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APPLICATION OF THE HYBRID-MIXED FINITE ELEMENT METHOD WITH STABILIZED NODAL ENRICHMENT ON 1D PROBLEM

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Abstract. *A new numerical method is proposed to estimate approximations of the displacements and stresses fields applying the Stabilized Generalized Finite Element Method (SGFEM) on Hybrid-Mixed Stress Formulation (HMSF) to a classical 1D structural mechanic problem – the bar problem under normal force. For the HMSF, three approximation fields are involved: stresses and displacement in the domain and displacement on the static boundary. In the combined HMSF-SGFEM approach the enrichment of stress and displacement domain field is provided by the product of the partition of unity (PoU) and a polynomials basis enrichment functions. The HMSF-SGFEM numerical simulation was implemented using MATLAB® routines. The performance of this new approach is illustrated and compared with classical Finite Element Method (FEM) by applying two discretized elements and the exact solution of this problem. The results show that combining these two nonconventional methods (HMSF-SGFEM) provided better matrix conditioning and a good convergence when is compared with the FEM results. In addition, considering both computational and numerical aspects, the HMSF-SGFEM can be easily extrapolated to 2D and 3D analysis of the elastic structural mechanics problems.*

Keywords: *Hybrid-Mixed Stress Formulation, Stabilized Generalized Finite Element Method, Nodal Enrichment, 1D Structural Mechanic Problem.*

1. INTRODUCTION

Over the past years, the nonconventional finite element methods, such as the Hybrid Stress Formulation (HSF), Hybrid-Mixed Stress Formulation (HMSF) introduced by Freitas et al. (1999), Generalized Finite Element Method (GFEM) originally presented in Oden, Duarte and Zienkiewicz (1998) and Stable Generalized Finite Element Method (SGFEM) produced by Babuška and Banerjee (2012) have been widely used to solve many problems of the structure mechanics and incompressible flows.

The main concept behind the HMSF is to approximate independently the stress and the displacement field in the domain and the displacement on the boundary. In Góis (2009), Góis and Proença (2012) and Haas and Góis (2016) the combination of the nonconventional methods, HMSF-GFEM, was applied to plane elasticity problems. More recently the HMSF was introduced in the topology optimization classical problem to verify its performance and limitation, Navarro and Góis (2017a) and Navarro and Góis (2017b).

In this work, to obtain a better sparse matrix and condition number of the solving system a stable nodal enrichment and a hybrid-mixed stress formulation are combined to generate a new class of finite element methods for 1D, 2D or 3D analysis.

2. HYBRID-MIXED STRESS FORMULATION – 1D STRUCTURAL MECANIC PROBLEM

To derive the Eq. (1), the Galerkin weighting relations for the equilibrium and compatibility conditions has taken to consider in 1D problem (*x* is the direction of analysis) . Where s_Ω represents nodal stress, $\mathbf{q}_\Omega$ is the nodal displacement in the domain and $\mathbf{q}_\Gamma$ is the nodal displacement on the boundary.

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{A}_\Omega & \boldsymbol{A}_\Gamma \\ \boldsymbol{A}_\Omega^T & 0 & 0 \\ -\boldsymbol{A}_\Gamma^T & 0 & 0 \end{bmatrix} \begin{pmatrix} s_\Omega \\ \boldsymbol{q}_\Omega \\ \boldsymbol{q}_\Gamma \end{pmatrix} = \begin{pmatrix} e_\Omega \\ -\boldsymbol{Q}_\Omega \\ -Q_\Gamma \end{pmatrix} \quad (1)$$

The elements of the sub-matrices and vectors from the linear equation system above (Eq. (1)) are represented by Eq. (2), Eq. (3), Eq. (4), Eq. (5), Eq. (6) and Eq. (7).

$$\boldsymbol{F} = F_{\boldsymbol{ij}} = \int_{\boldsymbol{0}}^{\boldsymbol{L}} \varphi_j \boldsymbol{f} \varphi_{\boldsymbol{i}} dx \quad (2)$$

$$\boldsymbol{A}_\Omega = A_{\boldsymbol{ij}\Omega} = \int_{\boldsymbol{0}}^{\boldsymbol{L}} \varphi_j \boldsymbol{f} \varphi'_{\boldsymbol{i}} dx \quad (3)$$

$$\boldsymbol{A}_{\boldsymbol{\Gamma}} = A_{\boldsymbol{ij\Gamma}} = -\varphi_{\boldsymbol{j}}(L)\varphi_{\boldsymbol{i}}(L) \quad (4)$$

$$\boldsymbol{Q}_\Omega = Q_{\boldsymbol{ij}\Omega} = -\int_{\boldsymbol{0}}^{\boldsymbol{L}} \varphi_j \boldsymbol{b} dx \quad (5)$$

$$\boldsymbol{Q}_{\boldsymbol{\Gamma}} = Q_{\boldsymbol{ij\Gamma}} = -\varphi_{\boldsymbol{j}}(L) t_{\boldsymbol{x}}(L) \quad (6)$$

$$e_{\Gamma u} = \int_{\Gamma u} \varphi_\Omega^T \, \bar{u} \, d\Gamma \quad (7)$$

To equations above φ represents the original polynomial shape functions used for the approximate fields in the problem (stress and displacement in the domain) and displacement on the boundary, $\boldsymbol{f}$ is the flexibility matrix, L is the domain length and $\boldsymbol{b}$ is the vector of body forces. The representation of the vector of surface forces is given by $t_{\boldsymbol{x}}$.

