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NUMERICAL IMPLEMENTATION FOR CONCEPTION OF STRUT AND TIE MODELS IN REINFORCED CONCRETE STRUCTURES

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Abstract. *The strut-and-tie models can be an excellent alternative for the design of reinforced concrete structures in regions with geometric or static discontinuity, replacing empirical procedures with a rational design methodology. The present article uses the topological optimization technique ESO (Evolucionary Structural Optimization) allied to the Finite Element Method for elastic-linear analysis of structures subjected to plane stress state. This way, it is possible to obtain optimized solutions in the structural design of a given problem. To validate the implemented methodology, optimal topologies of three distinct examples are presented, namely: bridge structure, deep beam with a large hole and Michell structure.*

Keywords: *strut-and- tie models, topology optimization, finite element method, reinforced concrete structures.*

1. INTRODUCTION

For regions or structural elements in which the Bernoulli Hypothesis does not adequately describe the structural behavior or stress distribution, other design alternatives can be used, such as the Finite Element Method (FEM) and the Strut-and-Tie Model (STM). The STM represents the generalization of the classical analogy of the truss beam model showm by Ritter and Morsch at the beginning of last century. This analogy associated with a reinforced concrete beam in an equivalent truss structure, where the discrete elements represent the fields of tensile stress (ties) and compression (struts).

Pioneering studies developed by Schalaich et al. (1987) allowed the systematic application of the strut-and-tie model to certain types of structural elements, such as deep beams, corbel and column, foundation blocks, among others.

However, the nonuniqueness of the topological model makes the design dependent on the experience and structural sensitivity of the designer. Therefore, the study of techniques that assist the designer in the automatic generation of the strut-and-tie model has gained impulse, especially the work of Liang et al. (2000), Liang et al. (2002).

It is in this context that the objective of this work is inserted, that is, to provide an efficient tool for the automatic generation of the strut-and-tie model via topological optimization (TO), defining the best configuration to be adopted for the analysis. We will adopt a layout optimization technique, initially proposed by Xie and Steven (1993) called Evolutionary Structural Optimization (ESO). The essence of the method is the gradual removal of less-requested regions, based on a penalty criterion based on von Mises equivalent stresses. That is, elements with tensions below a certain threshold are removed from the mesh at each iteration in a process called "hard-kill". This way, it is possible to obtain an optimal structure for a given remaining volume.

2. FORMULATION OF THE TRIANGULAR ELEMENT

The finite element implemented in this work is the two-dimensional triangular element of three nodes and two degrees of freedom per node, well-known as Constant Strain Triangle (CST), used in the numerical simulation by means of FEM, based on displacements, of structures under the plane state of tensions. The material is considered homogeneous, isotropic and linear.

According to the Principle of Virtual Work (PVW), applying a virtual deformation field compatible with the triangular element has: $\delta W_{int} = \delta W_{ext}$, that is, the external virtual work (δW_{ext}) is equal to the internal virtual work (δW_{int}). The δW_{int} can be written as:

$$\delta W_{\text{int}} = \iiint_V \sigma_{ij} \delta\varepsilon_{ij} dV \tag{1}$$

where, δ is the variational operator, σ_{ij} the real tensional state at any point in the element, and $\delta\varepsilon_{ij}$ is the state of virtual deformation at any point in the element. From the strain-displacement tensor of Green-Lagrange and neglecting the tensions in the plane of normal z, arrive at the Eq. (2) for the internal virtual work, where *t* is the thickness of the triangular element.

$$\delta W_{\text{int}} = t\iint_A [\delta u_{,x}\sigma_x + \delta v_{,y}\sigma_y + (\delta u_{,y} + \delta v_{,x})\tau_{xy}]dA \tag{2}$$

For three-node triangular element it is usual to adopt the interpolation functions in natural coordinates, which can be written as:

$$N_1 = \xi_1 = \xi = \frac{(y_2-y_3)}{2A}x + \frac{(x_3-x_2)}{2A}y + \frac{(y_3-y_2)x_2-(x_3-x_2)y_1}{2A} \tag{3a}$$

$$N_1 = \xi_2 = \eta = \frac{(y_3-y_1)}{2A}x + \frac{(x_1-x_3)}{2A}y + \frac{(x_3-x_1)y_1-(y_3-y_1)x_1}{2A} \tag{3b}$$

$$N_3 = \xi_3 = 1-\xi-\eta \tag{3c}$$

where x_1, x_2, x_3, y_1, y_2, y_3 are the Cartesians coordinates of three-node triangular element and A is the area of element.

