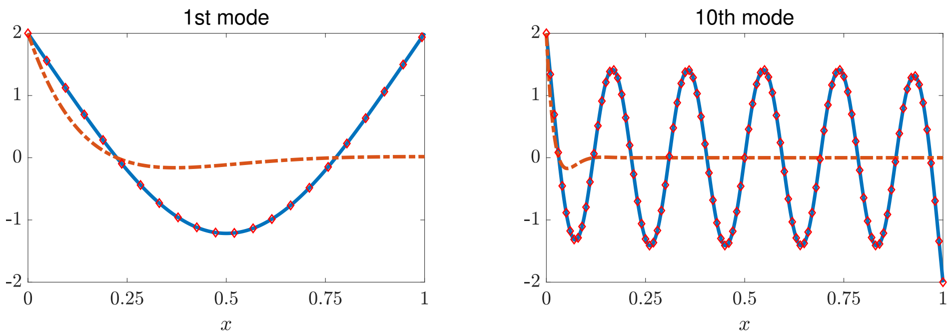

Figure 3. Mode shapes of the free-free beam.

$u_i(x)$ $s_i(x)$ $\bar{u}_i(x)$

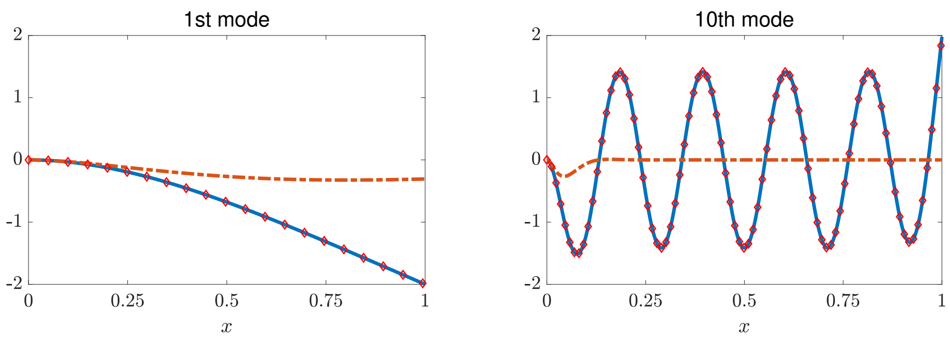

Figure 4. Mode shapes of the clamped-free beam.

$u_i(x)$ $s_i(x)$ $\bar{u}_i(x)$

It is shown that the remapped solutions $\bar{u}_i(x)$, computed by considering the inverse transformation $x = r$ and $u_i(x) = s_i(r)e^{k_i r}$, match the original solutions $u_i(x)$. The interesting feature here is that the solutions $s_i(r)$ slowly vary in space, as shown in Figs. 1 to 4, as opposed to the solutions $u_i(x)$. Hence, the solutions written in the mapped domain exhibit a smooth behavior and they can be accurately computed — i.e., by solving analytical equations (which was the case here), or by considering finite element approaches with coarse meshes — even at high frequencies when high-order modes are dealt with.

5. CONCLUSION

A Lie symmetry approach has been proposed to compute the vibration modes of Euler-Bernoulli beams. Within this framework, an alternative coordinate system $(r, s(r))$ is defined in which the modes shapes exhibit smooth variations. By considering this coordinate system, the vibration modes can be accurately described even at high frequencies when the wavelength is becoming small. Alternatively, the computation of these modes could have been undertaken with the finite element method using coarse meshes only. This opens new interesting prospects towards the analysis of more complex dynamic problems and wave propagation problems (1D and 2D periodic structures) for expressing the dynamic solutions on low-dimensional subspaces (small finite element models) issued from Lie symmetries. In this sense, the computation times could be sped up. Another prospect is on the development of a regularization procedure of the inverse mapping transformation to accurately describe the dynamic solutions in the physical space from those described in the reduced spaces.

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