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Revisiting the double pendulum engine

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Abstract.
We explore the potential of 0-1 test as a tool with fast results for dynamic system analysis. The system analyzed was the low gravity double pendulum $(0.98\ m/s)$*, in which, by its nature, the system has a high sensitivity to the initial conditions and thus a complex dynamics. We analyze the behavior of the 0-1 test to determine the regions that the system may go into chaotic and periodic regime. We also consider the relationships of the length* $l_1 = Ll_2$ *and the masses* $m_1 = Mm_2$ *for the 0–1 test analyzes. After analysis of the regions of the parameters* L *and* M *can be periodic or chaotic. We assign two results-based cases with the 0–1 test, a first periodic case and a chaotic case. That were confirmed with the Lyapunov exponent calculations, Bifurcation diagram and the FFT.*

Keywords: *Dynamics System, 0-1 Test, Lyapunov Exponent*

1. Introduction

The double pendulum is a simple example of a dynamic system, however, it has a complex behavior in which the initial conditions are highly sensitive. Thus, the equations of motion of the system are determined by differential equations that consider the angles formed by the rods, as well as their lengths and the masses that are connected at the ends of the rods **?**Stachowiak and Okada (2006). The fig.1 illustrates a double pendulum scheme.

For our analysis we consider that there is a relationship between the rod lengths, i.e. $l_1 = Ll_2$ and also the system masses, $m_1 = Mm_2$. Thus we can analyze the behavior of variation of these two parameters L and M for system dynamic analysis.

Thus, we will have two parameters that are linked to the system structure, we also consider that the system is under low gravity action, ie $0.98\ m/s^2$. Since the system is sensitive to initial conditions and also a system of complex dynamics, we will apply a qualitative analysis of the 0–1 test to determine the regions where the system is chaotic and periodic. Test 0–1 is a statistical analysis that considers a time series sampling and determines the values of K_{01} which for values close to $[0.8 : 1]$ the system has a chaotic character and for values between $[0 : 0.8]$ is presented periodically. Being a tool for faster dynamic analysis computationally, in relation to the Lyapunov exponent it considers can be calculated with the distances between two very close initial conditions. Or using the process with the Jacobian matrix of the system. Therefore, this paper aims to analyze the behavior of a double pendulum system using test 0-1, the Lyapunov exponent by analyzing the distances between two very close initial conditions, the bifurcation diagrams and the FFT of the systems. Thus, we have proven the power of the 0–1 test for dynamic system analysis.

2. Mathematical Modeling

The mathematical model used was based on the structure given by the Fig. 1, considering that the larger structure of the double pendulum.

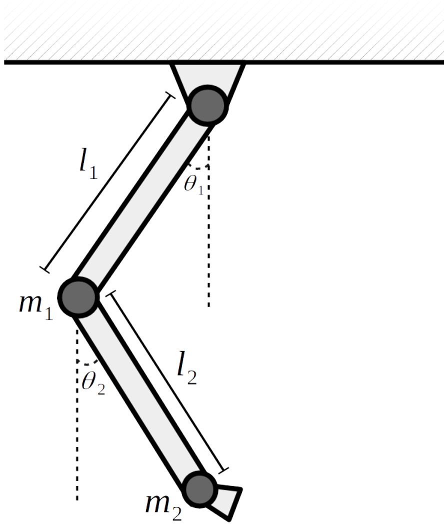

Figure 1. Scheme of double pendulum

The Eqs. 1 describe the angular movement of the arm structure and the angular velocity of the structure.

$$\begin{cases} (m_1+m_2)l_1\ddot{\theta}_1 + m_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + g(m1+m2)\sin(\theta) = 0 \\ m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 g\sin(\theta_2) = 0 \end{cases} \tag{1}$$

Rewriting the variables as: $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$ and $x_4 = \dot{\theta}_2$ and using $a = (m_1+m_2)L_1$, $b = m_2 L_2 cos(x_1 - x_3)$, $c = m_2 L_1 \cos(x_1 - x_3)$, $d = m_2 L_2$, $e = -m_2 L_2 x_2^4 \sin(x_1 - x_3) - g(m_1+m_2)\sin(x_1)$, and $f = m_2 l_1 x_4^2 \sin(x_1 - x_3) - m_2 g \sin(x_3)$ we can write this system as:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \frac{ed-bf}{ad-cb} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \frac{af-ce}{ad-cd} \end{cases} \tag{2}$$