Defining the vector of the nodal displacements by $\mathbf{q} = \{u_1 \quad u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3\}^T$ and representing the interpolation function in natural coordinates by vector column $\mathbf{\Phi}^T = (N_1 \quad N_2 \quad N_3)$, the approximate equations of the displacements associated to the nodal displacements **q** can be written in matrix form as Eq. (4).

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \mathbf{\Phi}^T & \mathbf{O}^T \\ \mathbf{O}^T & \mathbf{\Phi}^T \end{bmatrix}\mathbf{q} \tag{4}$$

In Eq. (4), **O** is a null column vector with three terms. Since the displacements *u* and *v* are functions of nodal displacements, their variations can be written from the following expression: $\delta a = \delta\mathbf{q}^T\left(\frac{\partial a}{\partial \mathbf{q}}\right)$, where $\mathbf{q}$ is the vector of nodal displacements, ∂ the differencial operator and a replaced by *u* and *v*. Substituting into Eq. (2), the virtual work of a triangular element is given by the following equation.

$$\delta W_{\text{int}} = \delta\mathbf{q}^T t\iint_A [\frac{\partial u_{,x}}{\partial \mathbf{q}}\sigma_x + \frac{\partial v_{,y}}{\partial \mathbf{q}}\sigma_y + (\frac{\partial u_{,y}}{\partial \mathbf{q}} + \frac{\partial v_{,x}}{\partial \mathbf{q}})\tau_{xy}]dA \tag{5}$$

The external virtual work is given by $\delta W_{ext} = \delta\mathbf{q}^T\mathbf{f}_{ext}$, where $\mathbf{f}_{ext}$ is the vector of external force, applied in the direction of the degrees of freedom of the element and nodal internal forces obtained from the external loading acting on the element. From the condition $\delta W_{ext} = \delta W_{\text{int}}$ follows that:

$$\delta\mathbf{q}^T t\iint_A [\frac{\partial u_{,x}}{\partial \mathbf{q}}\sigma_x + \frac{\partial v_{,y}}{\partial \mathbf{q}}\sigma_y + (\frac{\partial u_{,y}}{\partial \mathbf{q}} + \frac{\partial v_{,x}}{\partial \mathbf{q}})\tau_{xy}]dA = \delta\mathbf{q}^T\mathbf{f}_{ext} \tag{6}$$

The above expression must be valid for any compatible virtual displacement field ($\delta\mathbf{q}$), it follows that: $\mathbf{f}_{int} - \mathbf{f}_{ext} = \mathbf{0}$, where $\mathbf{f}_{int}$ is the vector of internal which is defined by:

$$\mathbf{f}_{\text{int}} = t\iint_A [\frac{\partial u_{,x}}{\partial \mathbf{q}}\sigma_x + \frac{\partial v_{,y}}{\partial \mathbf{q}}\sigma_y + (\frac{\partial u_{,y}}{\partial \mathbf{q}} + \frac{\partial v_{,x}}{\partial \mathbf{q}})\tau_{xy}]dA \tag{7}$$

The Eq. (7) can be written in the matrix form as:

$$\mathbf{f}_{int} = t\iint_A \begin{bmatrix} \sigma_x \mathbf{\Phi}_{,x} + \tau_{xy}\mathbf{\Phi}_{,y} \\ \sigma_y \mathbf{\Phi}_{,y} + \tau_{xy}\mathbf{\Phi}_{,x} \end{bmatrix} dA \tag{8}$$

Using the Newton-Raphson method for solving the problem $\mathbf{f}_{\text{int}} - \mathbf{f}_{\text{ext}} = \mathbf{0}$ it is necessary to determine the derivative of this expression in relation to the nodal displacements, thus obtaining the tangent stiffness matrix. Being constant $\mathbf{f}_{\text{ext}}$ in relation to nodal displacements, the tangent stiffness matrix is given by, $\mathbf{K} = \partial\mathbf{f}_{\text{int}} / \partial\mathbf{q}$ which after algebraic manipulations can be written according to Eq. (1).

