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ANALYSIS ON THE LONGITUDINAL WAVE PROPAGATION IN RODS WITH DISCONTINUITIES

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Abstract. *Structural vibration suppression has been the focus of several researches, mainly for developing high performance mechanical systems. Besides classical techniques, which usually involve passive, semi-active and active approaches, the use of periodicity concept for physical and geometric properties is an interesting alternative to get efficient design of systems for several problems. In general, periodic structures are composed by cells and the analysis of waves propagation is a fundamental task for specifying physical and geometric parameters to define the structure. In this sense, in this research is proposed to study the longitudinal transmitted and reflected waves in a structure composed by both left and right homogeneous parts and an intermediate part with periodic cell. Different geometric configurations and combinations of materials are evaluated through an analytical formulation written in the frequency domain to understand the relations between transmitted and reflected waves amplitudes in respect to the incident one. Based on these analysis are characterized the effects of stop band and pass band for different conditions.*

Keywords: *Periodic structures, combined materials, wave propagation, vibration suppression*

1. INTRODUCTION

The reduction of structural vibrations has been an important topic in several engineering applications. In traditional designs, different control techniques have involved the addition of viscoelastic materials and additional systems with actuators and sensors, featuring passive, semi-active and active configurations. Therefore, the propagation of waves in solids is of interest in several applications. The study of structures involving wave phenomena includes the response to impact loads and cracking propagation. Several researchers have shown that vibration reduction can be achieved using the concept of periodicity. These works discuss the design of structures with periodic physical and/or geometric properties. These structures involve elements or parts which are repeatedly connected and, when properly defined, ensure suppression of vibrations in predetermined frequency bands of interest. This idea has been applied in the design of satellite panels, train rails, and many other systems (NARISETTI, 2010; MANKTELOW, 2013).

Researchers like FAULKNER and HONG (1985) showed in their studies that an adequate introduction of discontinuities along a structure may result in the attenuation of the propagation of waves from one end of a cell to the beginning of another. Such attenuation is the result of the very interaction between incident, reflected and transmitted waves in the discontinuous areas. Periodic structures can be considered a special class of structures that undergo vibration formed by a macroscopic repetition of units (the so-called *cells*).

Since the cell is composed of parts of different dimensions (and/or materials), one can reflect part of the incident waves in the cell and, with this, create a filter effect in some frequency bands. These frequencies are called *stop bands*. Similarly, the frequencies bands with transmitted waves are called *pass bands* (BRILLOUIN, 1953; MANKTELOW, 2013).

The existence of these band gap is related to the contrast between the properties of the materials (modulus of elasticity/Young and specific mass) that make up the structure. According to SILVA *et al.* (2011), the wave propagation speed, which depends on the materials and the periodicity, is sufficient to estimate the frequency around which the largest band gap will occur. So, periodicity can be used to generate frequency ranges in which waves do not propagate or, at least, are considerably attenuated through the structure.

According ti the Bloch's theorem for periodic systems, for any structure with repetitive identical units, the change in wave amplitude across a unit cell does not depend upon the location of this cell within the structure. That means that waves propagating in a periodic structure can be analysed by the motion of a single cell (PHANI *et al.*, 2006).

The Bloch's parameters are complex numbers given by:

$$\Psi = \Psi^{Re} + i\Psi^{Im} \tag{1}$$

where Ψ is the *propagation constant*, the term Ψ^{Re} is the *attenuation factor* and Ψ^{Im} is the *phase constant*. The first one represents the amplitude decay of a wave propagation from one cell to the next one, and the second one is the measurement of the phase changing across one unit cell. As NOBREGA *et al.* (2016) showed, there are different responses from one cell to another ones in its neighboring only by a phase change equal to Ψ.

There is a relationship between the *propagation constant* and frequency, represented by $\omega(\Psi)$, it's called *dispersion relation* and is given by:

$$\omega(\Psi) = \omega(\Psi + 2m\pi) \tag{2}$$

where m is a positive integer number (BRILLOUIN, 1953). So, the frequency ω is a periodic function of the Bloch's parameter Ψ and the *first Brillouin zone* corresponds to the domain defined by $\Psi \in [-\pi, \pi]$.

If Ψ presents only the imaginary part, i.e., $\Psi^{Re} = 0$, then the waves propagate without attenuation (pass band zones). It means that, for the existence of stop bands, the real part of Ψ must be different from zero (SINGH *et al.*, 2004).

