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ITERATIVE CONDENSATION STRATEGY TO FATIGUE ANALYSIS OF VISCOELASTICALLY DAMPED

Lauren Karoline de Sousa Gonçalves
Ulisses Lima Rosa
Antônio Marcos Gonçalves de Lima
Universidade Federal de Uberlândia
laurenkaroline@ufu.br

Abstract. *Looking for structures' durability, safety, reliability and comfort have been motivated the study about increase of the fatigue life using viscoelastic dampers. However, analysis of dynamic responses and fatigue criteria of viscoelastic stochastic models could be prohibitive due to the computational time required to solve full models' formulations and the number of degrees of freedom involved. These models require the use of reduction basis particularly suitable that consider the frequency and temperature. In this sense, the purpose of this work is to evaluate computational gain of a proposed iterative reduction basis representing the stochastic behavior of a three-layer sandwich plate structure, treated with a passive constrained damping layer under Gaussian random loadings. Reduced results are then compared with the full model via numerical simulations. Numerical results are presented in terms of frequency responses functions (FRFs), stress responses (PSDs) and fatigue indexes estimated by Sines' criterion. With this approach, it is possible to highlight the robustness of the iterative condensation strategy.*

Keywords: *stochastic finite elements method, passive control, fatigue, reduction methods*

1. INTRODUCTION

In most of the situations, engineering systems are projected considering mean values of their physical and geometrical properties. However, it is common to occur variation in these properties due to inherent uncertainties of materials properties, environmental conditions and random loadings. In order to obtain real structural dynamic behaviors with reasonable accuracy, it is necessary to use stochastic finite element models (Ben Abdessalem et al., 2016). However, it is observed few models considering any type of uncertainty in literature (De Lima et al., 2014; Lambert et al., 2010; Pagnacco et al., 2012). This is because the stochastic simulations may become prohibitive due to high computational cost and the large number of degrees of freedom involved in the evaluation of stress responses and estimation of the fatigue index. Thus, it is necessary to apply a model condensation technique well suited to viscoelastic stochastic systems enabling the dynamic analyses.

Stochastic models' analysis has become more broadly applied due to the development of model condensation strategies and computers with robust processing and storage capacities, making it possible to solve models that require considerable computational resources (De Lima et al., 2010; Cunha-Filho et al., 2018). However, the choice of basis to approach a viscoelastically damped systems is not easy, due to temperature and frequency dependence of stiffness matrix (De Lima et al.; 2015). This study was motivated by the necessity of condensing stochastic viscoelastic systems using a proper reduction basis. A model reduction method proposed by Bobillot and Balmès (2002) and Cunha-Filho et al. (2018) was adapted and applied to the full model.

The main objective of this work is to evaluate the robustness of the iterative reduction basis and computational gain. Furthermore, evaluate fatigue failure of stochastic systems subjected to cyclic loading in the frequency domain. After a theoretical background, a numerical application is developed to compare the reduced deterministic and stochastics results with the full model and highlight the robustness of the iterative condensation strategy to insure their use in real projects.

2. ITERATIVE ENRICHED RITZ METHOD (IERM)

In order to build more realistic models, Stochastic Finite Elements Method (SFEM) is used. Uncertainties can be included in model parameters following non-parametric (Soize, 2000; Ritto et al., 2008) and parametric (De Lima et al., 2010) approaches. The numerical implementation is based on the theory of Khatua and Cheung (1973), which was also applied by De Lima et al. (2010).This section summarizes the finite element modeling of a moderately thin three-layer sandwich plate composed by a base-panel (1), a viscoelastic core (2) and a passive constraining layer (3), as depicted in Fig. 1.

