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ANALYTICAL TRANSIENT HEAT CONDUCTION THROUGH COMPOSITE HOLLOW CYLINDERS MEDIUM SUBJECTED TO MULTIPLE BOUNDARY CONDITIONS

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Abstract. *This study presents an analytical method of non-steady radial heat conduction in a multilayer cylinder medium with combined boundary conditions. The analysis developed here provides the capability to extend the applications for multiple given boundary conditions in closed-form solutions remodeling the original governing heat conduction equation into spatial state form using Laplace transformation solved for the s-domain, followed by the use of distributed transfer functions that result on Green's functions when the inverse Laplace transform is performed. In comparison to other methods, the mathematical framework presented just involves two by two matrixes procedures, which turns out simplifying efforts for the solution implementation for any number of layers. Further, a validation using computational finite element approach is presented along with the analytical method in order to verify the correctness and accuracy of the solution. The methodology described here can be useful for many emerging fields which both transient and steady-state heat transfer are important parameters to be accounted.*

Keywords: *Analytical Non-steady Heat Conduction, Composite Hollow Cylinder Medium, Laplace transform, Distributed Transfer Functions*

1. INTRODUCTION

Engineering applications that involves multi-layer components are many, such as civil industries (Zhang, *et al*., 2013), biomaterials (Gerhard, 1952), heat exchangers (Gu and O'Neal, 1995), electrical applications (Dobrzanski, *et al.*, 2007), and so on. Due to the advantage of combined thermal properties of different assembled materials, this configuration is commonly used in order to combine a set of desirable project requirements. Thereby, the knowledge of both transient and steady-state heat transfer behavior retains a fundamental role for the material which should be used to achieve the expected proposed results. With that scope, numerical results are highlighted among most engineers as a mean of solving problems of this nature faster, although much of the physics that occurs with the heat transfer phenomena remains vague. On the other hand, the analytical procedure tends to be more complicated, but provides more insight about the problem when compared to the numerical solutions (Torabi and Zhang, 2016). This work will combine both analytical and computational approaches, where the former will retain much more attention and the latter will be used for the validation of the analytical procedure developed here.

Several references present different ways to achieve the heat transfer solutions such as Separation of Variables Method (SVM) (Torabi and Zhang 2016; Jain, *et al.*, 2009), Laplace transform method (Carslaw and Jaeger, 1959; Lu, *et al*., 2005), generalized orthogonal and quasi-orthogonal expansion techniques (Gu and O'Neal, 1995; Özisik, 1980; Title, 1965), and so forth. Those methods reach accurate response along with computational finite difference method, but as the number of layers increases, the coefficient matrix for calculating the eigenvalues becomes larger, making the computational effort greater.

Although conduction heat transfer seems to be a well-founded subject, continued studies has been carried out in order to optimize the Laplace transform method by the fact that the inverse Laplace transformation is often a challenge to be obtained. With that scenario, the Distributed Transfer Function Method (DTFM) is introduced and broadly used in mechanical vibrations (Yang and Tan, 1992; Yang, 1994). Further, DTFM was also being used to find solutions of conduction heat transfer through composite plates (Yang, 2008) and composite hollow cylinders (Yang and Liu, 2017), which the latter development is discussed here. This technique demands the Laplace transform of the original second-order transient heat conduction governing equation, resulting in a s-domain expression, which will be converted to a first-order state equation when defined a consistent spatial state vector. Then, a unique solution is obtained through the use of the concept of state transition matrix that will be converted on a fundamental matrix which is nonsingular for all given time (Chen, 1999), reaching a later solution based on distributed transfer functions which is the Laplace transform of the

Green's function evaluated at s-domain of the boundary-initial value problems (Yang, 2008). Lastly, taking the inverse Laplace transform, the transfer functions became the Green's function evaluated in time, where by residue theorem (Carslaw and Jaeger, 1959; Morse and Feshbach, 1953) the final transient solution is achieved, being dependent only in terms of the fundamental matrix, the boundary-initial value problems and the residues, where the last is calculated from an expression given by the theory of complex analysis (Morse and Feshbach, 1953). Each one of these terms is dependent on the eigenvalues which can be solved disregarding both external and boundary disturbances of the system and finding the poles the transfer functions (Yang and Fang, 1994), achieving the final transient heat conduction solution.