3. 01- Test

The 0–1 Test is applied to a series of temporal data, and is based on statistical properties for the variable x_1 and x_3. Basically, the test considers the system of variables x_j in two new coordinates (p, q) and defined as:

$$p(n,\bar{c}) = \sum_{j=0}^{n} x(j)\cos(j\bar{c}) \tag{3}$$

$$q(n,\bar{c}) = \sum_{j=0}^{n} x(j)\sin(j\bar{c})$$

where $\bar{c} \in (0,\pi)$ is a constant. The average square displacement of the new variables $p(n,\bar{c})$ and $q(n,\bar{c})$ is given by:

$$M(n,c) = \lim_{n\to\infty} \frac{1}{N}\sum_{j=1}^{N}\left[(p(j+n,\bar{c}) - p(j,\bar{c}))^2 + (q(j+n,\bar{c}) - q(j,\bar{c})^2)\right] \tag{4}$$

where $n = 1, 2, \ldots, N$ and, therefore, we obtain the parameter K in the limit of a very long time:

$$K = \frac{cov(Y, M(\bar{c}))}{\sqrt{(var(Y)var(M(\bar{c}))}} \tag{5}$$

where array $M(\bar{c}) = [M(1,\bar{c}), M(1,\bar{c}), \ldots, M(n_{max},\bar{c})]$ and $Y = [1, 2, \ldots, n_{max}]$. In this way, the vectors x and y, we calculate the covariance $cov(x, y)$ and the variance $var(x)$, of n_{max} elements, are defined as,

$$cov(x,y) = \frac{1}{n_{max}} \sum_{n=1}^{n_{max}} [x(n) - \bar{x}]\,[y(n) - \bar{y}] \tag{6}$$

and

$$var(x) = cov(x,y) \tag{7}$$

where $\bar{x}$ and $\bar{y}$ are the average of $x(n)$ and $y(n)$, respectively. As a final result, the value of the searched parameter K is obtained taking the median of 100 different values of the parameter $\bar{c} \in (0, \pi)$, in Eq. 5. If the K value is close to 0 the system is periodic, on the other hand, if K value is close to 1 the system is chaotic. In all simulations we have chosen $n = 10000$ and $j = \frac{n}{100}, \ldots, \frac{n}{10}$.

Thus, we can statistically determine the regions for a set of system parameters, where we can find a possibly chaotic or possibly periodic system. In this way, being an efficient test to determine such behaviors of a system. Gottwald and Melbourne (2009, 2004).

4. Lyapunov Exponent

For a system of n ordinary differential equations with a hypersphere of initial conditions centered in $\vec{x}(t_0)$ Eckmann *et al.* (2018); Kantz (1994) . In this way, as time passes, this volume deforms. Assuming that, over the j-th size, the initial radius $d_j(t_0)$ has varied exponentially in time, so that the relation between $d_j(x_0)$ and the corresponding value at time t, given by $d_j(t)$, has a value:

$$d_j(t) = d_j(t_0)e^{\lambda_j(t-t_0)} \tag{8}$$

where $j = 1, 2, \dot{;} n$.

Rewriting Eq. 8 and considering $t \to \infty$

$$\lambda = \lim_{t\to\infty} \lim_{d_j(t_0)\to 0} \frac{1}{t} \ln\left[\frac{d_j(t)}{d_j(t_0)}\right] \tag{9}$$

Therefore, the values of λ_j obtained by Eq. 8 are the Lyapunov exponents Wolf *et al.* (1985).

5. Results

For numerical simulations, we use the following initial conditions $[0, \frac{\pi}{4}, 0, \frac{\pi}{4}]$ and parameters m_1, $m_2 = Mm_1$, l_1, $l_2 = Ll_1$. where $L \in [0.35; 0.7]$ and $M = [0.1; 1.0]$. We adopted the initial condition as a point at which the system was operating under low gravity, the recurrence of low gravity consists of a lower potential energy which affects the movement of the system. Thus, we used the 0-1 test which made a statistical analysis of the time series varying two system parameters, which is L for the pendulum shank length and M, the system mass. Thus, through test 0-1 we will have the possible regions in which the system is chaotic or periodic. For this, according to Bassinello *et al.* (2016) the regions in which $K_{01} \in [0.8 : 1.0]$. The systems can be found periodically. In the figure 2 represents the behavior of the test 01 K_{01} with the variation of the parameters L and M.