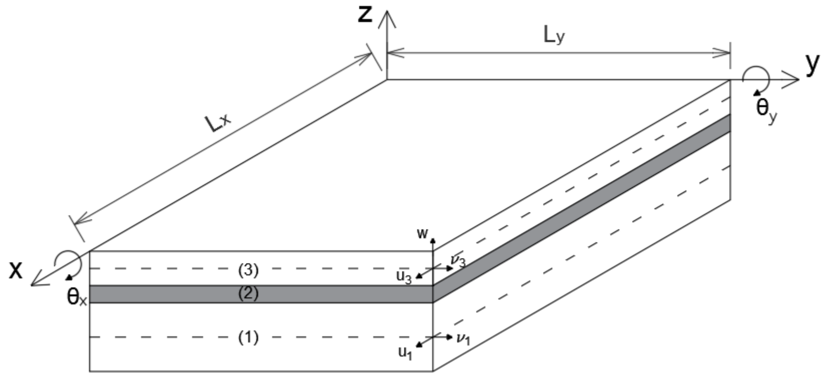


Figure 1: Three-layer sandwich plate

Sandwich plate dimensions are $654mm \times 527mm$, discretized by 10×10 plane rectangular finite elements mesh with four nodes and seven degrees of freedom (DOFs) per node, which are five in-plane displacements in directions $\left(u_1, v_{1,} u_3, v_3, w\right)$ and two cross-sections $\left(\theta_x, \theta_y\right)$. All materials were considered homogeneous, isotropic and with linear mechanical behavior. Kirchhoff's theory (Zienkiewicz and Taylor, 2005) and Mindlin's theory were used on the modeling of elastic layers and viscoelastic core, respectively. The mechanical properties of each layer are presented in Tab. (1).

Table 1. Mechanical properties of the sandwich plate.

Layer	Thickness (h)	Young's Module (E)	Poisson Ratio	Mass Density
Base plate	1.0	70	0.29	2700
Viscoelastic Core	0.13	-	0.49	950
Constraining Layer	0.25	70	0.29	2700

The stochastic viscoelastic sandwich system's behavior can be represented by an ordinary differential equation, as shown on Eq. (1):

$$\left[\boldsymbol{K}_1\left(\theta\right)+\boldsymbol{K}_3\left(\theta\right)+G\left(\omega,T,\theta\right)\boldsymbol{K}_2\left(\theta\right)-\omega^2\boldsymbol{M}\left(\theta\right)\right]\boldsymbol{U}\left(\omega,T,\theta\right)=\boldsymbol{F}\left(\omega\right) \tag{1}$$

where $\boldsymbol{M}\left(\theta\right)$ and $\boldsymbol{K}_k\left(\theta\right)$ with $k=1,2,3$ are the global FE random mass and stiffnesses matrices, respectively. $\boldsymbol{U}\left(\omega,T,\theta\right)$ is the stochastic response, and $G\left(\omega,T,\theta\right)$ is the random complex modulus to account for the uncertain temperature. The complex modulus and shift factor of the 3M ISD112^{TM} viscoelastic material used in this study are determined by Drake and Soovere (1984) formulation, following Eq. (2) for frequency and temperature intervals of, $1Hz \leq \omega \leq 10^6\ Hz$ and $210K \leq T_v \leq 360K$.

$$\mathrm{G}\left(\omega,T_v\right)=0.4307+\frac{1200}{1+3.24\times\left(\dfrac{i\omega_r}{1543000}\right)^{-0.18}+\left(\dfrac{i\omega_r}{1543000}\right)^{-0.6847}}\ \left[MPa\right] \tag{2}$$

where $\alpha\left(T\right)=10^{\left(-3758.4\times\left(\frac{1}{T}-0.00345\right)-225.06\times\log\left(0.00345\times T\right)+0.23273\times\left(T-290\right)\right)}$ is the *shift factor* and $\omega_r=\alpha\left(T\right)\omega$ is the so-called *reduced frequency*.

The solution of Eq. (1) requires the preliminary determination of elementary mass and stiffness matrices for elastics and viscoelastic layers from Eqs. (3) to (5). After that, global mass and stiffness matrices can be assembled by the system's connectivity, considering the behavior purely elastic and representing the initial strain state.

$$\mathbf{M}^{(e)} = \sum_{k=1}^{3} \rho^{(k)} h^{(k)} \int_{x=0}^{a} \int_{y=0}^{b} \mathbf{N}^T(x,y)\mathbf{N}(x,y)\,dydx \tag{3}$$

$$\mathbf{K}_e^{(e)} = \sum_{k=1}^{3} \int_{z=0}^{h^{(k)}} \int_{x=0}^{a} \int_{y=0}^{b} \mathbf{B}_k^T(x,y,z)\mathbf{C}_k\mathbf{B}(x,y,z)\,dydxdz \tag{4}$$

$$\mathbf{K}_v^{(e)}(\omega,T) = \int_{z=0}^{h^{(2)}} \int_{x=0}^{a} \int_{y=0}^{b} \mathbf{B}_2^T(x,y,z)\mathbf{C}_2\mathbf{B}_2(x,y,z)\,dydxdz \tag{5}$$

where $\mathbf{C}_k$ e $\mathbf{C}_2$ are the matrices of elastic and viscoelastic coefficients, respectively.

