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REVISITING AN ELLIPTIC NEARLY-ORTHOGONAL GRID GENERATION TECHNIQUE

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Abstract. *A nearly-orthogonal boundary-conforming grid generation technique based upon inhomogeneous elliptic partial differential equations is designed to address the mapping problem corresponding to the construction of a coordinate system fitted to a boundary of given shape, with a prescribed distribution of coordinate lines along all boundaries. The method for obtaining the inhomogeneous terms used to generate near-orthogonality along the boundaries is presented and discussed. The resulting methodology have been proven to be a viable alternative to deal with typical problems of elliptic grid generation systems, such as the tendency of mesh squeezing next to concave and mesh spreading next to convex boundaries. The technique is tested using a benchmark case and it is evaluated based upon standard academic criteria. The results suggest the technique to be flexible, efficient and reliable and, as such, an interesting asset, especially for applications to complex geometries.*

Keywords: *Boundary-conforming coordinate systems; Inhomogeneous elliptic partial differential equations; Nearly-orthogonal grid*

1. INTRODUCTION

Grid generation is the process to obtain an optimized collection of points in the sense that it minimizes the error in the discretization and solution of an equation or system of equations. Additionally, computational time relies on the solution of large algebraic systems and, as such, depends upon the locations of the non-zero elements, that is, the sparseness of the coefficient matrices. The sparseness, in its turn, depends both on the discretization schemes employed as well as on the arrangement of the nodal points. Sophistication of the discretization schemes is usually chosen a priori based upon the stability and accuracy requirements of a particular application. Mesh generation can provide for a well-structured arrangement of nodal points, granting for well-structured sparse matrices with narrow bandwidths. Such a feature can be well exploited by iterative solution methods (Gordon and Hall, 1973). With this objective, boundary-conforming coordinate systems have been used with success as grid generation systems (Thompson *et al.*, 1985). Ryskin and Leal (1983) argue that one additional advantage in the use of boundary conforming coordinate systems is that finite difference approximations for the equations of the problem of interest are solved on an uniform grid in the unit square $0 \leq \xi, \eta \leq 1$, and that uniform grids are preferable for the production of finite-difference schemes of high accuracy. Boundary conforming coordinate systems are obtained as the solution of boundary-value problems. Algebraic interpolation and conformal mapping are potential candidates. Such methods usually lead to great simplicity of implementation and low computational cost, although they may penalize applications for complex geometries (Ryskin and Leal, 1983). For example, grid folding has been reported in applications of algebraic interpolation methods by Gordon and Hall (1973), and subtle changes in geometry have been reported to dramatically alter the position of mapped boundary points in applications of conformal mapping (Ryskin and Leal, 1983). However, boundary value problems configure a classical category of problems in the field of partial differential equations, and obtaining the grid points as the solution of such a problem is a viable alternative (Steger and Sorenson, 1979). The authors argue that elliptic grids appear to be the most flexible for generating grids about surfaces. In fact, the smoothness inherent in harmonic functions, and the uniqueness of the solution guaranteed by the maximum principles, tend to force that discontinuities and/or strong slopes present in the boundaries do not propagate inside the domain, what is desired for diminishing truncation errors in finite difference approximation for derivatives. Thompson *et al.* (1985) emphasize that, although the extremum principles may be weakened, or may even not be achieved by using more general elliptic systems (inhomogeneous systems), the conditions necessary for the extremum principles consist on a set of sufficient, but not necessary, conditions for a one-to-one mapping. Thus, as systems in the absence of control functions will originate grids that are not in general orthogonal to the boundaries, as well as with no control over grid spacing, it becomes more convenient to use inhomogeneous elliptic generators.

Although some might consider grid generation a solved issue, this is not really true for some specialized applications, such as multiply-connected domains with heterogeneous media, and complex geometries. This is emphasized by the recent interest in the subject. Additionally, great contributions have been made recently. Kaul (2003) develops a "full automatic" methodology for grid generation based upon inhomogeneous elliptic partial differential equations in which the user is not required to specify any of the decay parameters. The method imposes boundary constraints based on an analogy to energy conservation that obviates the need for empirical decay parameter specification. Kaul (2010) extends this methodology to 3D applications. Since mesh generation usually involves both geometric and physical information of the application in question, and automatic grid generation methods consider only geometric aspects, we refrain from calling "automatic" such methods since the actual final mesh will ultimately be resolved by the analyst who is cognizant of its specific application. Zhang *et al.* (2008) present a method for applying Neumann-Dirichlet boundary conditions in 2D elliptic grid generation for complex geometries. The imposition of Neumann-Dirichlet orthogonality is proposed to be used in combination with Dirichlet orthogonality, with the definition of a priority criterion that defines to which boundaries each boundary condition is applied to. The method does not generally provide for orthogonal grids inside the domain, as may be inferred from statement made by the authors, but for boundary orthogonal meshes. Zhang *et al.* (2012) improve the methodology by the use of smoothing functions for relaxing the local orthogonality conditions in order to obtain softer grids. Bihlo and Haynes (2014) present an elliptic partial differential equations-based method for the construction of adaptive meshes in two dimensions. Wiesenberger *et al.* (2017) present a grid adaption method based on the monitor metric approach that uses weight functions and monitor metrics to control the distribution of cells across the domain. The method proposed is applicable to domains with boundaries defined by two contour lines of a two-dimensional function.