2. METHODOLOGY

Considering a multi-layer cylinder composed by *n* layers, as illustrated in figure 1,

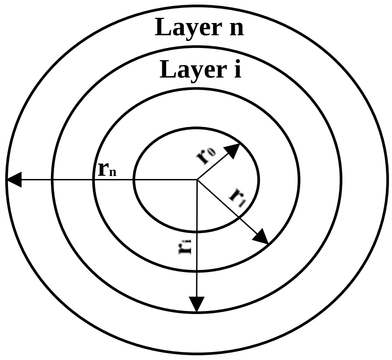

Figure 1. Composite multi-layer hollow cylinder.

The transient governing equation is declared by Eq.(1), accounting a dependent time-spatial internal heat generation within the composite hollow cylinder configuration.

$$\frac{\partial^2 T_i(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T_i(r,t)}{\partial r} + \frac{g_i(r,t)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i(r,t)}{\partial t}, \qquad r_{i-1} \le r \le r_i, \qquad t \ge 0 \quad (i = 1, 2 \dots, n) \tag{1}$$

Being T_i, α_i, k_i and g_i the temperature, material thermal diffusivity, material thermal conductivity and the heat source, respectively, of the *i-th layer*, r the radius and t a given time. Subjected to the following boundary-initial conditions,

$$a_1 \frac{\partial T_1(r,t)}{\partial r} + a_0 T_1(r,t) = \gamma_{in}(t)\,; \qquad r = r_0, \qquad t > 0 \tag{2a}$$

$$b_1 \frac{\partial T_n(r,t)}{\partial r} + b_0 T_n(r,t) = \gamma_{out}(t)\,; \qquad r = r_n, \qquad t > 0 \tag{2b}$$

$$-k_i \frac{\partial T_i(r_i,t)}{\partial r} = -k_{i+1}\frac{\partial T_{i+1}(r_i,t)}{\partial r}\,; \qquad r = r_i,\ \ i = 1,2,\dots,n-1, \qquad t > 0 \tag{2c}$$

$$T_i(r_i,t) = T_{i+1}(r_i,t); \qquad r = r_i,\ \ i = 1,2,\dots,n-1, \qquad t > 0 \tag{2d}$$

$$T_i(r,0) = \theta_i\,(r); \qquad r_{i-1} \le r \le r_i, \qquad t = 0 \tag{2e}$$

Where θ_i is an initial temperature profile of the *i-th layer*, and a_0, a_1, b_0, b_1 are boundary conditions dependent constants. The whole analysis predicts that the conduction heat transfer occurs only in radial direction which demands that the ratio between the cylinder length and the initial radius, L/r_0 must be greater than 10 (Incropera, *et al.*, 2003), so that the cylinder is considered as an infinite solid, which allows for an unidimensional approach. Also, the assembled materials used on the composite hollow cylinder medium presents isotropic properties.

2.1 Distributed Transfer Function Method Transient Solution

Applying the DTFM procedure as mentioned earlier, a solution in dimensional form has been derived as follows:

$$\{\eta(r,t)\} = \sum_{k=1}^{\infty}\left[U\left(r,-\beta_k{}^2\right)\right][R_k]\left([M_b]\{I_{g,k}(t)\} + \{I_{b,k}(t)\} + [M_b]\{I_{\theta,k}(t)\}\right) \tag{3}$$

Where $\left[U\left(r,-\beta_k{}^2\right)\right]$ is the fundamental matrix calculated by Eq.(4-6), $[R_k]$ is the residual matrix Eq.(7), $[M_b]$ and $[N_b]$ are the coefficient matrix of the inner and outer boundary-conditions given by Eq.(8), $\{I_{g,k}(t)\}$, $\{I_{b,k}(t)\}$ and $\{I_{\theta,k}(t)\}$ which are the constant matrixes that relates to the heat being generated within the composite hollow cylinder structure, boundary excitations and given temperature profile, respectively.