As previously mentioned, the computational cost is one limitation of the numerical implementation of stochastic systems, and the use of condensation models able to approach the dynamic responses are an option. In this sense, this study proposed an iterative method based on residual displacements and well adapted for viscoelastically damped systems. This basis $\mathbf{T}_{ERM}^{K} = \left[\phi^0\right]$ was initially formulated based on the number of modes shapes retained in the nominal basis ϕ^0 and solving the eigenvalue problem to frequency ω_K^0. Then, reduced dynamic stiffness matrix $\mathbf{Z}_R\left(\omega_k^0, T\right)$ can be obtained from Eq. (6).

$$\mathbf{Z}_R\left(\omega_k^0,T\right) = \mathbf{T}_{ERM}^T\left[\mathbf{K}_e + \mathbf{K}_v\left(\omega_k^0,T\right) - \left(\omega_k^0\right)^2\mathbf{M}\right]\mathbf{T}_{ERM} \tag{6}$$

Then, residues associated to the applied external force $\mathbf{R}_F\left(\omega_k^0,T\right)$ and displacement $\mathbf{R}_d\left(\omega_k^0,T\right)$ are obtained by Eq. (7) and Eq. (8), respectively.

$$\mathbf{R}_F\left(\omega_k^0,T\right) = \frac{\mathbf{T}_{ERM}\mathbf{Z}\left(\omega_k^0,T\right)\mathbf{T}_{ERM}^T\mathbf{F}_b}{\mathbf{Z}_R\left(\omega_k^0,T\right)} - \mathbf{F}_b \tag{7}$$

$$\mathbf{R}_d\left(\omega_k^0,T\right) = \frac{\mathbf{R}_F\left(\omega_k^0,T\right)}{\left[\mathbf{K}_e + \mathbf{G}_0\right]\bar{\mathbf{K}}_v} \tag{8}$$

Finally, the iterative condensation strategy can be obtained by associating the nominal basis enriched by real and imaginary parts of the displacement residues, as shown in Eq. (9).

$$\mathbf{T}_{ERM}^{k+1} = \left[\mathbf{T}_{ERM}^{k} \quad \mathrm{Re}\left(\mathbf{R}_d\left(\omega_k^0,T\right)\right) \quad \mathrm{Im}\left(\mathbf{R}_d\left(\omega_k^0,T\right)\right)\right] \tag{9}$$

3. STOCHASTIC SANDWICH PLATE DYNAMIC BEHAVIOUR

From global matrices reduced, frequency response function can be obtained from the relationship between displacement and applied loading. Thus, the stochastic FRF is formulated as follows:

$$\boldsymbol{H}(\omega,T,\theta) = \left[\boldsymbol{K}_1(\theta) + \boldsymbol{K}_3(\theta) + G(\omega,T,\theta)\boldsymbol{K}_2(\theta) - \omega^2\boldsymbol{M}(\theta)\right]^{-1} \tag{10}$$

The frequency-domain stress response of the viscoelastic system can be obtained considering stress and strain relations (de Lima et al., 2014). The stress responses can be constructed via power spectral densities (PSD) applying a white Gaussian noise random load $\boldsymbol{\phi}_f(\omega)$ as follows:

$$\boldsymbol{\phi}_s(\omega,T,\theta) = \left(\boldsymbol{C}_k\boldsymbol{B}(x,y)\right)\boldsymbol{H}(\omega,T,\theta)\boldsymbol{\phi}_f(\omega)\boldsymbol{H}^H(\omega,T,\theta)\left(\boldsymbol{C}_k\boldsymbol{B}(x,y)\right)^T \tag{11}$$

where $\boldsymbol{\phi}_s(\omega,T,\theta) = \begin{bmatrix} \boldsymbol{\phi}_{xx,xx} & \boldsymbol{\phi}_{xx,yy} & \boldsymbol{\phi}_{xx,xy} \\ & \boldsymbol{\phi}_{yy,yy} & \boldsymbol{\phi}_{yy,xy} \\ sym & & \boldsymbol{\phi}_{xy,xy} \end{bmatrix}$ is the stress response (PSD) of a time-domain random stress vector of the form, $\boldsymbol{s}(t,T,\theta) = \left[s_{xx}\ s_{yy}\ s_{xy}\right]^T$.