$$\mathbf{K} = t\iint_A \begin{bmatrix} \mathbf{\Phi}_{,x}\left(\frac{\partial \sigma_x}{\partial \mathbf{q}}\right)^T + \mathbf{\Phi}_{,y}\left(\frac{\partial \tau_{xy}}{\partial \mathbf{q}}\right)^T \\ \mathbf{\Phi}_{,y}\left(\frac{\partial \sigma_y}{\partial \mathbf{q}}\right)^T + \mathbf{\Phi}_{,x}\left(\frac{\partial \tau_{xy}}{\partial \mathbf{q}}\right)^T \end{bmatrix} dA \tag{9}$$

where $\mathbf{\Phi}_{,x}$ and $\mathbf{\Phi}_{,y}$ are the derivative of the form functions with respect to *x* and *y*, respectively, σ_x and σ_y are the normal stresses in the *x* and *y* directions, respectively, τ_{xy} is the shear stress, **q** is the nodal displacement vector, *t* is the thickness and *A* is the area of the element.

In the isoparametric representation, the Cartesian coordinates *x* and *y* are related to the coordinates ξ and η. Thus, to change the integration domain *dA* to *dξdη*, is used the relation $dA = det\boldsymbol{J}\, d\xi d\eta$ where **J** is the Jacobian matrix of the transformation of the *x* and *y* coordinates for the parametric coordinates ξ and η is given by Eq. (10). Thus, $det\boldsymbol{J} = 2A$.

$$J = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} \tag{10}$$

3. EVOLUTIONARY STRUCTURAL OPTIMIZATION

The ESO technique appears as an alternative to the mathematical rigor of classical optimization methods. It presents a simple theoretical base whose foundation consists in the insertion of voids in the structure through the gradual elimination of the less requested elements of the domain during the process of evolution. Therefore, in order to obtain the optimal configuration, a level of structural analysis dependent on a discrete domain should be added to the study, which makes the use of the Finite Element Method as step of the optimization algorithm.

In this work, the mathematical representation ESO is based on the concept of tension, that is, the maximum tension level in the structure, obtained by analysis via FEM, is taken as an indicator of the level of efficiency of each element. Elements with a low tension level are therefore systematically removed from the structure. At each iteration, new inefficient elements are eliminated from the mesh and the procedure is repeated until the field tension across the domain is practically constant and very close to the allowable tension of the material or that the minimum volume restriction is reached.

The removal criterion is made by comparing the von Mises tension of each element with the maximum von Mises tension across the entire structure. Therefore, at the end of each iteration all the elements that satisfy the inequality (11) will be eliminated. The form of removal of the element occurs by assigning low values for its Young's Modulus ($E=10^{-12}$). This way, the new mesh generation of the structure is avoided.

$$\sigma_e^{\upsilon M} < RR_i \sigma_{max}^{\upsilon M} \tag{11}$$

where $\sigma_e^{\upsilon M}$ is the tension of von Mises in the analyzed element, RR_i is rejection ratio $\left(0 < RR_i < 1{,}0\right)$ and $\sigma_{max}^{\upsilon M}$ is the maximum von Mises tension for each iteration. The rejection ratio is used to delay the removal process of the element. The removal cycle occurs until no more elements can be removed for a given RR_i value. When this occurs, a steady state is reached. The evolutionary process is redefined by adding to RR_i an ER evolution ratio. The rejection ratio is updated according to Eq. (12).

$$RR_{i+1} = RR_i + ER \quad i = 0,1,2,... \tag{12}$$

The initial value of the rejection ratio (RR_0) is defined empirically by the user. However, according to Querin (1997), to ensure better convergence, the RR_0 and ER values should be approximately 1%. Thus, to prevent RR_i from reaching very high values, the rejection ratio of each iteration must always be less than a pre-established maximum rejection ratio, RR_f. This prevents the removal of a very large region.