$$[U(r,s)] = \begin{cases} \left[\mu_1\left(r,-\beta_k{}^2\right)\right] \\ \left[\mu_i\left(r,-\beta_k{}^2\right)\right]\left[\mu_{i-1}\left(r_{i-1},-\beta_k{}^2\right)\right]\dots\left[\mu_1\left(r_1,-\beta_k{}^2\right)\right] \end{cases} \tag{4}$$

$$\left[\mu_i\left(r,-\beta_k{}^2\right)\right] = \left[u_i\left(r,-\beta_k{}^2\right)\right]\left[u_i\left(r_{i-1},-\beta_k{}^2\right)\right]^{-1} \tag{5}$$

$$\left[u_i\left(r,-\beta_k{}^2\right)\right] = \begin{bmatrix} J_0\left(\frac{\beta_k}{\sqrt{\alpha_i}}r\right) & Y_0\left(\frac{\beta_k}{\sqrt{\alpha_i}}r\right) \\ -\frac{\beta_k}{\sqrt{\alpha_i}}k_iJ_1\left(\frac{\beta_k}{\sqrt{\alpha_i}}r\right) & -\frac{\beta_k}{\sqrt{\alpha_i}}k_iY_1\left(\frac{\beta_k}{\sqrt{\alpha_i}}r\right) \end{bmatrix} \tag{6}$$

$$[R_k] = \frac{adjoint\left([M_b] + [N_b]\left[\mu_n\left(r_n,-\beta^2\right)\right]\left[\mu_{n-1}\left(r_{n-1},-\beta^2\right)\right]\dots\left[\mu_1\left(r_1,-\beta^2\right)\right]\right)}{det\left\{[M_b] - \frac{1}{2\beta_k}[N_b]\frac{d}{d\beta}\left[U\left(r_n,-\beta_k{}^2\right)\right]\right\}} \tag{7}$$

$$[M_b] = \begin{bmatrix} a_0 & \frac{a_1}{k_1} \\ 0 & 0 \end{bmatrix}; \; [N_b] = \begin{bmatrix} 0 & 0 \\ b_0 & \frac{b_1}{k_n} \end{bmatrix} \tag{8}$$

Being J_0 and Y_0 zero-order Bessel functions of first and second kind, respectively; J_1 and Y_1 first-order Bessel functions of first and second kind, respectively. The eigenvalues β_k are estimated solving Eq.(9) below by means of Rolle's theorem, which verifies the existence of an infinite number of roots for the eigenvalues in the form $\beta_1 < \beta_2 < \beta_3 < \cdots < \beta_n < \cdots$ (Özisik, 1968).

$$\beta = det\left([M_b] + [N_b]\left[\mu_n\left(r_n,-\beta^2\right)\right]\left[\mu_{n-1}\left(r_{n-1},-\beta^2\right)\right]\dots\left[\mu_1\left(r_1,-\beta^2\right)\right]\right) = 0 \tag{9}$$

Then, the constant matrixes are determined by Eq.(10-12) depending on which boundary value problem is being considered, and all necessary terms to solve the transient heat conduction are at hand. The final temperature and heat flux response are given by performing $T(r,t) = [1 \quad 0]\{\eta(r,t)\}$; $q(r,t) = [0 \quad -1]\{\eta(r,t)\}$.

$$\{I_{g,k}(t)\} = -\sum_{i=1}^{n}\left[U\left(r_{i-1},-\beta_k{}^2\right)\right]^{-1}\int_0^t\int_{r_{i-1}}^{r_i} e^{-\beta_k^2(t-\tau)}\left[\mu_i\left(\xi,-\beta_k{}^2\right)\right]^{-1}g_i(\xi,\tau)d\xi\,d\tau\begin{pmatrix}0\\1\end{pmatrix} \tag{10}$$

$$\{I_{b,k}(t)\} = \int_0^t e^{-\beta_k^2(t-\tau)}\begin{pmatrix}\gamma_{in}(\tau)\\ \gamma_{out}(\tau)\end{pmatrix}d\tau \tag{11}$$

$$\{I_{\theta,k}(t)\} = -e^{-\beta_k^2 t}\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\left[U\left(r_{i-1},-\beta_k{}^2\right)\right]^{-1}\int_{r_{i-1}}^{r_i}\left[\mu_i\left(\xi,-\beta_k{}^2\right)\right]^{-1}\theta_i(\xi)d\xi\begin{pmatrix}0\\1\end{pmatrix} \tag{12}$$

3. RESULTS AND DISCUSSION

3.1 Model validation

Two cases will be evaluated in order to validate the proposed methodology: Fist case will solve a numerical example considering a two-cylinder compared following a two-way validation with another analytical method proposed by

Mikhaillov, *et al*. (1983) and with Finite Element Method (FEM) provided by the software ANSYS, utilizing the properties illustrated at Tab.1.