Fatigue criteria used herein was the Sines criterion proposed by Sines (1959) for multiaxial loads. With this method, it is obtained a fatigue indexes (D_{Sines}) that is function of the square root of the stress deviatoric tensor second invariant . Sines' criterion is formulated by Eq. (12) which indicate fatigue failure for values greater than one. Khalij et al. (2010), Rosa and Lima (2016), Lambert et al. (2010) applied this method to fatigue analysis.

$$E\left[D_{Sines}\right] \approx \frac{E\left[\sqrt{J_{2a}(\theta)}\right]}{t_{-1}} \leq 1 \tag{12}$$

where $E\left[\sqrt{J_{2a}(\theta)}\right] \approx \sqrt{E\left[R_1^2(\theta)\right]+E\left[R_1^2(\theta)\right]+E\left[R_2^2(\theta)\right]+E\left[R_3^2(\theta)\right]+E\left[R_4^2(\theta)\right]+E\left[R_5^2(\theta)\right]}$ designates the square root of the amplitude of the second invariant of the random deviatoric stress tensor; statistical moments of the uncorrelated stationary Gaussian maximum measurements, $R_i(\theta)$, where $i=1\ to\ 5$.

Preumont (1985) presented a strategy for stationary Gaussian process, which the statistical moments of $R_i(\theta)$ are expressed as follows:

$$E\left[R_i(\theta)\right] = \sqrt{\lambda_0}\left(\mu_R + \gamma\beta_R\right) \quad for\ \ i = 1,\ldots \tag{13}$$

$$V\left[R_i(\theta)\right] = \lambda_0 \frac{\pi^2}{6}\beta_R^2 \qquad for\ \ i = 1,\ldots \tag{14}$$

where $\beta_R = 1\big/\sqrt{2\ln\left(\kappa_a N_p\right)}$ and $\mu_R = \sqrt{2\ln\left(\kappa_u N_p\right)}$ are, respectively, dispersion and the mode of $R_i(\theta)$, and $N_p = \frac{T_p}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}$ is the number of maxima associated with the components, $\boldsymbol{S}'(t)-\bar{\boldsymbol{S}}'$, evaluated in a period of time, t_p. λ_0 is spectral zero-order moments and λ_2 represents the spectral second-order moments of $\boldsymbol{S}'(t)$, and $\gamma = 0.5772$ is the Euler constant. Bandwidth parameters used in the process are defined as follows and dependent of the irregularity factor $\delta = \sqrt{1-\lambda_1^2/\left(\lambda_0\lambda_2\right)}$:

$$\kappa_u = \begin{cases} 1.5\left(1-e^{-1.8\delta}\right) & for \quad \delta < 0.5 \\ 0.94 & for \quad \delta \geq 0.5 \end{cases}, \ \kappa_a = \begin{cases} 7\delta & for \quad \delta < 0.5 \\ 4.05 & for \quad \delta \geq 0.5 \end{cases} \tag{15}$$

4. NUMERICAL APPLICATION

The numerical application described herein was performed on a three-layer sandwich plate treated with a passive constrained damping layer under Gaussian random loadings. The so-called Karhunen-Loève expansion is used as random fields discretization technique. The uncertain parameters were factored out of mass and stiffness matrices, and 10% uncertainty level was considered in the temperature and viscoelastic constrained core. The structure is a clamped-clamped-free-free rectangular plate made of aluminum, fully treated with a layer of 3M ISD112TM between base-plate and constrained-layer. A load was applied on the middle node and displacement was measured in the same point, under a frequency band of [0-100 Hz].