2. EQUATIONS DEFINING THE MAP

As discussed by Bishop and Goldberd (1980), manifolds configure the basic objects of study of mathematical models for many physical systems, upon which further structure may be defined to obtain whatever system is in question. A manifold has a dimension, and as a model for a physical system this is the number of degrees of freedom. Consequently, the number of degrees of freedom m in choosing the mapping functions is 2 in 2-D and 3 in 3-D. This means that m constraints can be imposed on the components of the metric tensor in order to obtain different characteristics in the constructed coordinate system. The metric tensor comes from Riemann's geometric postulate (Thompson *et al.*, 1985),

$$(\mathrm{d}s)^2 := \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \mathbf{a}_i \cdot \mathbf{a}_j \mathrm{d}\xi \mathrm{d}\eta, \tag{1}$$

where $\mathrm{d}s$ is the increment to the arc length and $\mathbf{r}$ is the vector of coordinates in M_p, that is $\mathbf{r} = (x, y)^t$. The (*covariant*) metric tensor a_{ij} is the symmetric tensor

$$a_{ji} = a_{ij} := \mathbf{a}_i \cdot \mathbf{a}_j. \tag{2}$$

Although the metric tensor (being symmetric) generally exhibit three independent components in 2-D, the m constraints referred to above are the maximum number that can be imposed if the space described by the resulting coordinate system is to be Euclidean. The condition that the space is Euclidean ("flat") corresponds to the Riemann curvature tensor of the coordinate system to be zero, thus the restriction to an Euclidean space imposes one constraint on a_{ij}, which lowers the number of freely specifiable constraints to two (Ryskin and Leal, 1983). Since the number of independent non-diagonal components of the metric tensor $a_{i,j}$ in 2-D is one, the orthogonality constraint ($a_{ij} = 0, i \neq j$) can be imposed to ensure the resulting coordinate system to be locally orthogonal.

The elliptic systems arise from the trivial observation that the Cartesian coordinates (x, y) in the physical plane are linear scalar functions of position. This means that ∇x and ∇y are constant valued vector fields, from which follows $\Delta \cdot \nabla x = 0$ and $\Delta \cdot \nabla y = 0$ everywhere in the physical space. In the Euclidean space, the operator $\Delta \cdot \nabla$ corresponds to the Laplacian operator and is usually denoted by ∇^2. This operator can be written in explicit form for any particular coordinate system, including the one we want to construct, that is

$$\nabla^2 \boldsymbol{\zeta} = 0, \tag{3}$$

where $\boldsymbol{\zeta} = (\xi, \eta)^t$. Inhomogeneous terms have been proposed to be used with such a system by Thompson *et al.* (1974) in order to obtain control over the mapping. In fact, Spekreijse (1995) argue that a elliptic grid generation technique has its relevance determined by the method used to calculate the appropriate inhomogeneous terms, which were called *control functions* by Thompson *et al.* (1974) and *forcing functions* by Eiseman (1979). It corresponds, of course, to using more comprehensive generator systems. The elliptic system in the presence of inhomogeneous terms is of the form

$$\nabla^2 \boldsymbol{\zeta} = \mathbf{Z}(\boldsymbol{\zeta}), \tag{4}$$

where $\mathbf{Z} = (P, Q)^t$.

Suppose that both the harmonic and the grid control maps are defined, such that the composite map exists (Spekreijse, 1995). Under this set of conditions, it is possible to demonstrate that the Laplacian of a generic function ϕ of class C^2 under the coordinate transformation becomes

$$\nabla^2\phi = a^{11}\phi_{\xi\xi} + 2a^{12}\phi_{\xi\eta} + a^{22}\phi_{\eta\eta} + \nabla^2\xi\phi_\xi + \nabla^2\eta\phi_\eta, \tag{5}$$

where $a^{ij}, i, j = \{1, 2\}$ are the contravariant tensor components. Chu (1971) has shown that the system of partial differential equations does not change its mathematical classification under the transformation. In fact, Eq. (5) is an asymmetric quasilinear elliptic partial differential equation (Evans, 2010). If in Eq. (5) ϕ is the vector of coordinates in the physical space, different elliptic generators may be obtained based on the choice for the terms $\nabla^2\xi$ and $\nabla^2\eta$. Using for example Eq. (4), one gets

$$a_{22}\mathbf{r}_{\xi\xi} - 2a_{12}\mathbf{r}_{\xi\eta} + a_{11}\mathbf{r}_{\eta\eta} + J^2(P\mathbf{r}_\xi + Q\mathbf{r}_\eta) = 0. \tag{6}$$

The covariant metric components are obtained from Eq. (1) as $a_{11} = \mathbf{r}_\xi \cdot \mathbf{r}_\xi$, $a_{12} = \mathbf{r}_\xi \cdot \mathbf{r}_\eta$, and $a_{22} = \mathbf{r}_\eta \cdot \mathbf{r}_\eta$. J is the Jacobian of the transformation, that is $J^2 = a_{11}a_{22} - (a_{12})^2$. Such a system has been widely used, as reported for example by Thompson *et al.* (1982) and Kaul (2003).