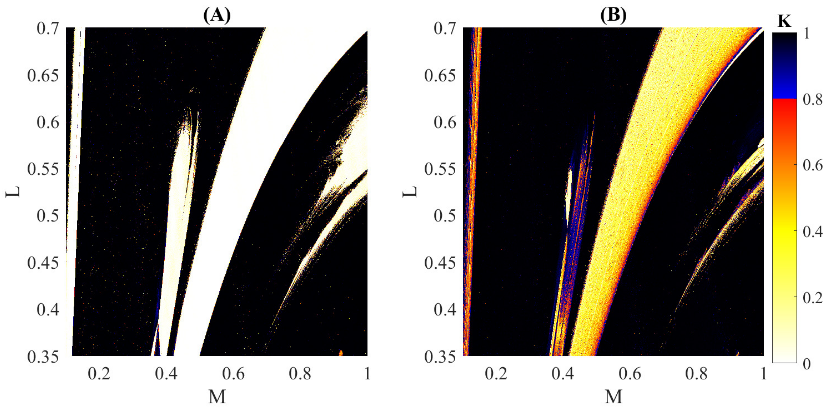

Figure 2. Graph of 0-1 Test bidimensioanl L e M of Eqs.2. (A) K_{u_2} and (B) K_{v_2}

Therefore, for a better analysis we have calculated the fork diagrams to the value of $L = 0.52$ cm of the system. Thus, we obtained the following fork diagrams contained in the figure 3 with $L = 0.52$ and sweeping the M parameter, i.e. the system mass ranging from 0.1 kg to 1 kg. We can observe a correspondence between the regions found in test 01 and the bifurcation bifurcation diagram. That is, we can observe the regions where the system is periodic and chaotic, there is agreement with the results obtained with the test-01. Figure 3 shows the behavior of test 01 for $L = 0.52$ and by scanning the $M \in [0.1 : 1.0]$ parameter, it can be seen that $0.11 < M < 0.141$, $0.411 < M < 0.452$, $0.46 < M < 0.47$, $0.532 < M < 0.673$ and $0.95 < M < 0.99$ have had periodic system behavior.

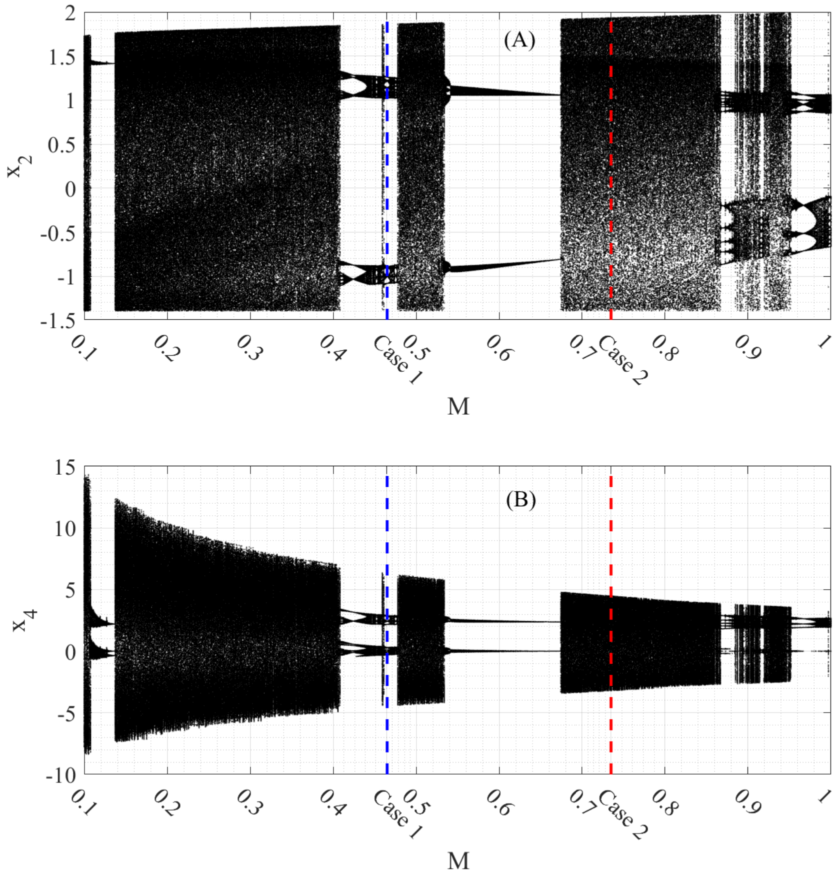

Figure 3. Bifurcation Diagram for of $L = 0.5248$ (A) x_2 vs M parameter and (B) u_4 vs L parameter. With Case 1 in dotted line blue and Case 2 in dotted line red.