Firstly, a reduced model using an iterative model reduction basis enriched by structural modifications was compared to full model. It was considered the operation temperature of viscoelastic material is equal to T=25ºC and a unitary impulse loading applied and measured on the central node. The deterministic results are shown in Fig. 2.

Analyzing Fig. 2, the ability to approximate responses using an iterative model reduction basis is noted and the maximum dispersion was 0.21% for frequency bands [0-100 Hz]. From FRFs and PSDs responses, it is obtained the distribution of the Sines' damage indicator for full and reduced models, assuming an equivalent endurance limit of $t_{-1} = 92MPa$ valid for 2.0×10^6 cycles (Figure 3). Sines' indexes of plate showed a difference around 1% between the full and reduced systems' maximum values. The deterministic system ensured good performance without fatigue failure.

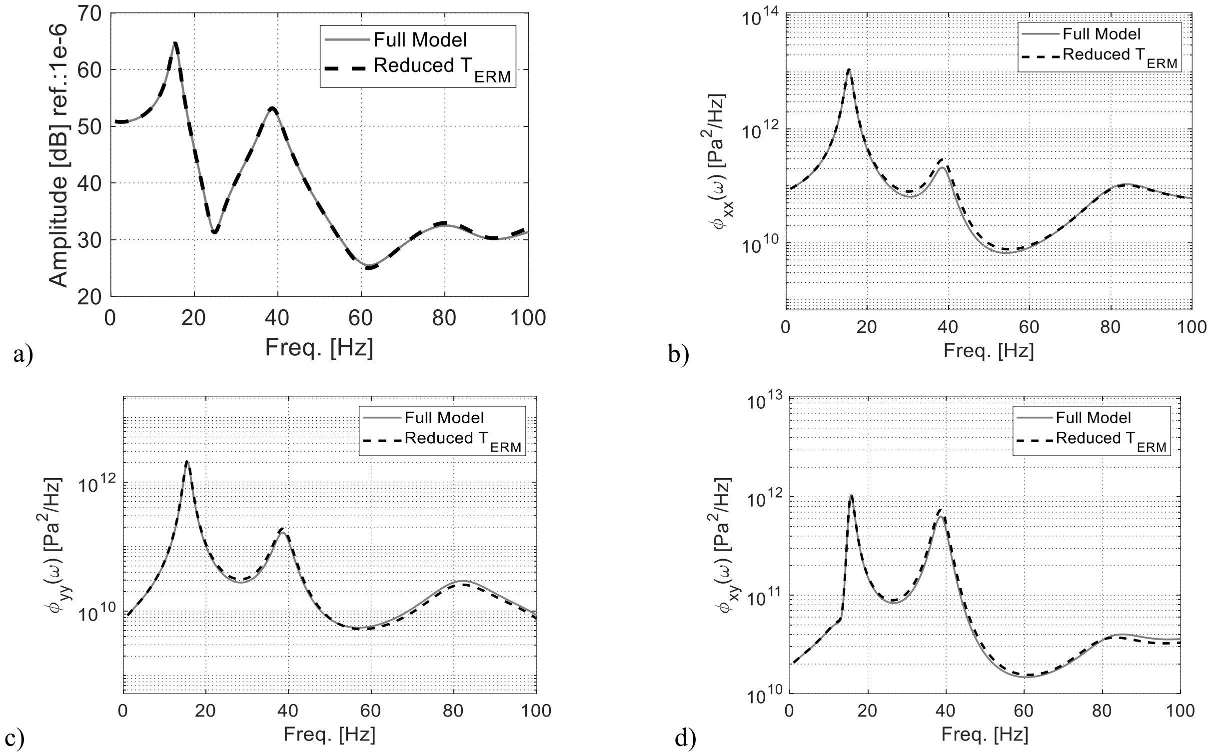


Figure 2: Amplitudes of full and reduced results of deterministic system: a) FRFs; b) stress response sxx; c) stress response syy; d) stress response sxy