In this context, the present work comprises the study of the transmission and reflection of longitudinal waves in a structure with discontinuity between homogeneous sides. The scope comprises evaluating different geometric configurations and combinations of materials using an analytical formulation to obtain the bands with attenuation at the amplitudes of the longitudinal waves. Relationships between transmitted by the incident amplitudes, as well as those reflected by the incident are considered. Besides this proposal, it is intended to contribute to design periodic structures connected to the boundaries of homogeneous sections.

2. METHODOLOGY

Considering a long and slender rod subjected only to axial tension and assuming that the effects of compression-expansion (Poisson's effects) at the ends can be neglected, by the free body diagram of a differential element dx, of the length of this bar, using Newton's second law, the equation of longitudinal motion of this element is given by (GRAFF, 1991):

$$-\sigma S + \left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) S + q(x,t) S dx = \rho S dx \frac{\partial^2 u}{\partial t^2} \tag{3}$$

where ρ is the material density and S is the cross-section area of the rod. In this equation, σ is the axial stress acting on a differential element; $u(x,t)$ is the longitudinal displacement (as a function of its position in a certain instant of time) and $q(x,t)$ is the body force per volume.

Considerations the material elastic behavior with constant Young's modulus (i.e., a rod with homogenous properties) and, in particular, considering the absence of body forces, the last equation can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \tag{4}$$

defining the phase velocity of the longitudinal waves by:

$$v = \sqrt{\frac{E}{\rho}} \tag{5}$$

Note that the phase velocity is independent of the wave frequency, so the waves in a rod are conservative and, therefore, the group velocity is equal to the phase velocity (DOYLE, 1997; GRAFF, 1991). For the solution of the wave equation, it is considered a free-wave propagation and a time harmonic motion (DOYLE, 1997). Thus, for convenience, considering that the axial displacement and force equations, $u_x(x)$ and $F_x(x)$, can be rearranged in matrix form, in a way that (BRENNAN, 1994; BRENNAN *et al.*, 1997)

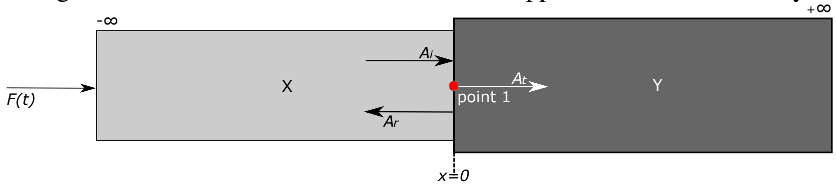
$$\left\{ \begin{array}{c} u(0,\omega) \\ F(0,\omega) \end{array} \right\} = \left[\begin{array}{cc} 1 & 1 \\ -jkSE & jkSE \end{array} \right] \left\{ \begin{array}{c} A_R \\ A_L \end{array} \right\} \tag{6}$$

in particular for $x = 0$. With the subscript R in A_R indicating right going, and A_L, with the subscript L indicating left going waves.

2.1 DISCONTINUOUS ROD

The case studied is a discontinuous structure with an axial force applied at its left boundary. Considering two semi-infinite rods connected in a way to cause a discontinuity of geometry, material or both, neglecting Poisson effects, which implies in the wave displacement $u(x)$ to be function only of the axial coordinates (SILVA *et al.*, 2011). In addition, stating that the properties of the study materials are uniform along their length, the model of Fig. 1 is shown.

Figure 1. Discontinuous rod with axial force applied at the left boundary.



The amplitude vectors are $\mathbf{a}_X$ (7) and $\mathbf{a}_Y$ (8).

$$\mathbf{a}_X = \{A_i \quad A_r\}^T \tag{7}$$

$$\mathbf{a}_Y = \{A_t \quad 0\}^T \tag{8}$$

where A_i, A_r and A_t are the incident, reflected and transmitted wave amplitudes, respectively. Note that, when we consider absence of force at the right bounday, there is a null amplitude in the vector $\mathbf{a}_Y$.

