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ANALYSIS VIA GENERALIZED INTEGRAL TRANSFORM TECHNIQUE FOR TEMPERATURE DISTRIBUTION AND EFFICIENCY OF EXTENDED SURFACES WITH VARIABLE THERMAL CONDUCTIVITY

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Abstract. *The present paper uses the Generalized Integral Transform Technique (GITT), a hybrid method that combines the principles of analytic and numeric solutions for partial differential equations resultant of the analysis of transient heat transfer problems A steady state problem of heat transfer from longitudinal extended surfaces with uni-dimensional formulation and variable thermal conductivity is studied, the boundary conditions are temperature constant at the base and adiabatic tip. Resulting in a non-linear partial differential equations (PDE) system, suitable for advanced mathematical tools, such as integral transformation technique in order to solve the PDE's and determine the temperature field and then its efficiency from the heat transfer from the fin. With the analysis of non-dimensional parameters, the importance of the form factor and the thermo-geometric coefficient on the temperature field and efficiency can be observed, and the impact of this non-linear formulation causes in all analysis. Then, multiple values for these parameters is necessary to understand what happens within a fin.*

Keywords: *Heat transfer, Extended Surfaces, Integral Transform Technique, GITT, Temperature-dependent property, Non-linear Problem*

1. INTRODUCTION

Heat dissipation is a fundamental phenomenon in the existence of the universe, and with the creation of elaborate machines, the need for devices that manipulated that dissipation increased. Aiming to do achieve that, fins are created, extended surfaces whose intent is to increase heat exchange between a piece of equipment and its surroundings.

A big reference on this topic is the book written by Kraus *et al.* (2002). In this book, numerous applications are detailed, as well as variations in parameters, like geometry, temperature-dependent thermal conductivity and convection, convective and radiative fins, forced convection, non-steady analysis, variant base temperature, *etc.*

Coşkun and Atay (2008) studied temperature gradients and the efficiency of rectangular fins with temperature-dependent thermal conductivity. Joneidi *et al.* (2009) presents a similar study, analysing rectangular fins with variable thermal conductivity, however, using a semi-analytic method, Differential Transformation (DTM). Khan and Aziz (2012) make an analysis focused of rectangular, convective and radiative fins with thermal dependent convective coefficient and heat generation. Turkyilmazoglu (2012) studies thermal diffusion in different fin profiles for thermal dependent convective and conductivity coefficients. Ghasemi *et al.* (2014) expands on the study of Joneidi *et al.*, analysing fins with temperature-dependent thermal conductivity as well as temperature dependent heat generation.

Kader *et al.* (2016) uses Lie point symmetry method to obtain exact solutions for the temperature gradient in rectangular fins, taking into account temperature-dependent thermal conductivity and convective coefficient, and using different formulations for the conductivity. Patra and Ray (2016) calculate fin efficiency using the Homotopy Perturbation Sumudu Transform Method (HPSP), with temperature-dependent thermal conductivity.

In this paper, temperature gradients are evaluated in uni-dimensional fins in steady state using the Generalized Integral Transform Technique - GITT (Cotta (1993)). The materials parameters, as well as the fins and the fluids, are varied to achieve different temperature distributions and efficiency values and the impact of this non-linear differential equations causes in all analysis. The thermal conductivity is assumed to be a function of the temperature. To obtain these results the symbolic computing software *Wolfram Mathematica* is used.

2. MATHEMATICAL MODELS

We begin using the equation of conservation of energy adapted to fins, presented in Eq. (1). Where A_c is the cross-sectional area of the fin and P is the perimeter. T_∞ represents the temperature of the ambient fluid. This equation includes the non-linearity associated with the thermal conductivity $k(T)$ and the convective coefficient h. The boundary conditions are presented in Eq. (2). The first states that the temperature of the base of the fin is a constant T_b. Whereas the second one implies that there is no heat exchange through the tip of the fin.