Fig. 4 presents the Lyapunov exponent of the structure using eq. 9, we can see that when comparing with the fork diagram, there is a correspondence between regions where the system is chaotic with $\lambda_{x_1} > 0$ and periodic with $\lambda_{x_1} < 0$. Lyapunov exponents were calculated using the equation **??**, that is, based on the divergences of two very close initial conditions. Approximation of initial conditions was around 10^{-9}. The regions that presented themselves as periodic were: $0.1 < M < 0.13$, $0.4 < M < 0.45$, $0.461 < M < 0.473$, $0.5353 < M < 0.6739$ and $0.952 < M < 0.992$

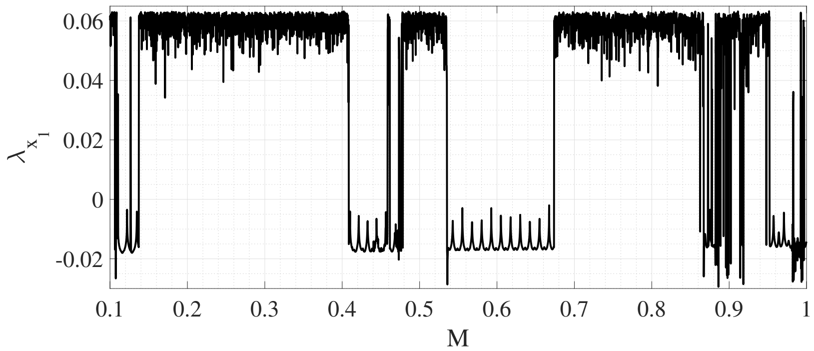

Figure 4. Lyapunov exponent for range M.

Fig.4 presents the Lyapunov exponent of the structure using eq. 9, we can see that when comparing with the fork diagram, there is a correspondence between regions where the system is chaotic with $\lambda_{x_1} > 0$ and periodic with $\lambda_{x_1} < 0$. Lyapunov exponents were calculated using the equation **??**, that is, based on the divergences of two very close initial conditions. Approximation of initial conditions was around 10^{-9}. The regions that presented themselves as periodic were: $0.1 < M < 0.13$, $0.4 < M < 0.45$, $0.461 < M < 0.473$, $0.5353 < M < 0.6739$ and $0.952 < M < 0.992$

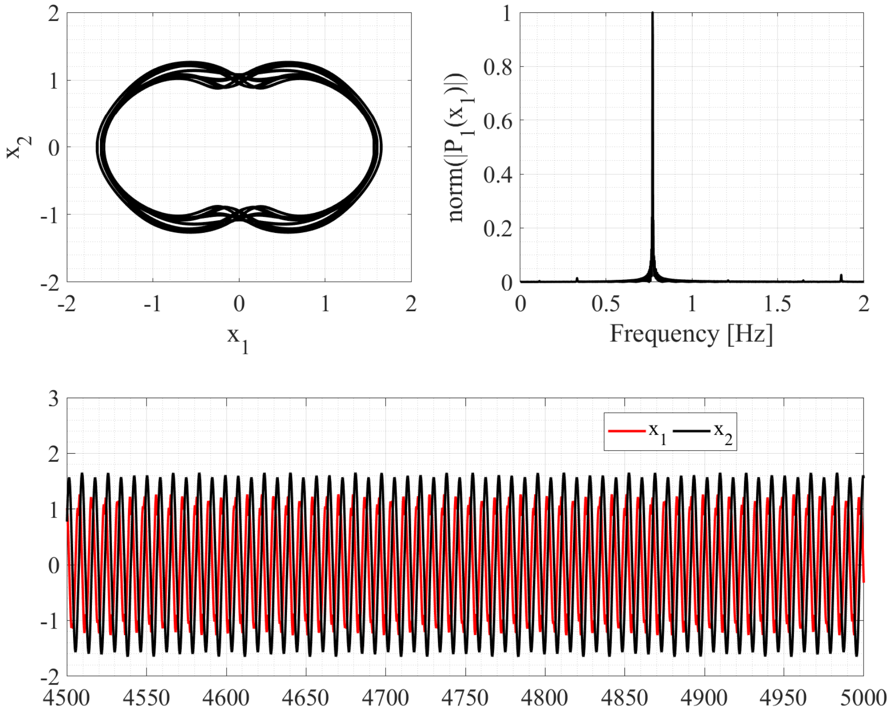

Figure 5. (A) Phrase Portrait, (B) FFT and (C) Time Series. With $L = 0.4658$, $M = 0.5248$.