Mathematically, ESO can be written as:

$$D(j) = \begin{cases} D_0, if \ j \in \Gamma \\ 0, \ if \ j \in \Gamma' \end{cases} \tag{13}$$

where D (j) is the constitutive matrix of the point $j \, \varepsilon \, \Omega$; D_0 is the initial constitutive matrix of design defined for each finite element, $\Omega = r + r'$ is the domain of the structure, such that $r = (\Omega / (\sigma_e /> = RR_i))$ is the set of elements that will not be removed and $r' = \Omega - r = (\Omega / (\sigma_e / <RR_i))$ is the set of elements that will be removed from the structure, all in the i^{th} iteration.

Therefore, the ESO algorithm presents the following steps:

1st step: discretize the domain for analysis with a refined mesh of finite elements;

2nd step: solve the linear elastic problem, applying displacement boundary conditions and external or body forces;

3rd step: Determine the von Mises stress distribution for each element and maximum at each iteration;

4th step: Remove the elements that satisfy inequality 3, within a predefined volume limit (p%);

5th step: Repeat os step 2, 3 e 4 until the steady stage is reached;

6th step: at iteration i^{th}, if $RR_i < RR_f$, update rejection ration according Eq.(4) and repeat steps 2, 3 e 4. Otherwise, do not update RR and plot final topology.

The Fig. 1 schematically illustrates the ESO algorithm:

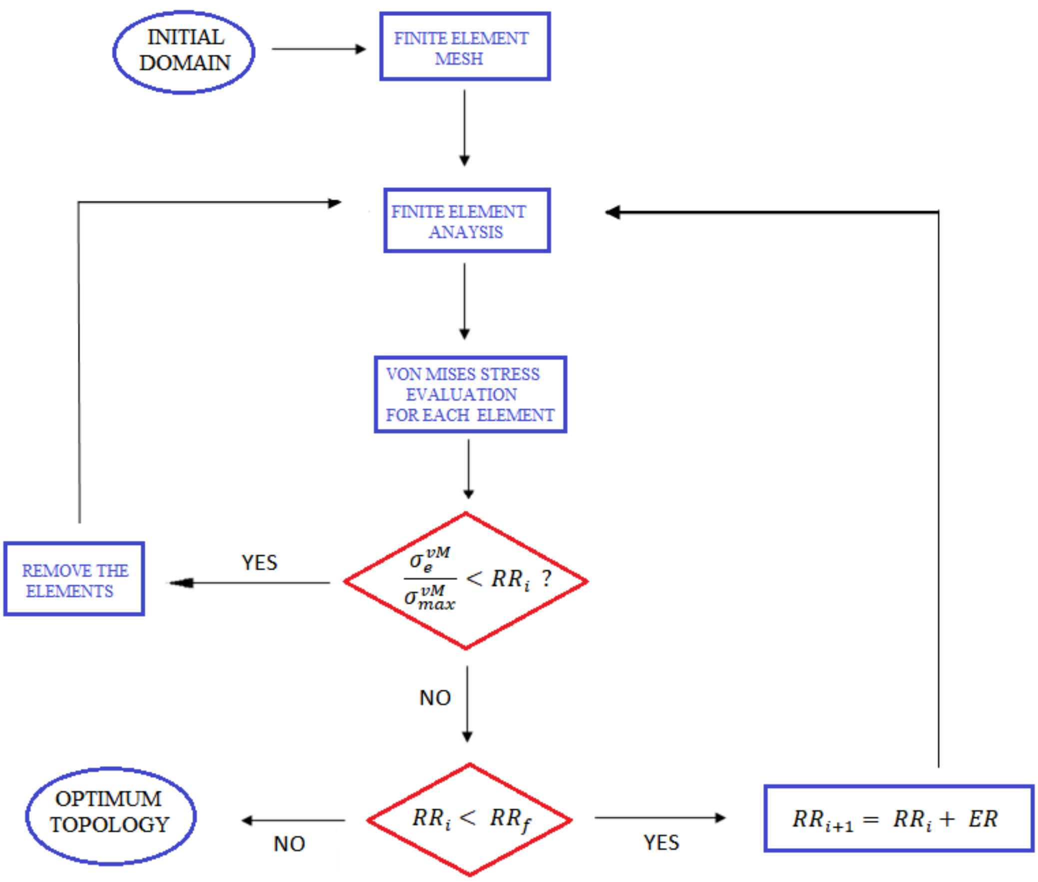

Figure 1. Flowchart for ESO procedure.