If no inhomogeneous terms are used (this is, $\mathbf{Z} = 0$), the equations become a system of Laplace's equations, whose solution is called a set of harmonic functions. In the case of inhomogeneous elliptic equations, the solution is called subharmonic if $\mathbf{Z} > 0$ and superharmonic if $\mathbf{Z} < 0$. According to Mastin and Thompson (1978a,b), the maximum and minimum principles, this is, that extremum (maximum for subharmonic and minimum for superharmonic functions) of a solution may not occur inside the domain, form important properties of elliptic systems. They prevail under the requirements that the solution is of class C^2, at least continuous on the boundaries, and that the domain is bounded. These properties inherent in elliptic systems enable the solution to present no extremum inside the domain, assuring uniqueness of the solution. Therefore, it is possible to insure the mapping between the physical and computational domains to be one-to-one, guaranteeing non-overlap of lines of the same curvilinear coordinate. In addition, Lewy (1936) demonstrates that, if the mapping is harmonic, the Jacobian of a one-to-one transformation does not vanish. In fact, Heinz (1959) shows that mappings satisfying the system of Eqs. 3 have Jacobians which not only do not vanish, but may have a minimum positive value under the refereed circumstances.

As discussed in several texts, such as Thompson *et al.* (1974) and Thompson *et al.* (1985), another important property to grid generation based upon elliptic systems is the intrinsic smoothness. It means that elliptic generators yield grids smoother than the initial grids, in which eventual grid foldings in the initial grid are smoothed out. In fact, Thompson *et al.* (1985) show through variational calculus that grids obtained using Laplace's equations are the smoothest possible; such characteristics are also expected for grids obtained with Eq. (6), although less pronounced. Such a property tends to force discontinuities and/or strong slopes present in the boundaries not to propagate inside the domain. Smoothness is also desirable, as argue Noack and Anderson (1990), for diminishing truncation errors in the solution. Another, usually undesired, characteristic of grids obtained from Laplace's equations, extensively discussed by Thompson *et al.* (1985) and particularly well summarized by Thompson (1987), consists in the fact that, by the very nature of the equations, lines tend to concentrate in regions next to convex boundaries, and spread in concave regions. Thompson *et al.* (1985) emphasize that, although the extremum principles may be weakened, or may even not be achieved by using more general elliptic systems, the aforementioned conditions form a set of sufficient, but not necessary, conditions for a one-to-one mapping. Furthermore, Khaymaseh *et al.* (1999) argue that systems in the absence of control functions will originate grids that are not in general orthogonal to the boundaries, as well as with no control over grid spacing. Thus, it becomes more convenient to use inhomogeneous elliptic generators.

3. CONTOUR ORTHOGONALITY

3.1 General Aspects

Khaymaseh *et al.* (1999) report that experience in the field of computational simulation has been demonstrating that grid quality in terms of smoothness and orthogonality may have strong impact in the numerical solution. Particularly in the field of computational fluid dynamics, orthogonality or near-orthogonality next to boundaries is usually desirable. According to Wiesenberger *et al.* (2017), although it is possible to represent Neumann boundary conditions in a curvilinear nonorthogonal grid, the accuracy of the discretization deteriorates. In case of a boundary corresponding to a wall, orthogonality and reduced grid spacing next to the wall are necessary in order to diminish the errors in the calculation of the large gradient of properties present in the boundary layer. Thompson *et al.* (1985) demonstrate that grid skewness increases the truncation error in numerical calculation of derivatives using the finite difference method. Still according to the authors, implementation of algebraic turbulence models becomes more reliable, once the information normal to the boundary is usually needed in such models. Similarly, algorithms that solve the parabolized Navier-Stokes equations require coordinate lines to approximate streamlines, and the normal to them, especially next to solid boundaries.

According to Khaymaseh *et al.* (1999), boundary orthogonality can be effected in two ways. In *Dirichlet-Neumann Orthogonality*, control functions are not necessary. In this case, the points along the boundary are allowed to slide until boundary orthogonality is achieved, and the elliptic system has iterated to convergence. This approach is typically recommended for non-physical contours, such as far field boundaries, since it usually modifies the grid spacing near the boundary. In *Dirichlet Orthogonality*, control functions (then called orthogonal control functions) are used to insure orthogonality next to the boundaries while the initial distribution of points along the boundary is kept. Zhang *et al.* (2008) explain that, compared to Dirichlet orthogonality, Dirichlet-Neumann orthogonality is more difficult to use especially for cases with highly irregular boundaries. It has been reported for example boundary curvature differences obtained by the imposition of Dirichlet-Neumann orthogonality in applications to academic geometries widely used in the literature, as well as difficulties in the control of the nodal movements along boundaries. Both limitations are not tolerable for most applications (Zhang *et al.*, 2008). As discussed by Kaul (2003), strictly orthogonal grids can only be generated with partial control over the mesh spacing, while nearly orthogonal grids allow for total control of the mesh spacing; such distinction is what is also called in the literature respectively as strong and weak constraint formulations (Eça, 1996). Still according to Kaul (2003), this is in keeping with the observation that with Dirichlet-Neumann boundary conditions, the grids generated near the boundaries are strictly orthogonal at the boundaries, but only nearly orthogonal with Dirichlet boundary conditions prescribed at those boundaries.