3. STABLE GENERALIZED FINITE ELEMENT METHOD

The GFEM and SGFEM combine the structure of the classic FEM with the nodal enrichment technique, in this section the methodology for build the bases of approximation of the SGFEM is presented.

In SGFEM displacement based approach, a PoU functions created a confirming approximation which can be improved by a nodal enrichment strategy. In this work the polynomial functions are adopted to represent the enrichment fields and the shape functions. In Eq. (8) and Eq. (9) the representative enrichment H_i are shown.

$$H_i = (x - x_i)^2 - \sum_{j=1}^{2} \varphi_j \, (x - x_i)^2 \quad (8)$$

$$\varphi_i^* = \varphi_i H_i \quad (9)$$

4. NUMERICAL RESULTS

4.1 The bar problem formulation - 1D Structural Mechanic Problem

Consider a homogeneous bar - 1D Structural Mechanic Problem, presented in Fig. (1), which has a normalized parameters *P = 1* (vector of surface force), *E = 1* (Young´s Modulus), *A = 1* (Cross Sectional Area), *L = 1* (domain length) and *q = 1* (body force).

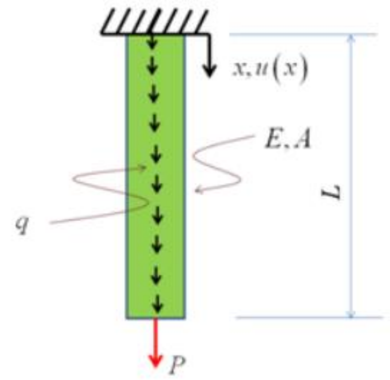

Figure 1. Bar under normal force – 1D Structural Mechanic Problem.

It is important to mention that in all numerical simulations only the first node, associated with the displacement boundary condition, wasn´t enriched. Considering a discretization with three linear elements (two nodes), for example, only nodes 2, 3 and 4 are enriched. According to this condition, the enriched functions will always be φ_2^*, φ_3^* and φ_4^*.

4.2 Numerical Analysis to Enrichment HMSF

Figure 2 presents a comparison between the exact solution and the HMSF-SGFEM to displacement and stress in the domain x. As can be seen, the enriched solution presents a better solution in the stress domain once the solution is recovered.

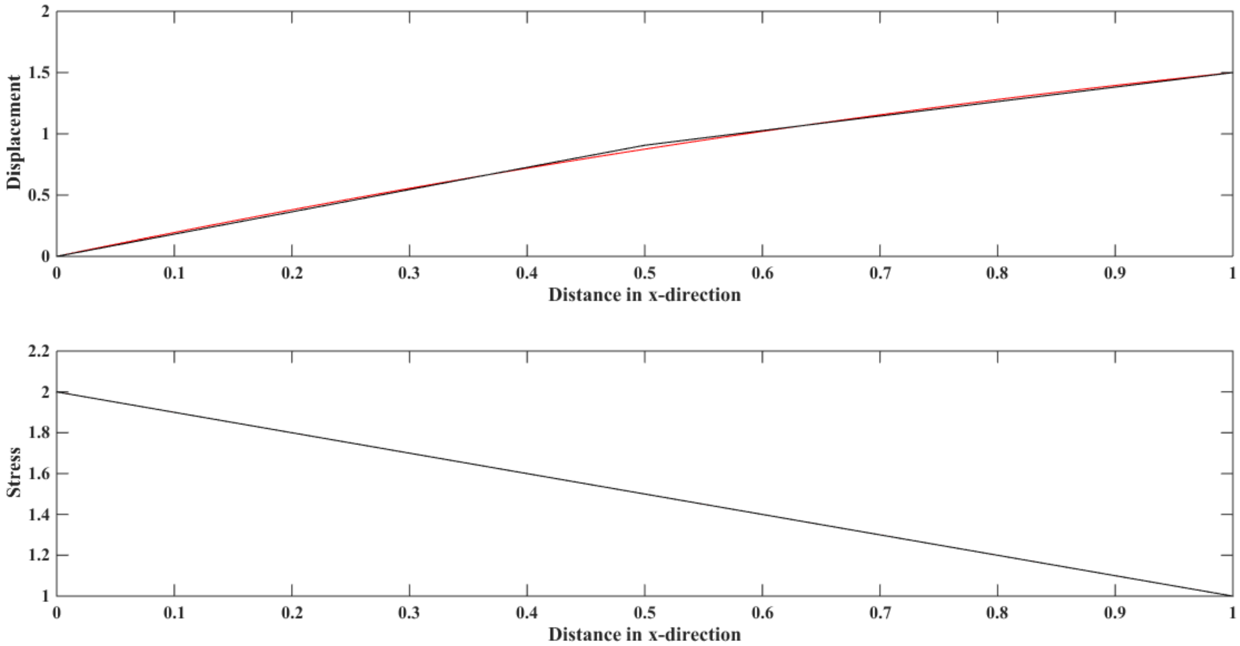

Figure 2. Comparison between the exact solution and HMSF-SGFEM in displacement and stress domain for two discretized elements.