4. VON MISES CRITERIA

The removal criterion is based on the von Mises stress of each element in the ith iteration, calculated in the centroid of each element, using the nodal stress values. In terms of the principal stresses, the von Mises stresses can be written according to Eq. (14):

$$\sigma_e^{vM} = \frac{\sqrt{2}}{2}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \quad (14)$$

For the case of plane tension state, with $\sigma_3 = 0$, it is:

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{esc}^2 \quad (15)$$

where σ_1 and σ_2 are the principal stress and σ_y is the material yielding tension.

5. EXAMPLES

5.1 Introduction

The formulation described in conjunction with MEF was implemented, considering an elastic-linear analysis in plane stress state. The discretization of the two-dimensional region is made from triangular finite elements. This option is due to the fact that this element requires a sufficiently detailed discretization of the continuum, thus allowing to define the compression and traction regions of the strut-and-tie model with more refinement.

5.2 Bridge Structure

The Fig. 2a presents a problem proposed by Liang et al. (2002) that applied the method called PBO (Performance-Based Optimization) to define the optimal topology of the structure shown in Fig. 2b. To do so, he used a mesh of 90x30 quadrilateral elements and four nodes.

It is a bridge with a central tray subjected to a uniformly distributed load with concentrated forces of 500kN per node. It was considered in the present analysis, Young's modulus of material E = 200GPa, Poisson's ratio equal to υ = 0.3 and thickness equal to t = 30 cm. It was adopted a mesh with 5400 three-node triangular elements.

During the evolutionary process, it was not allowed to remove the elements in which the external loading is applied as well as the elements where the boundary conditions are imposed.

The optimum topology shown in Fig. 2c was obtained using RR = 1%, ER = 1.5% and a final volume equal to 35%, in which the compression fields (strut) are represented in blue and the tensile fields (tie) in red.

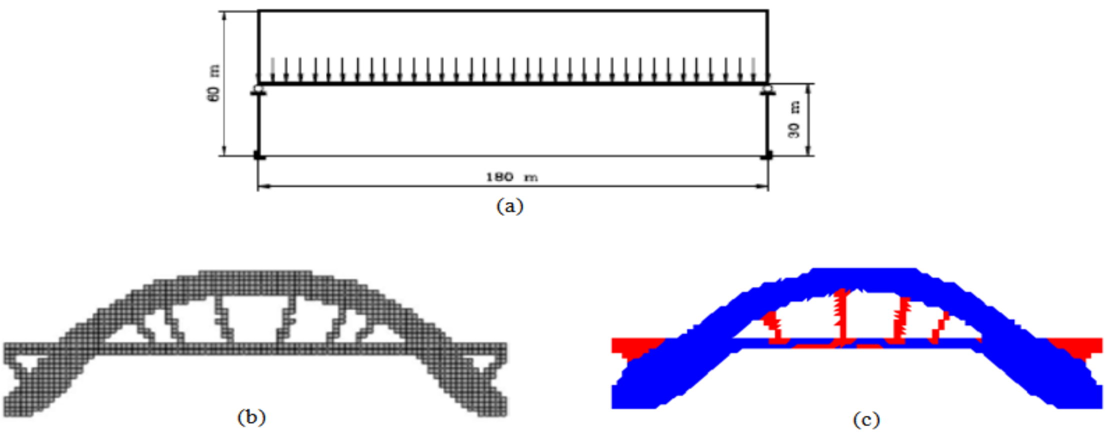

Figure 2. (a) Design domain of a bridge structure (b) Optimal topology obtained by Liang et al. (2002) (c) Optimal topology obtained with the present formulation.

5.3 Deep Beam with large hole

This example corresponds to a structural element extracted from Schlaich et al. (1987) and also studied by Liang et al. (2000). It is a deep beam with a large hole.

Its geometry, applied load and boundary conditions are shown in Fig. 3a. The Young's modulus of material was taken to be equal to E = 20820 MPa, Poisson's ratio equal to $\upsilon = 0.15$.