So, for the equality at point 1, considering displacement and force continuities and working with the rates transmitted above incident and reflected above incident, the final equations are Eq. (9) and Eq. (10).

$$\frac{A_t}{A_i} = \frac{2}{1 + \left(\sqrt{\frac{\rho_{YX}}{E_{YX}}}\right)^{-1} S_{YX}E_{YX}} \tag{9}$$

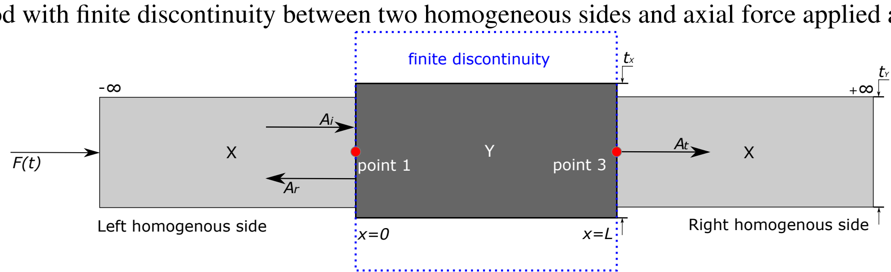
$$\frac{A_r}{A_i} = \frac{1 - \sqrt{\frac{\rho_{YX}}{E_{YX}}} S_{YX}E_{YX}}{1 + \sqrt{\frac{\rho_{YX}}{E_{YX}}} S_{YX}E_{YX}} \tag{10}$$

where $k = \omega\sqrt{\frac{E}{\rho}}$, and S_{YX}, E_{YX} and ρ_{YX} indicate, respectively, the relations between the areas, Young's modulus and material densities to the parts X and Y, i.e., the left under the right sides of the rod.

2.2 ROD WITH A FINITE DISCONTINUITY

In the case of a finite discontinuity of length L between two homogeneous sides (Fig. 2), note that there is an infinite amount of waves being reflected from one end to the other that remain contained within of this cell.

Figure 2. Rod with finite discontinuity between two homogeneous sides and axial force applied at the left end.



The transmission and reflection equations for a finite discontinuity between homogeneous rods are presented below.

$$\frac{A_t}{A_i} = \frac{4k_{YX}S_{YX}E_{YX}e^{j2\pi L_{\lambda_Y}}}{(1 + k_{YX}S_{YX}E_{YX})^2 e^{2j2\pi L_{\lambda_Y}} - (k_{YX}S_{YX}E_{YX} - 1)^2} \tag{11}$$

$$\frac{A_r}{A_i} = \frac{-k_{YX}^2 S_{YX}^2 E_{YX}^2 e^{2j2\pi L_{\lambda_Y}} + k_{YX}^2 S_{YX}^2 E_{YX}^2 + e^{2j2\pi L_{\lambda}} - 1}{(1 + k_{YX} S_{YX} E_{YX})^2 e^{2j2\pi L_{\lambda_Y}} - (k_{YX} S_{YX} E_{YX} - 1)^2} \tag{12}$$

Equations (11) and (12) can be obtained by solving a matrix equation written based on the force and displacement continuities, i.e., $\mathbf{h}_X^{point1}$ equal to $\mathbf{h}_Y^{point1}$ and $\mathbf{h}_Y^{point3}$ equal to $\mathbf{h}_X^{point3}$, which imply to the following equation:

$$\begin{bmatrix} H_{X11} & -\tilde{H}_{12} \\ H_{X21} & -\tilde{H}_{22} \end{bmatrix} \begin{Bmatrix} A_t \\ A_r \end{Bmatrix} + \begin{bmatrix} -\tilde{H}_{11} & H_{X12} \\ -\tilde{H}_{21} & H_{X22} \end{bmatrix} \begin{Bmatrix} A_i \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

Then, for algebraic simplicity:

$$\gamma \mathbf{a}_o + \mu \mathbf{a}_i = \mathbf{0} \Longleftrightarrow \mathbf{a}_o = -\gamma^{-1} \mu \mathbf{a}_i \tag{14}$$

where the input and output wave amplitude vectors are $\mathbf{a}_i$ and $\mathbf{a}_o$, respectively. The matrices γ and μ are calculated by

$$\gamma = \left[(\mathbf{H}_X)_1 \quad \left(-\tilde{\mathbf{H}}\right)_2 \right] \tag{15}$$

$$\mu = \left[\left(\tilde{\mathbf{H}}\right)_1 \quad (-\mathbf{H}_X)_2 \right] \tag{16}$$

where $\tilde{\mathbf{H}} = \tilde{\mathbf{T}}_{CELL} \mathbf{H}_X$ with the subscripts 1 and 2 indicating the columns of the matrices between parentheses. (BRENNAN, 1994; BRENNAN *et al.*, 1997).