Several methods for obtaining grid orthogonality (or near-orthogonality) next to boundaries are described by Khaymaseh *et al.* (1999), and a particularly interesting one for imposing Dirichlet orthogonality was designed by Khaymaseh and Mastin (1996). This method differs from the one proposed by Sorenson (1980), which originally proposed Dirichlet Orthogonality, by skipping the need for the user to specify the normal spacing to the boundary. In Khaymaseh and Mastin's approach, normal spacing from the initial grid is held.

3.2 Orthogonal Functions

The method proposed here for obtaining grid near-orthogonality next to boundaries is based on the method designed by Khaymaseh and Mastin (1996) but, instead of using the generation system used by the authors, one would use Eq. (6). For determining the orthogonal functions, orthogonality condition is imposed, $a_{12} = \mathbf{r}_\xi \cdot \mathbf{r}_\eta = 0$, such that the Jacobian of the transformation becomes $J^2 = a_{11}a_{22} - (a_{12})^2 = a_{11}a_{22}$.

The system of Eqs. (6) is now given by

$$a_{22}\mathbf{r}_{\xi\xi} + a_{11}\mathbf{r}_{\eta\eta} + a_{11}a_{22}(P\mathbf{r}_\xi + Q\mathbf{r}_\eta) = 0. \tag{7}$$

Taking the scalar product of (7) with $\mathbf{r}_\xi$, one will get

$$a_{22}(\mathbf{r}_{\xi\xi} \cdot \mathbf{r}_\xi) + a_{11}(\mathbf{r}_{\eta\eta} \cdot \mathbf{r}_\xi) + a_{11}a_{22}(P\mathbf{r}_\xi \cdot \mathbf{r}_\xi + Q\mathbf{r}_\eta \cdot \mathbf{r}_\xi) = 0. \tag{8}$$

Using the orthogonality condition again, one can then solve for P to get

$$P_O = -\left[\frac{\mathbf{r}_{\xi\xi} \cdot \mathbf{r}_\xi}{(a_{11})^2} + \frac{\mathbf{r}_{\eta\eta} \cdot \mathbf{r}_\xi}{a_{11}a_{22}}\right]. \tag{9}$$

For Q_O, the procedure is analogous. The orthogonal functions are calculated on the boundaries of the domain, and the quantities necessary for calculating P_O and Q_O demand for a special procedure. This is described in the next section.

3.3 Methodology

3.3.1 Initial Grid: The Parabolic Generator

The iterative procedure necessary in elliptic grid generation methods require an initial condition to start with. Although algebraic generators are usually chosen for the initial grid, due to its low computational cost and ease of implementation, such methods may incur in grids with overlap of lines of the same curvilinear coordinate, which may not be problematic for elliptic generator of Laplace type, but may characterize a source of problems for an inhomogeneous generator since, as pointed by Spekreijse (1999) and Fletcher (1996), in this case the range of initial conditions that would lead to convergence is substantially reduced.

The method chosen for obtaining the initial grid was the parabolic grid generation technique proposed initially by Noack and Anderson (1990), but used as described in Uller and Azevedo (1991). This method was chosen based on its characteristic which combines smoothness and mesh generation capacity with all contours specified, inherent in elliptical generators, with the efficiency of hyperbolic generators. Thus, the option for the parabolic grid as the initial condition will have all of its advantages, such as smoothness and no grid folding, at an encouraging computational cost.

One property of the parabolic generator has encouraged a modification to the methodology proposed by Khaymaseh and Mastin (1996) for obtaining contour near-orthogonality. Noack and Anderson's parabolic generator does not allow the user to control the grid normal spacing next to all boundaries. Thus, since the initial grid spacing next to boundaries is maintained in Khaymaseh and Mastin's method, an inadequate grid spacing may be obtained when both techniques are combined. Therefore, Khaymaseh and Mastin's method has been modified to overcome this problem.

3.3.2 "Left" and "Right" Boundaries

For $\xi = 1$ ("left" boundary), derivatives in the other coordinate, $\mathbf{r}_\eta$ and $\mathbf{r}_{\eta\eta}$, and $a_{22} = \|\mathbf{r}_\eta\|$ are calculated using centered finite difference approximations that only need points along the boundary, which are fixed and do not change during the iterative process.