For the modeling of the element in the developed program was considered a mesh composed of 6600 finite elements of the type CST and 3452 nodes. The optimal topology presented in Fig. 3b was obtained considering the following parameters for ESO optimization: RR_0 = 4.0%, ER = 2.0% and final volume equal to 52%, in which the compression fields (strut) are represented in blue and the traction fields (tie) in red.

The Fig. 3c shows the optimal topology found by Liang et al. (2000) via ESO, and in Fig. 3d the respective model of strut-and-tie in which the dashed lines correspond to the compression regions while the continuous ones represent the traction regions.

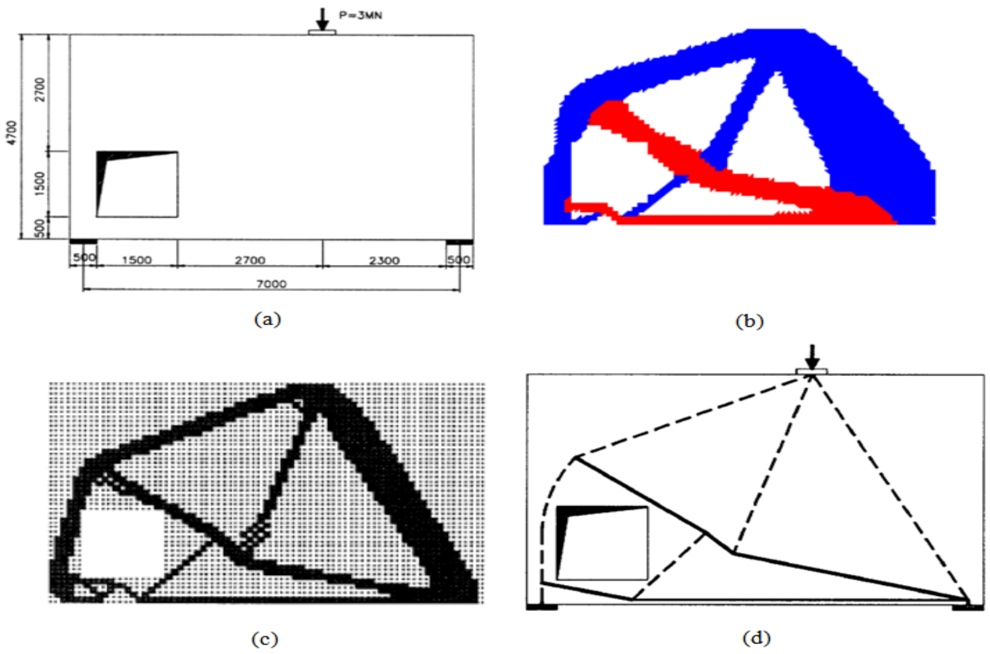

Figure 3. (a) Deep beam with large hole (b) Optimal topology obtained by the present work (c) Optimal topology obtained by Liang et al. (2000) (c) Strut-and-tie model proposed by Liang et al. (2000)

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5.4 Michell Structure

This example is to illustrate the evolution of the structure during the process, whose optimum topology represents the desired strut-and-tie model. The Fig. 4 shows the simply supported Michell structure under a concentrated load of F=400N. The design domain is divided in 7680 three-node triangular elements. This example was studied by Xie and Steven (1993).

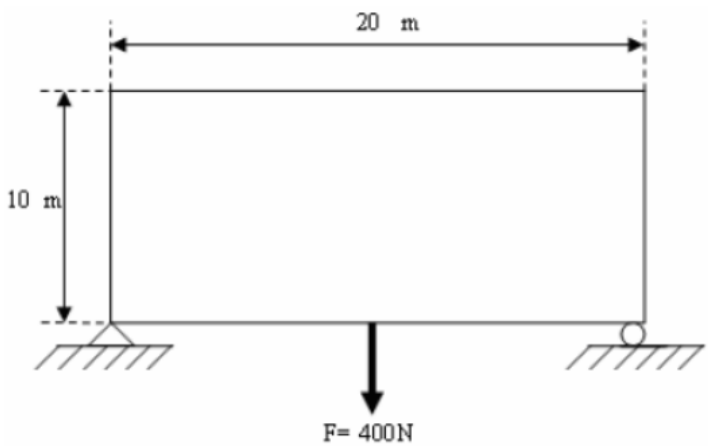

Figure 4. Design domain of simply supported Michell structure

The evolutionary began with rejection ratio (RR) equal to 1% and the evolution ratio (ER) equal to 0.75%. The Young's modulus of material was taken to be equal to E = 200 GPa, Poisson's ratio equal to $\upsilon = 0.15$, final volume equal to 40% and the volume removed by iteration of 1,75%. The Fig. 5 shows the evolution of the structure, where VR represents the total volume removed until the iteration.