3. RESULTS AND DISCUSSIONS

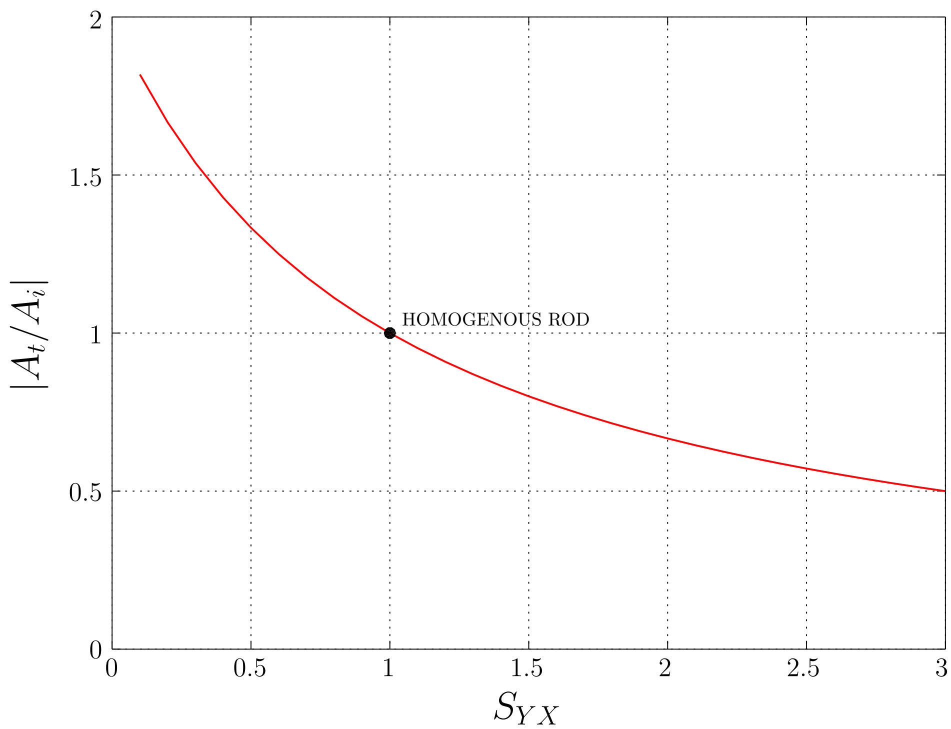
This section of the paper presents numerial simulations for analisys of the change in cross-sectional area of the rod and which are its consequences for the longitudinal propagating waves in the structure.

3.1 DISCONTINUOUS ROD

For a condition of discontinuity in a rod (Fig. 1), the transmitted and reflected wave amplitudes are independent of frequency. Therefore, the study is done according to the ratio of cross-sectional areas and/or material properties, by Eqs. (9) and (10).

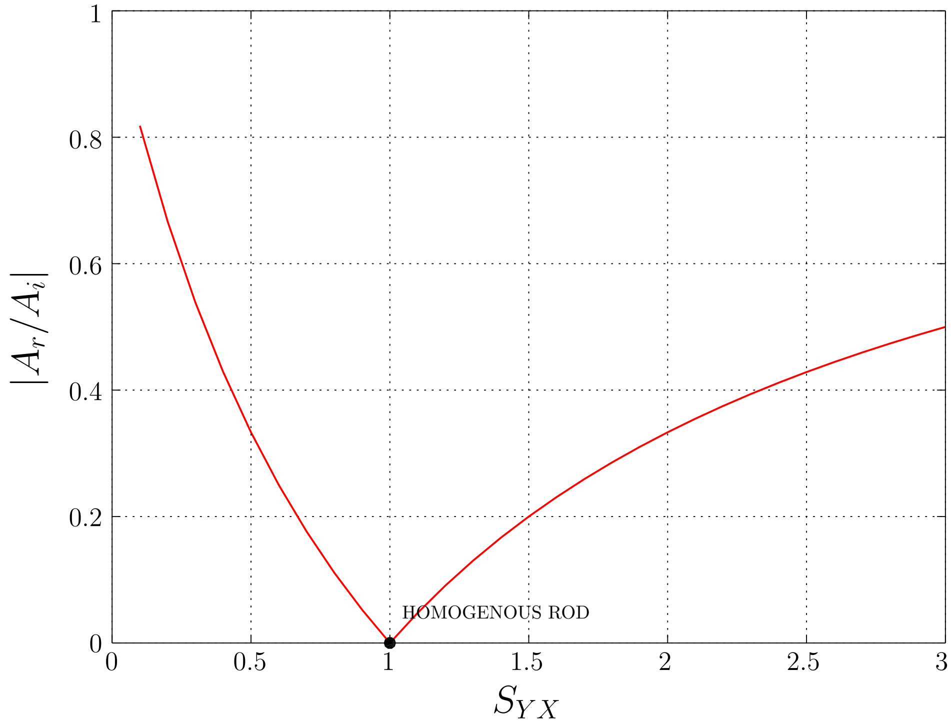
The Fig. 3 shows the relationship between the transmitted and incident wave amplitudes as a function of the change in cross-sectional area. Note that the transmitted wavelength is greater than the incident $\left(\frac{A_t}{A_i} > 1\right)$ when the ratio between the areas is less than 1 ($S_{YX} < 1$). The Fig. 4 shows the amplitude of the reflected longitudinal wave increasing, since, once it is reduced in amount of transmitted, there is an increase of the reflected one.