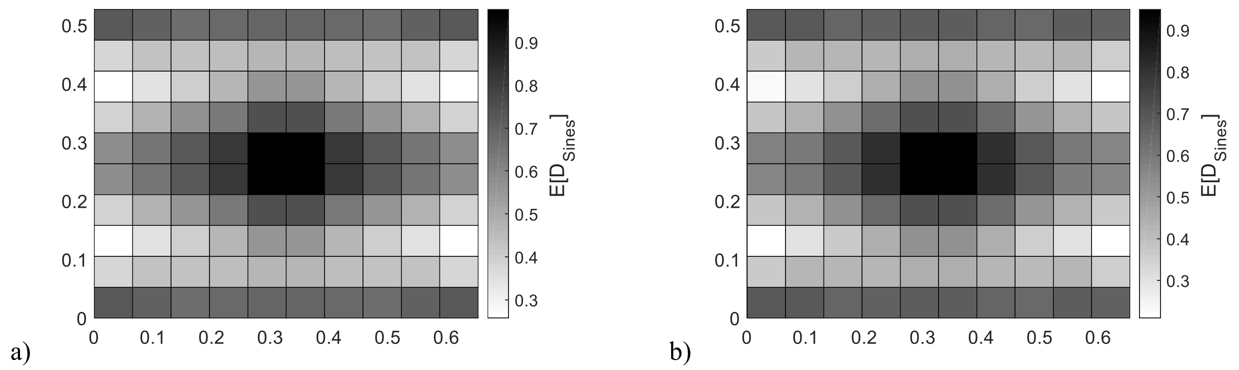


Figure 3: Sines' indexes: a) Full system; b) Reduced system

In relation to computational cost, Table 2 shows a quantification of the efficiency of this basis with a mean computational gain of around 98% for the calculation of frequency and stress responses of the determinist model. This result was obtained using a computer with Intel(R) Core(TM) i7-6700K of 4GHz, 16GHz RAM, GEFORCE NVIDIA 9800GT 1GB DDR3 and Windows 10 Pro 64 bits operational system.

Table 2. Computational cost to obtain the responses of full and reduced models.

	FRF	PSDs	GDLs
Full Model	62.91 s	925.75 s	847×847
Iterative Reduction Model	0.41 s	7.44 s	59×59

By applying modal assurance criteria (MAC), it is possible to compare the modal shape of reduced-order $(\boldsymbol{\Phi}_{NR})$ and complete $(\boldsymbol{\Phi}_{R})$ models (Cunha Filho et al., 2018). The comparison of modal shapes is established herein by Eq. (16), and the results are shown in Fig. 4.

$$MAC = \frac{\left|\boldsymbol{\Phi}_R^T \boldsymbol{\Phi}_{NR}^T\right|^2}{\left(\boldsymbol{\Phi}_R^T \boldsymbol{\Phi}_R\right)\left(\boldsymbol{\Phi}_{NR}^T \boldsymbol{\Phi}_{NR}\right)} \tag{16}$$

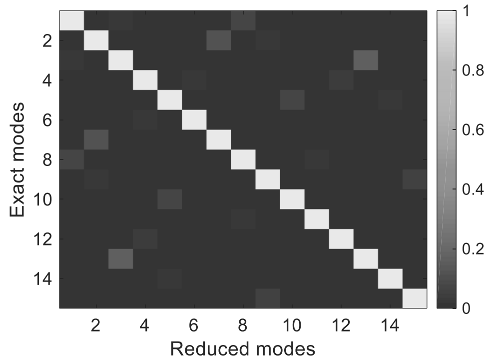


Fig. 4 - MAC associated with reduced-order and complete models by an iterative basis $\boldsymbol{T}_{ERM}$

As has been remarked a good capability of iterative reduction method in approach the deterministic results, a moderately thin three-layer sandwich plate element was formulated to obtain the behavior considering uncertainties. The interest now lies in implementing a viscoelastic stochastic model that considers random loadings and uncertain design variables. An SFEM was performed to analyze the system considering 10% uncertainties level in the thickness of viscoelastic core, constraining layers and temperature.