Approximations for $\mathbf{r}_\xi$, $\mathbf{r}_{\xi\xi}$ and $a_{11} = \|\mathbf{r}_\xi\|$ can not be calculated using information on the boundaries only, and need information regarding the normal spacing to the boundary. In order to calculate them, the orthogonal projection through the boundary is calculated for every point of the first internal line, with exception to corner points; such mirrored points are the ghost points. To achieve that, an one-sided approximation for the first derivative $(\mathbf{r}_\xi)_{1,j}$ using only the internal point $\mathbf{r}_{2,j}$ and the boundary point $\mathbf{r}_{1,j}$ is calculated:

$$(\mathbf{r}_\xi)_{1,j} = \mathbf{r}_{2,j} - \mathbf{r}_{1,j} + \mathcal{O}(\Delta\xi). \tag{10}$$

The ghost points are obtained by the orthogonal projection of the points present on the first internal line through the boundary line. In order to obtain such points, the finite difference approximation for the derivative in the normal direction to the boundary is necessary. The derivative in the normal direction to the boundary $\vec{r}_\xi^{\,ort}$ can then be obtained by the orthogonal projection in the direction of the unitary normal vector to the boundary, $\vec{n}$, this is,

$$\mathbf{r}_\xi^{ort} = \mathbf{n}\mathbf{n}^t\mathbf{r}_\xi, \tag{11}$$

where the unitary normal vector $\mathbf{n}$ can be shown to correspond to

$$\mathbf{n} \equiv \frac{\mathbf{r}_\xi}{\|\mathbf{r}_\xi\|} = \frac{y_\eta\mathbf{i} - x_\eta\mathbf{j}}{\sqrt{x_\eta^2 + y_\eta^2}}. \tag{12}$$

Using this result, one gets

$$\mathbf{r}_\xi^{ort} = \mathbf{n}\mathbf{n}^t\mathbf{r}_\xi = \mathbf{n}(\mathbf{n}\cdot\mathbf{r}_\xi) = (y_\eta x_\xi - x_\eta y_\xi)\frac{y_\eta\mathbf{i} - x_\eta\mathbf{j}}{a_{22}}. \tag{13}$$

It should be noted that for calculating $\mathbf{r}_\xi^{ort}$ the information needed only regards to the boundary points, which do not change, and to the internal points, in this case from past iteration. Such modification on the method proposed by Khaymaseh and Mastin (1996) is a manner to work around the characteristic of the parabolic grid (used as initial grid) that implies in grids with not adjustable grid spacing normal to all boundaries; Khaymaseh and Mastin's method conserves the initial grid spacing next to all boundaries. With the new approach, the spacing becomes adjustable by means of the calibration of the orthogonal functions. This can be done either before or during the iterative process: in case of a more experienced analyst, familiar with the expected behavior of the orthogonal functions in some cases, it can be performed previously to the iterative process; otherwise, the user may, by performing a short number of iterations, be able to tell if the grid is converging to a good enough result or if adjustments are necessary. The whole procedure will be clarified later, in a forthcoming article with the full methodology.

Continuing the orthogonal projection operation, the ghost point is found by Taylor expanding the first derivative on the normal direction to the boundary, replacing it with $\mathbf{r}_\xi^{ort}$ and solving for $\mathbf{r}_{0,j}$, that is,

$$\mathbf{r}_{ghost,j}^{(1)} = \mathbf{r}_{1,j} - (\mathbf{r}_\xi^{ort})_{1,j}. \tag{14}$$

On the evaluation of the orthogonal control functions, $\mathbf{r}_{\xi\xi}$ should be calculated using second order centered difference approximation using a ghost point, a boundary point and the first internal point, updated iteratively. The value of $\mathbf{r}_\xi$ used for calculating P_O and Q_O is found by first order finite difference approximation, using the first internal point form the past iteration, namely

$$(\mathbf{r}_\xi)_{1,j}^n = \mathbf{r}_{2,j}^{n-1} - \mathbf{r}_{1,j} + \mathcal{O}(\Delta\xi), \tag{15}$$

where n is the iteration number. The metric coefficient a_{11} is calculated as

$$a_{11} = \left[(x_\xi^{ort})_{1,j}\right]^2 + \left[(y_\xi^{ort})_{1,j}\right]^2. \tag{16}$$

For the $\xi = M$ ("right") boundary, where M is the number of points in ξ direction, the process follows the one adopted for $\xi = 1$: derivatives in η coordinate are calculated by second order centered finite difference approximations using only boundary points, and do not change during the iterative process. Derivatives in ξ direction do not depend only on boundary points, and are calculated as for $\xi = 1$ contour:

$$(\mathbf{r}_\xi)_{M,j}^n = \mathbf{r}_{M,j} - \mathbf{r}_{M-1,j}^{n-1} + \mathcal{O}(\Delta\xi). \tag{17}$$

Equation (17) is then used to calculate $\mathbf{r}_\xi^{ort}$, which is used to find the ghost point as

$$\mathbf{r}_{ghost,j}^{(M)} = \mathbf{r}_{M,j} + (\mathbf{r}_\xi^{ort})_{M,j}. \tag{18}$$

The second derivative $\mathbf{r}_{\xi\xi}$ is evaluated using second order centered difference approximation as before, using the internal point update iteratively, a boundary point and the ghost point. a_{11} is also calculated as before, but now using information at $\xi = M$.