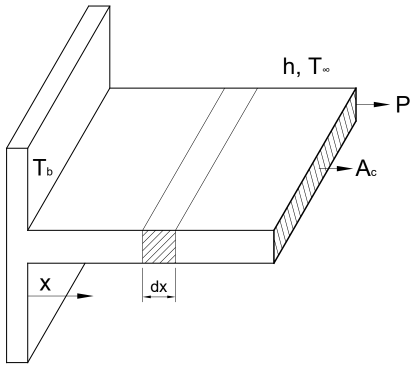

Figure 1. Schematic representation of a rectangular fin.

$$A_c \frac{\mathrm{d}}{\mathrm{d}x}\left[k(T)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\right] - Ph(T(x) - T_\infty) = 0 \tag{1}$$

$$T(0) = T_b; \quad \left.\frac{\mathrm{d}T(x)}{\mathrm{d}x}\right|_{x=L} = 0 \tag{2}$$

The thermal conductivity is defined as a linear function of the temperature, as presented in Eq. (3). Where k_a is the thermal conductivity at ambient temperature and λ is a term the defines the variation of k with the temperature.

$$k(T) = k_a[1 + \lambda(T(x) - T_\infty)] \tag{3}$$

From the temperature field, determined by solving Eqs. (1) and (2), it is possible, by Newton's law of cooling, calculate the heat transfer rate from the extended surface by:

$$Q = \int_0^L P(T(x) - T_\infty)dx \tag{4}$$

The fin efficiency is defined as the ratio of the heat transfer rate, Eq. (4), to heat transfer rate from the fin if there is no thermal resistance, i.e. the whole extended surface is entirely in base temperature, representing what the literature know as ideal heat transfer, where occurs a maximum temperature gradient between the fin and the environment. The efficiency is represented by:

$$\eta = \frac{Q}{Q_{\text{ideal}}} = \frac{\int_0^L P(T(x) - T_\infty)dx}{PL(T_b - T_\infty)} \tag{5}$$

The following dimensionless groups are used:

$$\theta = \frac{T - T_\infty}{T_b - T_\infty}; \quad X = \frac{x}{L}; \quad M = \sqrt{\frac{PhL^2}{A_c k_a}}; \quad \beta = \lambda(T_b - T_\infty) \tag{6}$$

Therefore, the original differential equation can be reduced to:

$$\beta\left(\frac{\mathrm{d}\theta(X)}{\mathrm{d}X}\right)^2 + [1+\beta\theta(X)]\frac{\mathrm{d}^2\theta(X)}{\mathrm{d}X^2} - M^2\theta(X) = 0 \tag{7}$$

With the respective boundary conditions:

$$\theta(0) = 1; \quad \frac{\mathrm{d}\theta(X)}{\mathrm{d}X}\Big|_{X=1} = 0 \tag{8}$$

And the efficiency rewritten as:

$$\eta = \int_0^1 \theta(X)dX \tag{9}$$

3. INTEGRAL TRANSFORMATION

To solve this non-linear ODE problem, we used the GITT methodology. An auxiliary eigenvalue problem which provides the eigenfunctions and the respective orthogonality property, to be used as a basis for the integral transformation procedure, must be chosen. Then, is possible to represent the temperature field as expansions on these eigenfunctions. We may also, in order to improve the computational performance of this mathematical procedure, set homogeneous boundary conditions by implying a filtering scheme which homogenises the boundary conditions given by Eq. (8). The following equation is proposed using the Generalized Integral Transform as:

$$\theta(X) = \overline{\theta_i}(X)^* + F(X) \tag{10}$$

The integral transform pair for performing the integral transformation and inversion operations is given by:

$$\theta(X) = \sum_{i=1}^{\infty}\tilde{\psi}_i(X)\overline{\theta_i} + F(X); \quad \overline{\theta_i} = \int_0^1 (\theta(X) - F(x))\tilde{\psi}_i(X)dX \tag{11}$$

Where:

$$\overline{\theta_i}(X)^* = \sum_{i=1}^{\infty}\tilde{\psi}_i(X)\overline{\theta_i} \tag{12}$$

With the equations above, the filter expression is naturally $F(X) = 1$, ensuring a new set of ODE, replacing $\theta(X)$, given by Eq.(10), in Eq. (7) and (8), with completely homogeneous boundary conditions for $\overline{\theta_i}(X)^*$:

$$\beta\left(\frac{\mathrm{d}\overline{\theta_i}(X)^*}{\mathrm{d}X}\right)^2 + \beta\overline{\theta_i}(X)^*\frac{\mathrm{d}^2\overline{\theta_i}(X)^*}{\mathrm{d}X^2} + [1+\beta]\frac{\mathrm{d}^2\overline{\theta_i}(X)^*}{\mathrm{d}X^2} - M^2\overline{\theta_i}(X)^* - M^2 = 0 \tag{13}$$

$$\overline{\theta_i}(0)^* = 0; \quad \frac{\mathrm{d}\overline{\theta_i}(X)^*}{\mathrm{d}X}\Big|_{X=1} = 0 \tag{14}$$