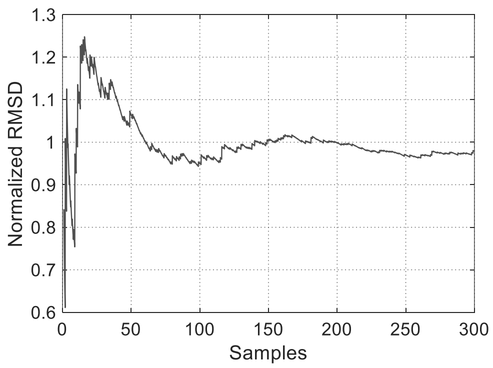
In order to validate results, it was important to estimate an optimal number of samples. For this purpose, a convergence analysis via normalized root-mean-square deviation (RMSD) of FRFs was performed. Figures 5(a) shows normalized RMSD. Eq. (17) shows how it is calculated:

$$RMSD = \sqrt{\frac{1}{Ns}\sum_{i=1}^{Ns}\left|\boldsymbol{H}\left(\omega,T,\theta\right)-\boldsymbol{H}\left(\omega,T\right)\right|^2} \tag{17}$$

where random, $\boldsymbol{H}\left(\omega,T,\theta\right)$, and nominal, $\boldsymbol{H}\left(\omega,T\right)$, FRFs of the viscoelastic system.

Figure 5(a) shows that a satisfactory convergence is noted about 300 samples, thus leading to the conclusion that the sample number used was representative. From the FRFs envelopes in Fig. 5(b), the representativeness of the iterative Ritz basis used can be observed. Furthermore, it was analyzed the computational time dispended during responses calculation in the process concerning 300 samples. The application of the iterative Ritz basis provided a computational gain of 92.5% in comparison with the full model.

Following the stress responses and Sines' indexes of the plate were determined. These analyses were performed only for reduced SFEM due to computational cost prohibitive of full model. Envelopes of random PSDs are composed by its maximum, mean and minimum values, in the frequency domain, via Power Spectral Densities (PSD) for each one of the plate's elements. The loading is a white Gaussian noise with $\phi_f = 85\times10^3\ Pa^2/Hz$ applied in the central node. Amplitudes of the stress response of critical element are shown on Fig. (6).



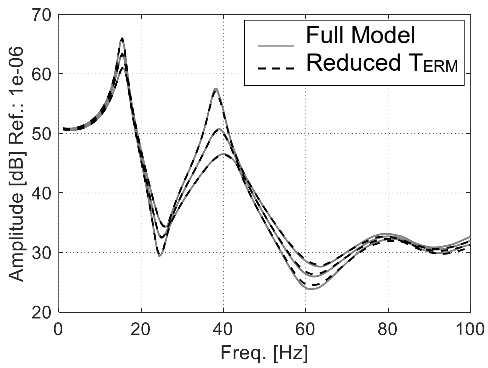


Fig. 5 - Numerical stochastic results: a) RMSD normalized; b) Full and reduced FRFs