Table 1. Thermal properties and geometrical parameters for the two-layer hollow cylinder configuration.

Two-layer cylinder properties	$r_0 = 1,\ r_1 = 2,\ r_2 = 4, k_1 = 4,\ k_2 = 1, \alpha_1 = 4,\ \alpha_2 = 1$

Defining prescribed temperatures $T_0 = 1$ and $T_2 = 0$ at the inner and outer surface, respectively, we have a boundary condition of excitation response form, described by Eq. (11). Given the geometrical and thermophysical parameters enables the determination of the eigenvalue problem given in Eq. (9), where the first ten computed eigenvalues are expressed at Tab. 2.

Table 2- First ten eigenvalues of the two-layer hollow cylinder configuration.

k	Root(β_k)	Eigenvalue($-\beta_k{}^2$)
1	1.31659	-1.73341
2	2.49771	-6.23857
3	3.60429	-12.99093
4	4.88884	-23.90084
5	6.27688	-39.39925
6	7.64644	-58.46814
7	8.85750	-78.45539
8	9.93127	-98.63014
9	11.18202	-125.03757
10	12.56321	-157.83437

All presented cases were solved by running MATLAB on a Windows desktop computer with an Intel Core i7 processor and 16 GB memory. For this case, 3978 eigenvalues were necessary for each considered time $t = 0.1$ s, $t = 0.8$ s, and $t \to \infty$, where the elapsed time took 55.8 seconds to complete the iterative solution procedure. Determining the residues and temperature profiles by means of Eq. (7) and Eq. (3), respectively, one can verify at Fig.2 that the three-way aforementioned validation scheme is in good agreement with the present DTFM methodology discussed in this work, being possible to expand the analysis to more complex cases.

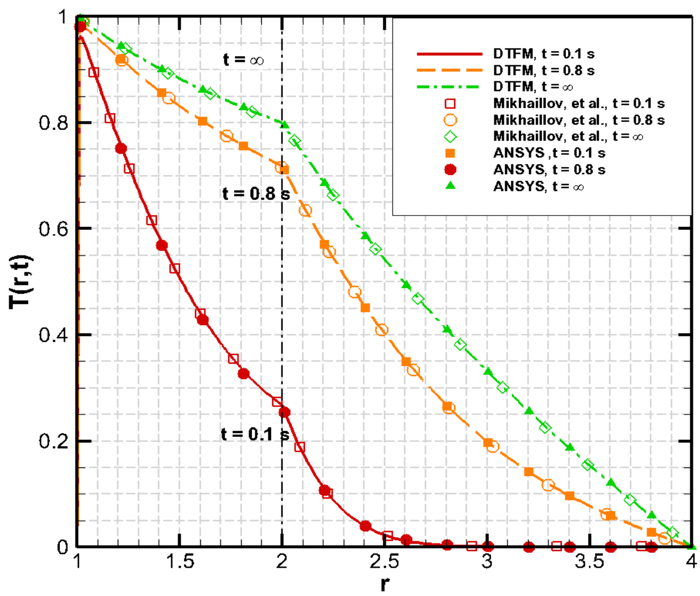

Figure 2. Temperature profile on the two-layer hollow cylinder with prescribed inner and outer temperature surface.

The second case will treat a heat conduction problem through a seven-layers composite hollow cylinder validated by means of FEM approach, with geometric and thermophysical properties given at Tab.3.

Table 3-Thermal properties and geometrical parameters for the seven-layer hollow cylinder structure.