Figure 3. Transmitted waves amplitudes in relation to longitudinal incident waves in functin of the cross-sectional area ratio variation, with constant materials properties.



The changing in cross-sectional area effects directly the wave propagation because of the impedance of the rod. The supression occurs because the impedance of the segment Y is greater than the impedance of segment X. This way, the segment Y represents bigger resistance to the wave propagation than the segment X and, as a consequence, the amplitude of wave in the segment Y becomes smaller. On the other hand, with $S_{YX} < 1$, the impedance of the side Y of the rod is smaller than the X side, what explains the increase of the amplitude of transmitted waves, as can be seen in Fig. 3.

Figure 4. Reflected waves amplitudes in relation to longitudinal incident waves in functin of the cross-sectional area ratio variation, with constant materials properties.



3.2 INFINITE ROD WITH DISCONTINUITIES

Considering rods with rectangular cross-sectional area and constant width, different cross-sectional heights are considered, for area ratios $t_{YX} = 1.2$, $t_{YX} = 1.5$ and $t_{YX} = 2$, to obtain the transmission and reflection graphics of longitudinal wave amplitude propagating in the rod in function of the frequency.

Analyzing Figs. 5 and 6, note that the smaller the area ratio $t_{YX} = t_Y/t_X$, the greater the tendency of the wave amplitude transmitted to be of 100%. This is because the cross-section variation is one of the characteristics that favor the suppression of vibrations. Increasing values of t_{YX} indicate greater differences between the sides Y and X of the rod and, consequently, smaller transmission of longitudinal wave amplitude. So, as can be observed, the phases of transmitted and reflected waves are the opposite.

The gap in between the begining and the end of the wave is what is called *band gap* and the maximum value os amplitude at Fig. 6 means the optimum supression for this case of cross-sectional area, i.e., there is a frequency value, in the midle of the *stop band*, that allows the reflection of incident longitudinal wave, according to which ratio $t1_{YX}$ considered.

Figure 5. Transmitted waves amplitudes in relation to longitudinal incident waves.