Then, once we calculate the expression for $\overline{\theta_i}(X)^*$, we are able to determine the temperature field $\theta(X)$ of the original problem. To achieve such profiles for the temperature field, we selected a auxiliary eigenvalue problem, as mentioned before, to determine the eigenfunctions that is used in Eq.(12). Based on a simple Sturm-Liouville formulation that has a straightforward analytic solution, which comprises some information from the ODE, we have:

$$\frac{\mathrm{d}^2\psi_i(X)}{\mathrm{d}X^2} + \mu_i^2\psi_i(X) = 0 \tag{15}$$

$$\psi_i(0) = 0; \quad \frac{\mathrm{d}\psi_i(X)}{\mathrm{d}X}\Big|_{X=1} = 0 \tag{16}$$

The associated orthogonality propriety of the eigenfunctions given by Eqs. (15) and (16) is:

$$\int_0^1 \psi_i(X)\psi_j(X)dX = 0, \text{for i} \neq \text{j}; \quad \int_0^1 \psi_i(X)\psi_i(X)dX = N_i, \text{for i=j} \tag{17}$$

Where we defined the normalized eigenfunctions as:

$$\tilde{\psi}_i(X) = \frac{\psi_i(X)}{\sqrt{\int_0^1 \psi_i(X)^2 dX}} = \frac{\psi_i(X)}{\sqrt{N_i}} \tag{18}$$

Once $\psi_i(X)$ is calculated, we are able to obtain its eigenvalues μ_i from the boundary conditions represented by Eq. (16), and the integral transformation technique may continue. Replacing $\overline{\theta_i}(X)^*$ given by Eq.(12) into Eq.(13), multiplying by $\tilde{\psi}_j(X)$ both sides followed by integrating over the domain [0,1] in the X direction we are able to calculate the transformed potentials $\overline{\theta_i}$. The orthogonality propriety of the eigenfunctions is used to simplify the overall equation, removing some of the summations.

The final solution for $\overline{\theta_i}$ can only be obtained through numerical calculations by using the *NSolve* routine of *Wolfram Mathematica*. To recover the desired temperature field $\theta(x)$, the transformed potentials $\overline{\theta_i}$ obtained by GITT, are employed into the Eq. (11), and the expansion is truncated at a specific value N that guarantee a proper convergence.

4. RESULTS

The first step in analysing the results was taking a look at the convergence ratio of the eigenvalues used in the $\psi_i(X)$ function. The results for the temperature were compared using different numbers of eigenvalues, this comparison is presented in Tab. 1 and Tab. 2. This analysis was observed for four different positions along the fin and for two set of parameters. It can be seen that there are no alterations on the fifth decimal number after twenty-five eigenvalues. Therefore, aiming to achieve maximum accuracy with a good computational time, through this paper the number chosen was thirty eigenvalues.

Table 1. Convergence test for temperature field assuming $M = 1$, $\beta = -0.5$

	$X = 0.1$	$X = 0.4$	$X = 0.7$	$X = 1.0$
$N = 05$	0.98866	0.96153	0.94517	0.93970
$N = 10$	0.98857	0.96149	0.94517	0.93973
$N = 15$	0.98858	0.96149	0.94517	0.93972
$N = 20$	0.98859	0.96149	0.94517	0.93972
$N = 25$	0.98859	0.96149	0.94517	0.93972
$N = 30$	0.98859	0.96149	0.94517	0.93972

Table 2. Convergence test for temperature field assuming $M = 3$, $\beta = 1$

	$X = 0.1$	$X = 0.4$	$X = 0.7$	$X = 1.0$
$N = 05$	0.97814	0.92524	0.89288	0.88198
$N = 10$	0.97799	0.92517	0.89288	0.88204
$N = 15$	0.97801	0.92517	0.89289	0.88203
$N = 20$	0.97802	0.92517	0.89289	0.88203
$N = 25$	0.97801	0.92516	0.89289	0.88203
$N = 30$	0.97801	0.92516	0.89289	0.88203

The results for the efficiency were also compared using the same numbers of eigenvalues of the temperature analysis, this comparison is presented in Tab. 3 and Tab. 4. Once more, there are no alterations on the fourth decimal number after twenty-five eigenvalues.