Radius	$r_0 = 0.5,\ r_1 = 1.5,\ r_2 = 3, r_3 = 5,\ r_4 = 6.5, r_5 = 8.5,\ r_6 = 9.5, r_7 = 10.5$
Thermal Conductivity	$k_1 = 2,\ k_2 = 3, k_3 = 5,\ k_4 = 5, k_5 = 3,\ k_6 = 2, k_7 = 1.5$
Thermal Diffusivity	$\propto_1 = 6,\ \propto_2 = 8, \propto_3 = 10,\ \propto_4 = 14, \propto_5 = 11,\ \propto_6 = 9, \propto_7 = 7$

Prescribing a heat flux profile at the inner surface of the form

$$\gamma_{in}(t) = q_0\,(1 - e^{-0.2t}) \tag{13}$$

It fulfills the requirements to perform the eigenvalue problem solving Eq. (9), casting the computed eigenvalues at Tab. 4.

Table 4- First ten eigenvalues of the seven-layer hollow cylinder configuration.

k	Root(β_k)	Eigenvalue($-{\beta_k}^2$)
1	0.52886	-0.27970
2	1.61140	-2.59664
3	2.38287	-5.67809
4	3.28664	-10.80201
5	4.16964	-17.38592
6	5.97242	-25.22852
7	6.80319	-35.66989
8	7.91119	-46.28353
9	8.69527	-62.58700
10	9.74188	-75.60784

Here, 20 eigenvalues presented good result convergence for each $t = 0$ s, $t = 2$ s, $t = 7$ s, $t = 13$ s, and $t = 40$ s, with solution elapsed time of 21.1 seconds. Equation (13) describes the boundary condition problem as a boundary excitation on the inner surface, leading to the determination requirement Eq. (11). Hence, computing the residues and transient response cast by Eq. (7) and Eq. (3), the temperature profile throughout the seven-layered system is illustrated at Fig. 3.

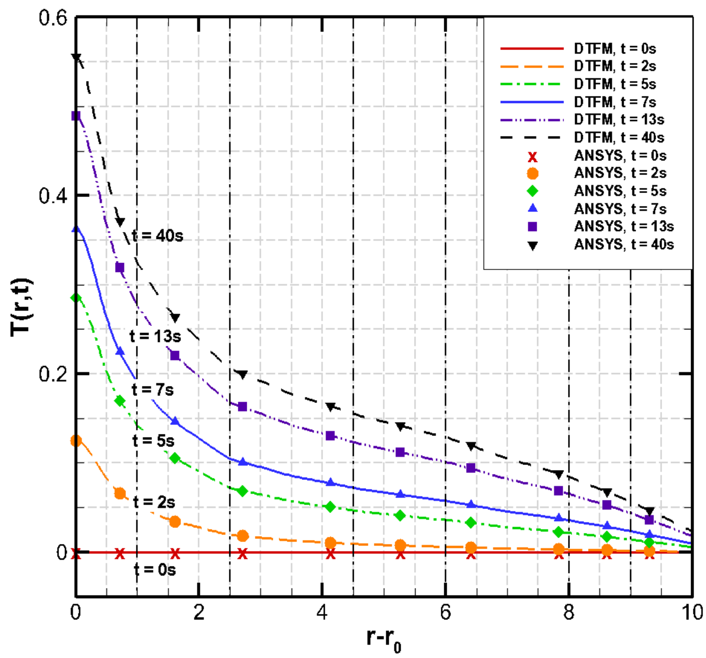

Figure 3. Temperature profile on the seven-layer hollow cylinder composite with prescribed heat flux on the inner surface.

Still, as observed at Fig. 3, the correctness of the presented method is good agreement with the computational approach, evidencing the strong potential of DTFM to describe transient conduction heat transfer with reliable solutions.

3.2 Results

As a matter to challenge the correctness solution between DTFM and FEM approach, a case with multiples boundary conditions was proposed. This case involves five hollow cylinder layers in perfect contact with each other subjected to an inner flux at the inner surface, a volumetric heat source on the second layer and a thermal profile throughout the whole system. But instead of the second case discussed on the previous subsection, a heat flux profile with cusps was arranged, so that the function becomes non-differentiable, which implies a non-straightforward solution procedure when compared with section two development.

Firstly, following the solution procedure of the validation cases, Tab. 5. defines the geometrical and thermophysical properties of the five-layer composite hollow cylinder structure

Table 5- Thermal properties and geometrical parameters for the five-layer hollow cylinder structure.