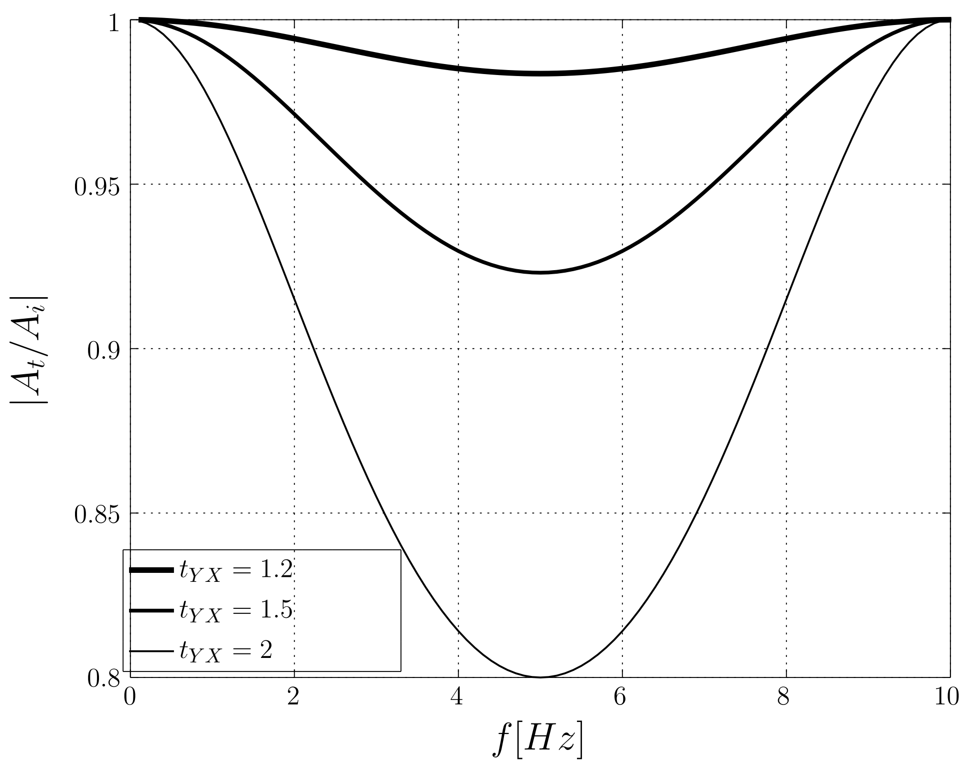
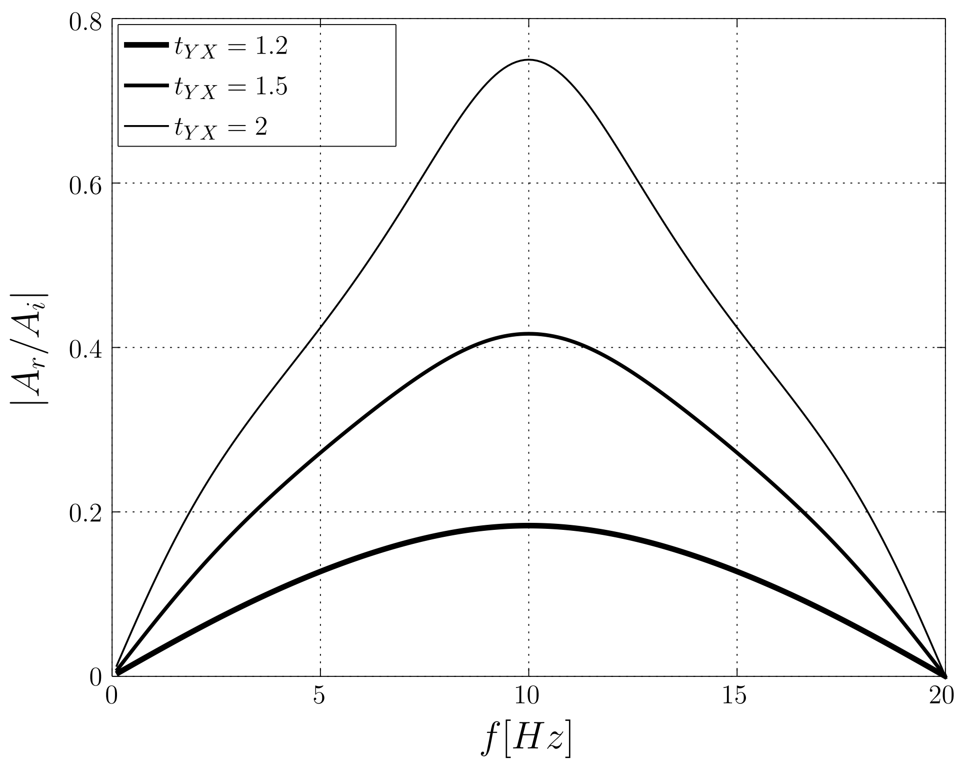


Figure 6. Transmitted waves amplitudes in relation to longitudinal incident waves.



4. CONCLUSIONS

The plots for both descontinuity in a rod and a finit cell between left and right homogenous sides show that the increasing ratio between cross-sectional areas along a rod favors the suppression of vibrations, causing reflection of the portion of the incident wave coming from the application of an axial force at the end of the rod. Increasing values of S_{YX}, or t_{YX}, indicate greater differences between the sides Y and X and, consequently, smaller transmission of longitudinal waves amplitudes. But it is important to take into consideration that this presented work only analyzes the propagation of longitudinal waves, thus, it is not considered the flexural waves behavior at the discontinuity, what would require another formulation and different considerations, following the same path. Besides, the formulation presented is developed from matricial solutions, but the results are given by the analitical equations, and for a number of cells above one, it is interesting to work directly with the matrices, because the formulation becomes a lot more complex.

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