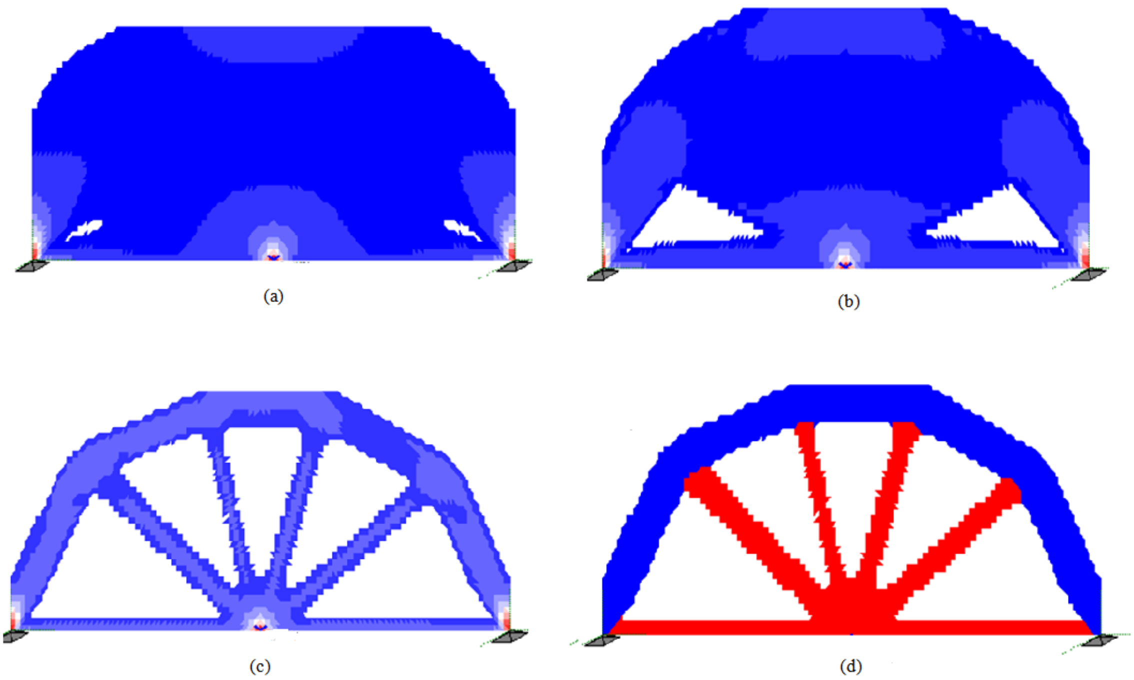

Figure 5. (a) Iteration 21, RR = 2,5%, VR = 5,6% (b) Iteration 69, RR = 4,75%, VR = 18,6% (c) Iteration 200, RR = 10,75%, VR = 60,0% (d) Strut and tie model

6. CONCLUSION

The objective of the article is the presentation of a numerical formulation to verify the flow of stresses in reinforced concrete structures under the focus of the strut-and-tie models. The topological optimization algorithm, called Evolutionary Structural Optimization, was used for this purpose. In general, it is an evolutionary procedure of element removal together with elastic analysis using the finite element method in plane stress state analysis.

An extended domain is initially defined and, iteratively, the method searches for an optimum topological configuration in which the members are indicated as strut and ties. Therefore, the present work assists the designer in conception of the strut-and-tie model, and thus the efforts in the members may be evaluated.

Finally, it can be stated that the three numerical examples demonstrated good accurancy with the results found in the literature and illustrate the effectiveness of the proposed optimization procedure.

7. ACKNOWLEDGEMENTS

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