The second case, we consider the following parameter $L = 0.52$, and following Lyapunov's exponent, test 01 and the bifurcation diagram the system proved chaotic. Thus, fig. 6. (A) represents the phase map, 6 (B) represents the FFT graph, we can observe the resulting multi-frequencies of the system. In fig. 6 (C) represents the time series of the system.

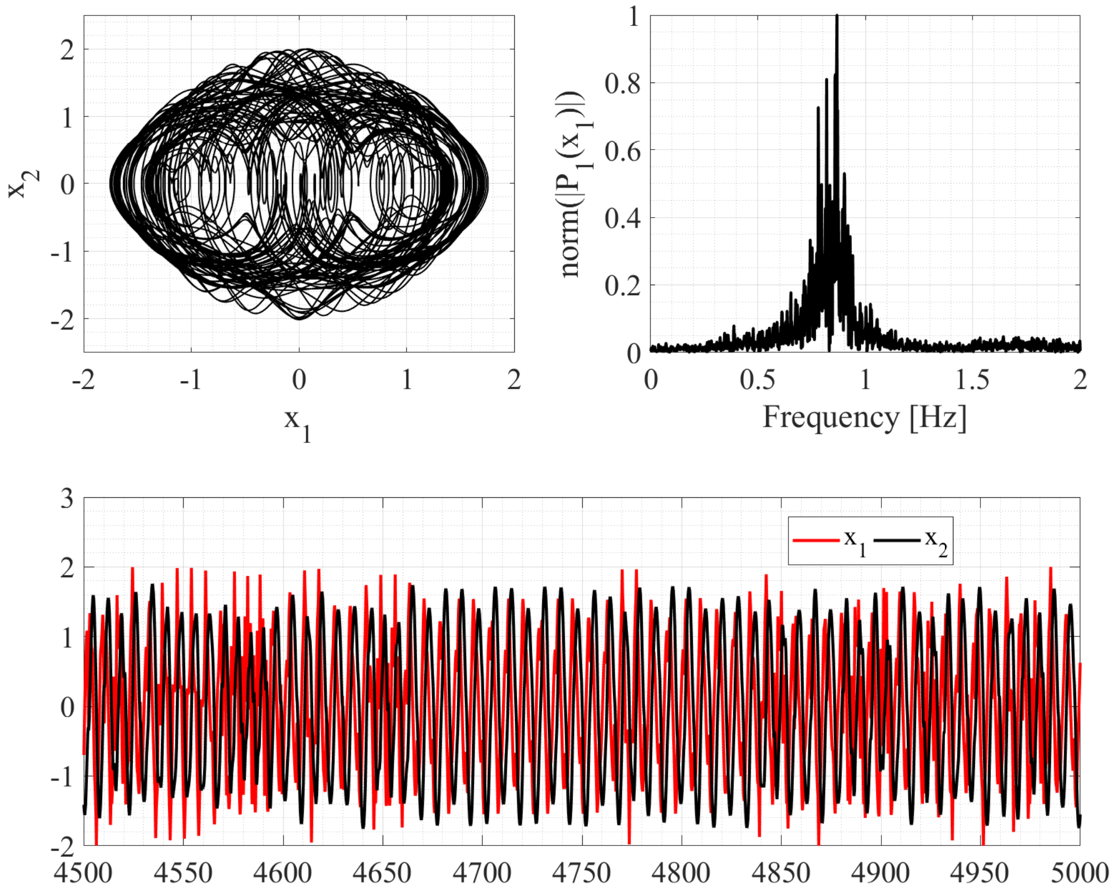

Figure 6. Representation of trajectory. (A) Phrase Portrait, (B) FFT and (C) Time Series. With $L = 0.7350$ and $M = 0.5248$.

6. Conclusion

In this paper, we investigated the simple pendulum system by applying the 0-1 test to identify regions with periodic and chaotic potentials. Thus, we choose a value of $L = 0.52$ for a scan of the mass M of the system. With this, we analyze the Lyapunov exponent, where for $L = 0.52$ and by scanning the M parameter we determine the periodicity intervals that the system has, along with the bifurcation diagrams, which have the same intervals. This was confirmed with the time series of the systems and the Fourier transforms. Therefore, we conclude that the 0-1 test is a great tool to obtain the parametric regions that the system has a chaotic or periodic regime. In which, they were confirmed with classical dynamic analysis tools such as the Lyapunov exponent, bifurcation diagram and the Fourier Transform Spectrum.

7. ACKNOWLEDGEMENTS

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