Radius m	$r_0 = 0.04445,\ r_1 = 0.047625,\ r_2 = 0.10795, r_3 = 0.12065\ \ r_4 = 0.14605\ r_5 = 0.25$
Thermal Conductivity $\mathrm{W/m°C}$	$k_1 = 24.09,\ \ k_2 = 79.7, k_3 = 24.9,\ k_4 = \ 0.53, k_5 = 2.11$
Thermal Diffusivity $10^{-7}\ \mathrm{m^2/s}$	$\propto_1= 69.94,\ \ \propto_2= 258.60, \propto_3= 69.94,\ \propto_4= 3.583, \propto_5= 8.96$

An initial condition of $\theta_i\ = 60\ °\mathrm{C}$ temperature profile is stablished. The inner surface heat flux profile was determined by applying the Inverse Heat Conduction Problem (IHCP) (Özisik, 1980) due a volumetric heat source within the inner hollow cylinder, which amount have same intensity from that one verified by a second heat generation at the second layer of 30E+06 W/m³, however those heat sources occurs at different times. The first heat will be computed for a given time range of $0 \leq t_{g0} \leq 300$ s, whereas the second $300\ \mathrm{s} \leq t_{g2} \leq 600$ s. Performing IHCP is necessary, once DTFM framework developed for composite hollow cylinder structure does not allow to declare an internal heat source on the first layer, since mathematical singularities are observed for Y_0 when $r_0\ = 0$. Therefore, the verified heat flux at the inner surface is illustrated on Fig. 4.

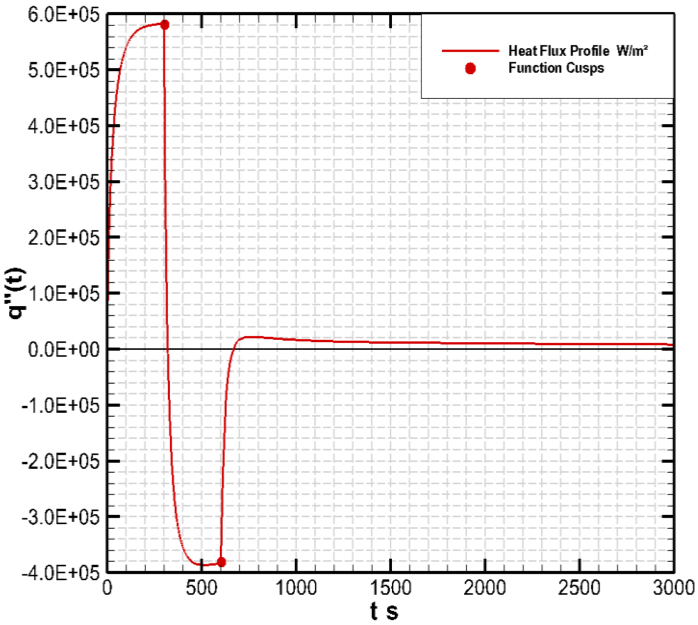

Figure 4. Resultant heat flux profile at the inner surface resultant from an IHCP applied for a volumetric heat source of 30E+06 W/m³.

Symbology observed at Fig.4. indicates the function cusps, which demands and adaptative step-wise procedure for the solution is proposed, since no exact regression quadrature are able to thorough describes the given profile.

a) Divide the heat flux profile in several segments between cusps, from the very plot beginning until the last identified point.
b) With each function segment established, regressions turn out to be feasible. For this work, three Gaussian regression were used for each identified segment;

c) The first regression is due to the occurrence of the first heat source, within $0 \leq t_{g0} \leq 300$ s. At this point, the second source have not happened yet, so we solve the boundary conditions of boundary excitation and given temperature profile, which is given by Eq. (11) and Eq. (12), respectively.
d) After solving the eigenvalue problem and find residues, the transient solution is computed and the output temperature profile for $t_{g0} = 300$ s must be used as input for the next observed regression;
e) For the second regression, we will have three boundary condition occurring at the same time; the list time step from step (d), the second segment of the heat flux profile and the volumetric heat source at the second layer being activated lasting from $300 \text{ s} \leq t_{g2} \leq 600$ s, being necessary to introduce the constant matrix expressed by Eq. (10); now, $t_{g2} = 600$ s is the input for the last regression;
f) No heat source is identified for the last regression, where only the temperature profile determined on (e) and the remnant of the heat flux profile are performed.
g) Repeat items (d), (e) and (f) until all desired boundary conditions and required regression are done.