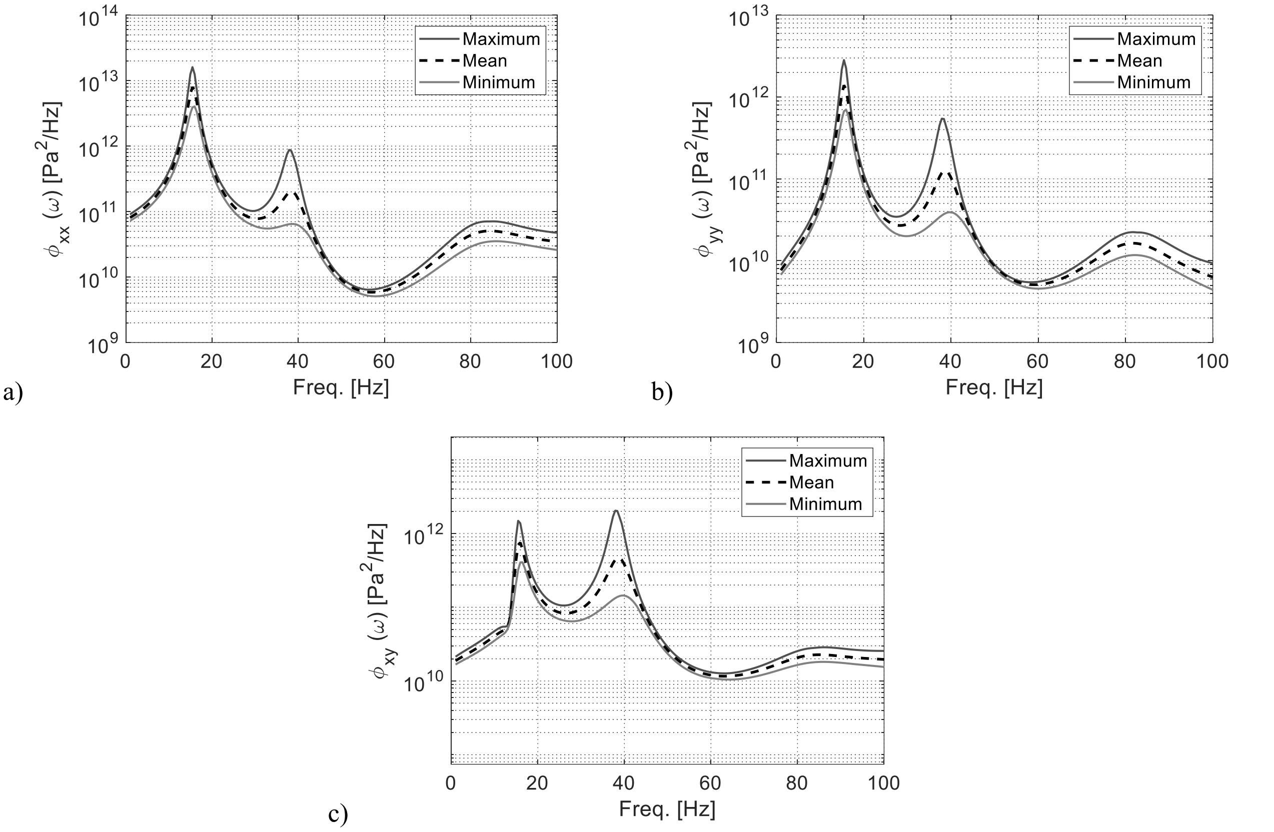


Fig. 6 - Envelopes of the amplitudes of the stress response: a) sxx; b) syy; c) sxy

Figure 7 shows Sines' indexes of the plate, which must be lower than one to indicate non-failure. The fact that these values may vary in the presence of uncertainties highlights the importance of performing this kind of analysis, taking random effects into account in the modeling of dynamic systems.

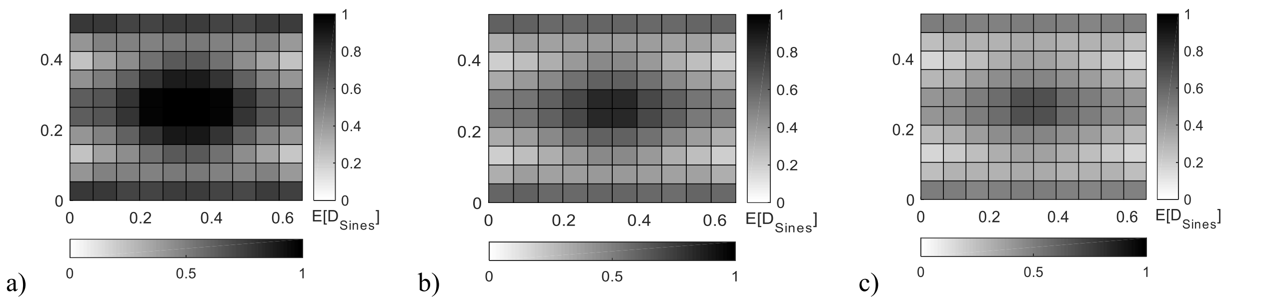


Fig. 7 - Distribution of the Sines' damage indicator: a) maximum; b) mean; c) minimum

The critical points of the plates can be clearly perceived in terms of the fatigue failure. Statistical expectation for Sines' indexes showed that in the center have values greater than one, thus it noticed which elements are more susceptible to failure after a certain number of cycles.

5. CONCLUDING REMARKS

A condensation model involving stochastic full models' formulations have been essential due to computational time involved and the number of degrees of freedom. From the iterative condensation strategy proposed for viscoelastically damped systems, it was observed a good ability to approximate reduced responses with complete model.

By including uncertainties level in the thickness of viscoelastic core, constraining layers and temperature, the application of the iterative Ritz basis provided a computational gain of 92.5% in comparison with the full model.

Furthermore, the fatigue responses showed the importance of considering uncertainties in three-layer sandwich plate structure treated with a passive constrained damping layer under Gaussian random loadings.

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