3.3.3 "Top" and "Bottom" Boundaries

For $\eta = constant$ boundaries, ξ and η change roles. Now $\mathbf{r}_\xi$, $\mathbf{r}_{\xi\xi}$ and a_{11} are calculated with central difference approximations using only boundary (fixed) points. In this case, $\mathbf{r}_\eta^{ort}$ needs to be calculated:

$$\mathbf{r}_\eta^{ort} = (-y_\xi x_\eta + x_\xi y_\eta)\frac{-y_\xi \mathbf{i} + x_\xi \mathbf{j}}{a_{11}}, \tag{19}$$

where for $\eta = 1$ boundary

$$(\mathbf{r}_\eta)_{i,1}^n = \mathbf{r}_{i,2}^{n-1} - \mathbf{r}_{i,1} + \mathcal{O}(\Delta\eta), \tag{20}$$

and for $\eta = N$,

$$(\mathbf{r}_\eta)_{i,N}^n = \mathbf{r}_{i,N} - \mathbf{r}_{i,N-1}^{n-1} + \mathcal{O}(\Delta\eta). \tag{21}$$

The ghost points are determined by

$$\mathbf{r}_{i,ghost}^{(1)} = \mathbf{r}_{i,1} - (\mathbf{r}_\eta^{ort})_{i,1} \tag{22}$$

and

$$\mathbf{r}_{i,ghost}^{(N)} = \mathbf{r}_{i,N} + (\mathbf{r}_\eta^{ort})_{i,N}, \tag{23}$$

where N is the number of points in η direction. $\mathbf{r}_{\eta\eta}$ and a_{22} are evaluated following the same logic as for $\mathbf{r}_{\xi\xi}$ and a_{11}, respectively, in the previous section.

3.4 Orthogonal Functions Interpolation

As described, the orthogonal functions are calculated on the boundaries of the domain. However, boundary points are fixed and it is not over such points that orthogonal functions operate to produce the orthogonality condition, but over points belonging to the lines on the vicinity of the respective boundary. Hence, orthogonal functions have to be interpolated from the values calculated on the boundaries, as described above, into the domain. The technique chosen for interpolation is the bivariate interpolation by a Boolean sum of projection operators method proposed by Gordon and Hall (1973), popularly known in the literature as transfinite linear interpolation. Implementation is performed as follows. Next to $\xi = constant$ boundaries

$$\mathbf{Z}_O(\xi,\eta) = \frac{m-\xi}{m-1}\mathbf{Z}_O(1,\eta) + \frac{\xi-1}{m-1}\mathbf{Z}_O(m,\eta), \tag{24}$$

and next to $\eta = constant$ boundaries

$$\mathbf{Z}_O(\xi,\eta) = \frac{n-\eta}{n-1}\mathbf{Z}_O(\xi,1) + \frac{\eta-1}{n-1}\mathbf{Z}_O(\xi,n). \tag{25}$$

In the equations above, $\mathbf{Z}_O = (P_O, Q_O)^t$, m is the number of lines chosen by the user for interpolation in ξ direction, and n the number of lines chosen by the user for interpolation in η direction. It is worth mentioning that, for regions next to corners, the interpolation may not be performed, and special procedures are used, as will be clarified in the next section.

3.5 Boundary Point Distribution

According to Sorenson (1980), for many applications in solving fluid flow equations, the ability of the method in allowing the user to control boundary point distribution is of great importance. Eça (1996) reports that favorable boundary point distribution can greatly improve mesh quality. However, it may be difficult to predict such an appropriate point distribution along boundaries, which is the reason why Zhang *et al.* (2008) argue that, in practice, the most direct and popular boundary nodal distribution is uniform; the authors explain that such approach may work well in simple domains but not in complex ones. Still according to Zhang *et al.* (2008), it has been showed that inappropriate nodal distribution may cause serious skewness interiorly. According to Fletcher (1996), the boundary point distribution should be monotonic and smooth, since that if spacing changes too fast, it may cause convergence problems and grid folding. Perhaps the main limitation in the approach proposed by Kaul (2003) is regarding the boundary point control, which is not possible in such approach, and may be necessary for some applications.

The methodology proposed allows for the control of the distribution of points along any boundary by the user. The choice may have impact over the convergence rate, or produce skewness in the interior of the domain. Fletcher (1996) argues that an effective stretching function for control of point distribution on the boundaries is the normalized hyperbolic tangent stretching function presented by Roberts (1971) and generalized by Eiseman (1978). The argument is that this function makes concentration locally uniform. Such functions have been implemented with success. Thompson *et al.* (1982) presents a list of options about stretching functions studies that can be used instead of the one implemented.