Defined the geometrical and boundary conditions for the problem, as well as the adaptative procedure determined, one is able to perform the eigenvalue problem described by Eq. (9). The then first determined eigenvalues are given at Tab. 6.

Table 6- First ten eigenvalues of the five-layer hollow cylinder configuration.

k	Root(β_k)	Eigenvalue($-\beta_k^2$)
1	0.00258063	-0.00000666
2	0.01198663	-0.00014368
3	0.03054600	-0.00093306
4	0.05455644	-0.00297641
5	0.07303805	-0.00533456
6	0.09007835	-0.00811411
7	0.11396798	-0.01298870
8	0.13650761	-0.01863433
9	0.15191382	-0.02307781
10	0.17293085	-0.02990508

For each regression, 54 terms were necessary to confer good convergence when performing Eq. (3), with a total elapsed time of 377.74 s to complete the iterative solution. The resultant temperature profile is illustrated at Fig. 5 below.

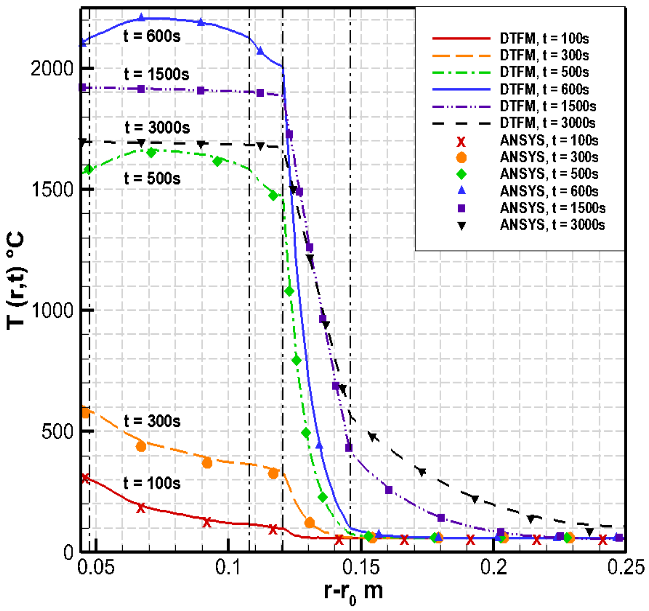

Figure 5. Temperature profile for a five-layer composite hollow cylinder subjected to prescribed heat flux at the inner surface, given initial temperature profile and volumetric heat generation at the second layer

About the solution correctness, the proposed adaptative scheme does not achieve the perfect match as observed on the validation schemes, where functions that describes the boundaries problem are differentiability for the whole domain. However, it was verified a maximum error of 5,40 % between DTFM and FEM analysis for the worst-case scenario, which is a permissible result dispersion, since it was caused due to the unperfect regression between the curve fit adopted with Fig. 4 given profile.

4. CONCLUSION

Aiming the reliability of the method along with others transient heat conduction approach, DTFM showed to be an accurate toll for predict temperature profiles in composite hollow cylinders configuration with any number of layers. When differentiable functions are used, one can expect tiny or absence of errors when compared with FEM results, while on the treatment of non-differentiable functions are handled, one can verify the existence of small error amount, since this error is induced due to curve fitting procedure.

This method involves only 2x2 matrixes operations, which turn out to be straightforward the replicability and reproducibility of DTFM. Still, simple matrix manipulations make DTFM one among the available analytical method one that demands less computational effort.

Obtained results for the case involving multiple boundary conditions can be applied for real cases on emerging fields, since day-to-day industry challenges tend to imply more complex parameters that must not be neglected.

Forthcoming work using DTFM are currently in progress, which aims to remove several considered assumptions made for this work, accounting contact conductance between layers, thermal stress analysis, evaluation of the effects when using material properties varying with the temperature profile and so on.

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