Table 3. Convergence test for efficiency assuming $M = 3$

	$\beta = -0.5$	$\beta = -0.2$	$\beta = 0.2$	$\beta = 0.5$	$\beta = 1.0$
$N = 05$	0.272700	0.309679	0.353193	0.382462	0.426303
$N = 10$	0.271419	0.309064	0.352897	0.382262	0.426178
$N = 15$	0.271256	0.308997	0.352867	0.382242	0.426165
$N = 20$	0.271213	0.308981	0.352860	0.382237	0.426162
$N = 25$	0.271197	0.308975	0.352857	0.382235	0.426161
$N = 30$	0.271191	0.308973	0.352856	0.382234	0.426160

Table 4. Convergence test for efficiency assuming $\beta = 0.2$

	$M = -0.5$	$M = -0.2$	$M = 0.2$	$M = 0.5$	$M = 1.0$
$N = 05$	0.788230	0.510764	0.353193	0.266539	0.213828
$N = 10$	0.788194	0.510631	0.352897	0.266016	0.213013
$N = 15$	0.788191	0.510617	0.352867	0.265962	0.212930
$N = 20$	0.788190	0.510614	0.352860	0.265949	0.212909
$N = 25$	0.788189	0.510613	0.352857	0.265944	0.212902
$N = 30$	0.788189	0.510612	0.352856	0.265942	0.212898

To fully understand the temperature distribution across the fin, different parameters - according to Eqs. (6) - were used to simulate different conditions. In Fig. 2 and Fig. 3 the graphics presents the temperature profile for each parameters variation. For Fig. 2, variation of the thermo-geometric factor M, and for Fig. 3, the impact of the linear coefficient of the thermal conductivity β.

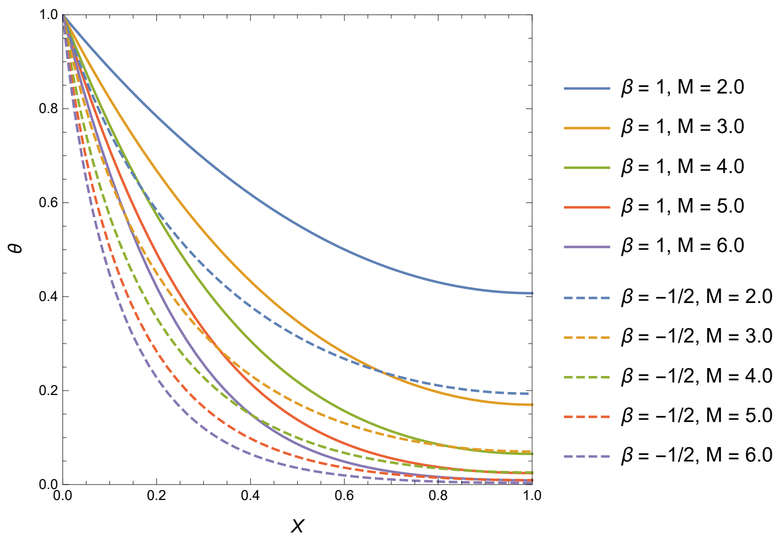

Figure 2. Temperature distribution in straight fins with different thermo-geometric factors.

When analysing the variation of the thermo-geometric factor one can observe that all of five values simulated, result in similar profiles. But for a given β, when increasing M, the temperature inside the fin drops to lower values, that is an expected behaviour, once we have M directly proportional to h (convective coefficient). If the convective effect becomes more relevant, so the temperature decreasing is proportionally relevant, tending to the outside fluid temperature for each point inside the fin. For negative values of β, resulting in lower thermal conductivity, this effect is even more intense, leading to lower temperatures in earlier points of the fin, due to higher thermal resistances.

For studying the effects of a wide range of β, it was used both positive and negative values, simulating increasing and decreasing thermal conductivity. Similar to the other case, all values produce distributions that look alike. We could note, the same arguments above, now is clear to see, the influence of M and β, where a combination of higher β and lower M will represent a smooth drop in the fin temperature field, leading to higher efficiency and ideal for the fin purposes.