4. RESULTS

4.1 Evaluation Criteria

As proposed by Zhang *et al.* (2008), the grids are evaluated based upon two standard academic criteria, namely, orthogonality and smoothness, represented respectively by the following indicators: grid orthogonality and aspect ratio. Orthogonality is assessed based upon maximum deviation from orthogonality (MDO),

$$\mathrm{MDO} = \max(\theta_{i,j} - 90^\circ), \tag{26}$$

and average deviation from orthogonality (ADO),

$$\mathrm{ADO} = \frac{1}{M-2}\frac{1}{N-2}\sum_{j=2}^{N-1}\sum_{i=2}^{M-1}\max(\theta_{i,j} - 90^\circ), \tag{27}$$

where $\theta_{i,j}$ corresponds to the angle between the coordinate lines i and j, that is

$$\theta_{i,j} = \left[\cos^{-1}\left(\frac{a_{12}}{\sqrt{a_{11}a_{22}}}\right)\right]_{i,j}. \tag{28}$$

In Eqs. (27) and (28) $\theta_{i,j}$ should be used in degrees. Naturally, both MDO and ADO are evaluated in degrees. Similarly, maximum grid aspect ratio (MAR),

$$\mathrm{MAR} = \max\left[\max\left(f_{i,j}, f_{i,j}^{-1}\right)\right], \tag{29}$$

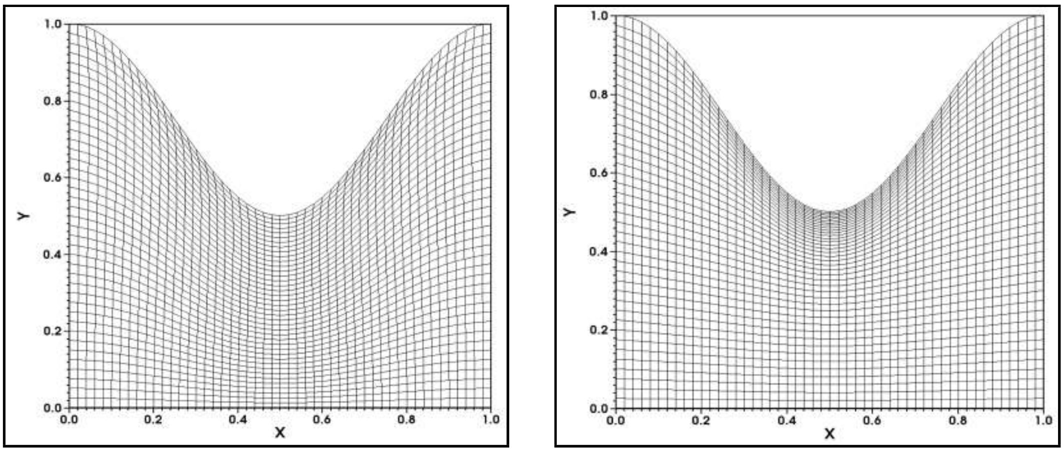
and average grid aspect ratio (AAR),

$$\mathrm{AAR} = \frac{1}{M-2}\frac{1}{N-2}\sum_{j=2}^{N-1}\sum_{i=2}^{M-1}\max\left[\max\left(f_{i,j}, \frac{1}{f_{i,j}}\right)\right], \tag{30}$$

are used to measure grid aspect ratio. In Eqs. (29) and (30), $f_{i,j}$ is called *distortion function* (Ryskin and Leal, 1983) and corresponds to the ratio between the magnitudes of the tangent vectors to coordinate lines. In Eqs. (27) and (30), M is the number of grid lines in ξ direction, and N is the number of grid lines in η direction.

4.2 Example Cases

Figures 1 to 3 present the grid obtained with the present method for an academic geometry widely used in the literature (Khaymaseh *et al.*, 1999), this is, a benchmark case. The domain is defined by $0 \leq x \leq 1$, and $0 \leq y \leq 0.75 + 0.25\sin(\pi(0.5 + 2x))$. Uniform distribution of boundary points was adopted. In the cases presented, the grid size is 50×40 elements. The grid presented in Fig. 1a was obtained with the Parabolic generator, and the evaluation criteria are the following. Maximum aspect ratio of 2.0, average deviation from orthogonality is of $27\,\mathrm{deg}$, average aspect ratio is of 1.5, maximum deviation from orthogonality of $69\,\mathrm{deg}$. The grid presented in Fig. 1b was obtained with the Laplacean generator, and the evaluation criteria are the following. Maximum aspect ratio of 5.2, average deviation from orthogonality is of $57\,\mathrm{deg}$, average aspect ratio is of 1.8, maximum deviation from orthogonality of $21\,\mathrm{deg}$. Finally, the grid presented in Fig. 2 was obtained with the current method, and in this case maximum aspect ratio of 4.9 and maximum deviation from orthogonality of $34\,\mathrm{deg}$ are reported; average deviation from orthogonality is of $17\,\mathrm{deg}$, and average aspect ratio is of 2.1. The results show that grid quality greatly improves with the current method compared to the Parabolic and Laplace generators. For the case of 2, orthogonality and aspect ratio are illustrated by means of color field representations in Fig. 3. As may be seen in Fig. 3, deviation from orthogonality at the boundaries is minimum, except close to the right and left top corner regions. Deviation on corners is expected due to the conformability requirement. Some mild deviation may also be observed in the vicinity of the top middle region. This is explained by the need for smoothing to neutralize the tendency of pronounced mesh squeezing at the concave boundary, reported for example in Zhang *et al.* (2008). Thus, strict orthogonality has been abdicated in order to improve grid spacing control and smoothness, which is noticeable in the whole mesh. Additionally, It should be be mentioned that none of the cases (Laplacean, parabolic or the nearly-orthogonal grid) exhibited grid folding.



a) Parabolic grid. b) Laplacean grid.