4.3 Error Analysis

To evaluate the proposed method (HMSF-SGFEM), the Superconvergent Patch Recovery (SPR) was implemented, but was not necessary to recover the displacement field once the exact solution is known. Also, the energy norm error, Eq. (10), present in Zienkiewicz and Zhu (1987) are included in SPR.

$$\|e\| = \left[\int_{\Omega} (u - \tilde{u})^2 \, d\Omega\right]^{1/2} \tag{10}$$

A comparison in terms of the error estimator energy norm between FEM, HMSF, and HMSF-SGFEM is presented in Fig. 3 which shows that the HMSF-SGFEM has a better solution in terms of the energy norm error. The main feature of this new approach is recovered a better displacement field in the domain when compared with this same field obtained by the HSMF without enrichment. To the stress field, both HMSF and HMSF-SGFEM recover the exact solution of the 1D Structural Mechanic Problem.

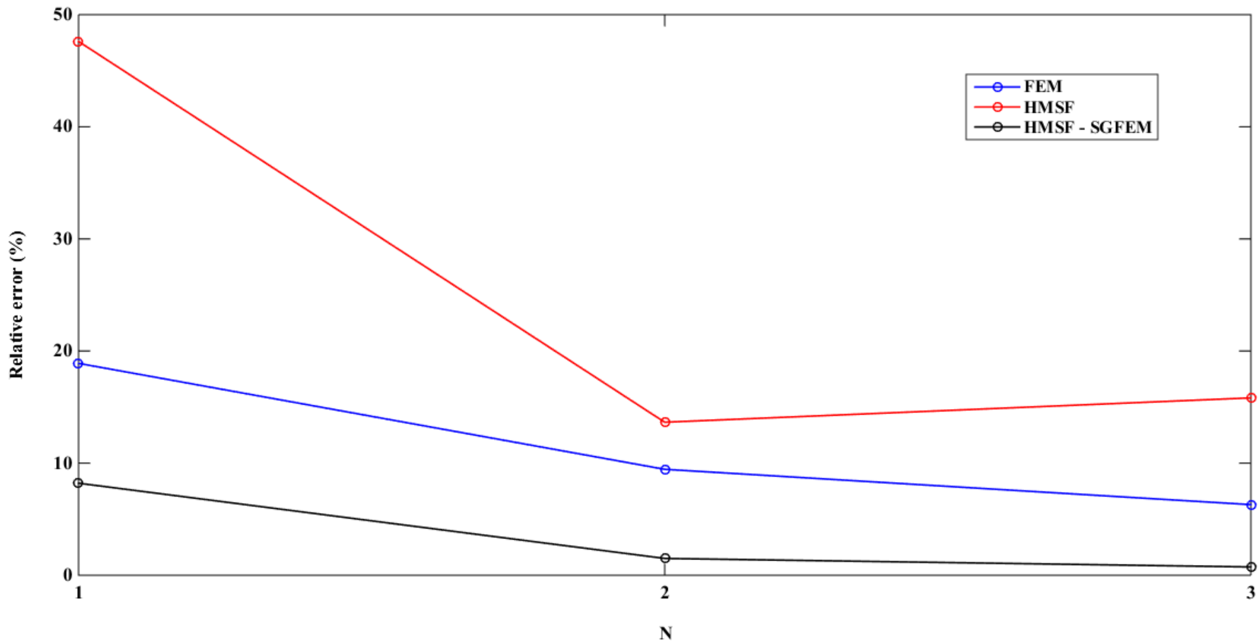

Figure 3. The convergence rates of the HMSF, FEM and HMSF-SGFEM where N holds the number of degrees of freedom of the discretization. It should be noted that the rate slopes of FEM and HMSF-SGFEM are approximately the same.

In Table 1, the conditioning of the coefficient matrix in Eq.(1), for two discretized elements, shows that stable enrichment has better conditioning then the enrichment applied in HMSF-GFEM, which prevents a severe loss of accuracy in the linear system as expected.

Table 1. Comparing the matrix conditioning of two elements discretization

Methods	Conditioning of the stiffness matrix
FEM	6.85E+0
HMSF	3.60E+8
HMSF-GFEM	4.10E+8
HMSF-SGFEM	3.81E+8

5. CONCLUSION

The results obtained has shown to be in agreement with the classical 1D bar problem. The chosen stable generalized finite element method applied to nonconventional formulation has proven to be more efficient and stable then the classical hybrid-mixed stress formulation. The next step on the study of HMSF-SGFEM is the application to 2D and 3D plane elasticity problems, also stability and convergence should be improved.

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