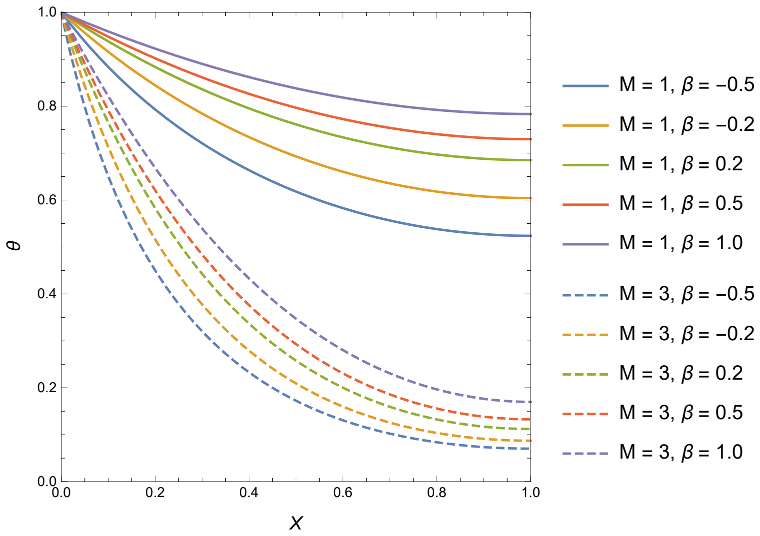

Figure 3. Temperature distribution in straight fins with different linear coefficients for the thermal conductivity.

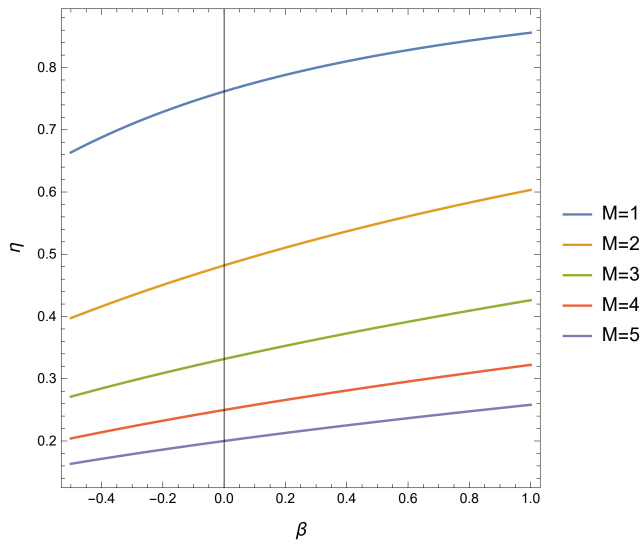

Figure 4. Efficiency for different thermo-geometric factors

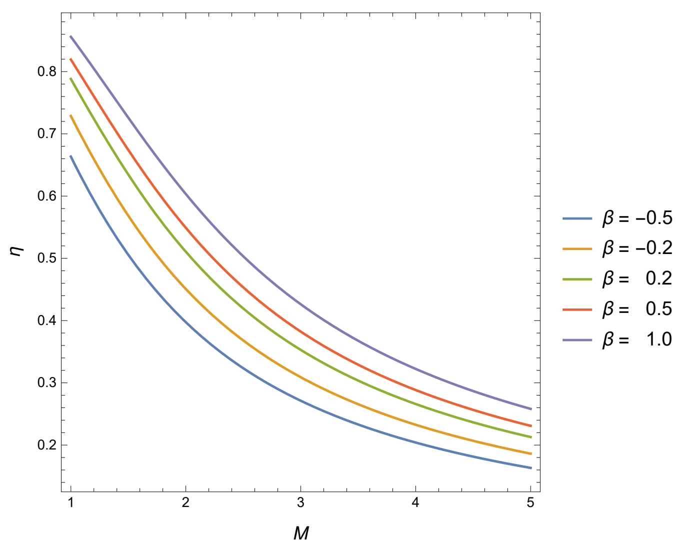

Figure 5. Efficiency for different β.

In Fig. 4 and Fig. 5, we studied the efficiency of the fin. Once again, we could note a similar analysis, for a given M we have higher efficiency for higher β, and for higher values of M we have lower efficiency i.e., higher conductivity and/or lower convective effects, we have better efficiency according perfectly what is shown in Fig.2 and Fig. 3.

5. CONCLUSIONS

In the current paper, the Generalized Integral Transform Technique was used to determine the temperature distribution in straight fins with temperature-dependent thermal conductivity, as well as their efficiency. The technique was able to solve the PDEs despite their non-linearity with good precision and fast convergence. It was possible to observe the effects of varying proprieties to the temperature distribution and efficiency. Measuring the impact of the temperature-dependent thermal conductivity on the overall distribution of temperature.

It can also be observed that the reduction of M can significantly increase the efficiency of the fin, whereas β has little effect. Similar effect can be observed in the temperature distribution. Therefore, it is usually better to change β instead of changing M. However, since both are dependent on the material used, those changes are not easy to achieve. And since changing M also depends on the geometry of the fin and convective coefficient, change β may be the better course of action.

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