Figure 1. Parabolic and Laplacean grids for an academic geometry widely used in the literature.

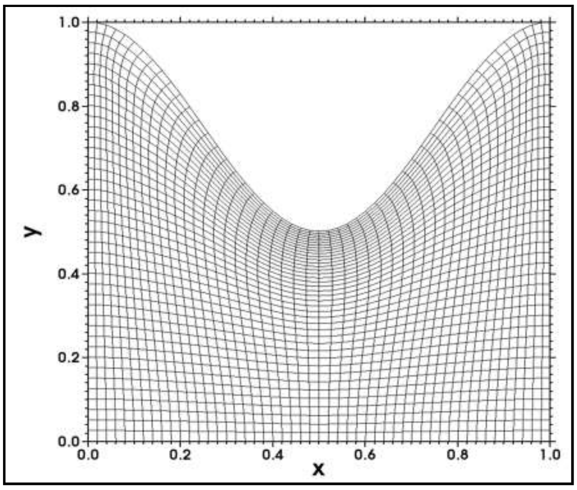
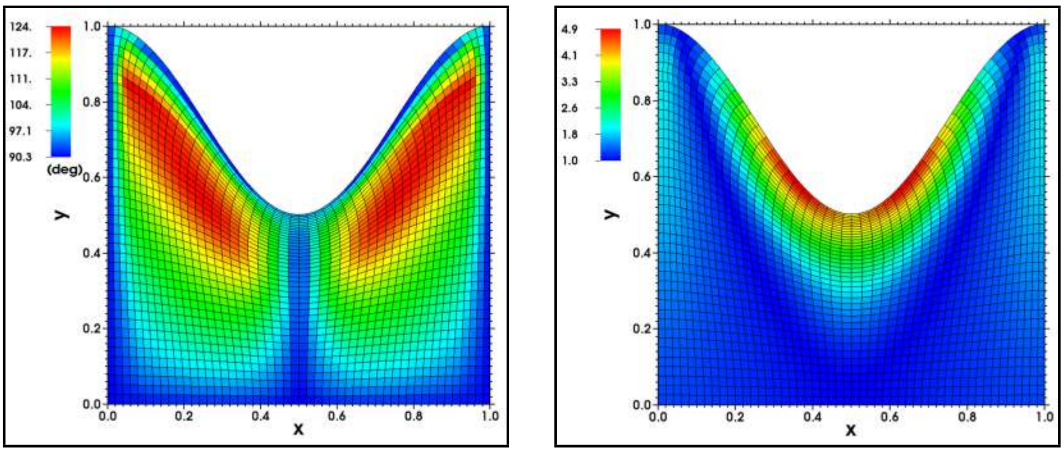


Figure 2. Nearly-Orthogonal grid for an academic geometry widely used in the literature.



a) Color field representation of orthogonality. b) Color field representation of aspect ratio.

Figure 3. Color field representation of orthogonality and smoothness for grid of Fig. 2

5. CONCLUSIONS

In CFD analysis and mesh generation, complex geometries remain a challenge. In this study, that is the mainly concern. The mapping problem corresponding to the construction of a coordinate system fitted to a boundary of given shape, with a prescribed distribution of coordinate nodes along this boundary is addressed. The method proposed consists of an inhomogeneous elliptic system in order to betake inherent attributes of such systems, such as smoothness and controllability to address the subject. Additionally, the method provide for the construction of a boundary-fitted coordinate system nearly-orthogonal to the boundaries in two dimensions; the method can be extended to 3D. As a drawback, the meshes obtained are not in general orthogonal over the entire domain, which may be desired in some applications. On the other hand, the flexibility obtained with the method is certainly an interesting asset, especially for complex geometries. Moreover, the method allows the user to deal with typical problems of elliptic grid generation systems, such as the tendency of mesh squeezing next to concave boundaries, and mesh spreading next to convex ones. Also, grid folding and other limitations prohibitive for most applications, such as severe skewness, have been addressed with success. As commented, inhomogeneous elliptic generation systems are sensible to initial conditions, reason why a parabolic generation method has been chosen, which has proved to work fairly well. The method shall be submitted to more tests, such as the impact of the boundary point distribution both on grid quality and convergence rate, other benchmark cases, as well as other practical applications. The full methodology shall be presented in details in a forthcoming article.

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