



25<sup>th</sup> ABCM International Congress of Mechanical Engineering  
October 20-25, 2019, Uberlândia, MG, Brazil

## COB-2019-2132

### A Cosserat Rod model for a cantilever beam that exhibits large displacements

#### H.E. Goicoechea

Mechanical Engineering Department, Pontifícia Universidade Católica do Rio de Janeiro  
Instituto de Física del Sur (IFISUR), Departamento de Ingeniería, Universidad Nacional del Sur  
CONICET, Av. L. N. Alem 1253, B8000CPB, Bahía Blanca, Argentina  
egoicoechea@ifisur-conicet.gob.ar

#### R. Lima, R. Sampaio

Mechanical Engineering Department, Pontifícia Universidade Católica do Rio de Janeiro  
robertalima@puc-rio.br, rsampaio@puc-rio.br

#### F.S. Buezas

Instituto de Física del Sur (IFISUR), Departamento de Física, Universidad Nacional del Sur  
CONICET, Av. L. N. Alem 1253, B8000CPB, Bahía Blanca, Argentina  
fbuezas@ifisur-conicet.gob.ar

**Abstract.** A cantilever beam subjected to an external eccentric load is modelled as a Cosserat rod. The results are employed to test the capability of the implementation of the Cosserat rod model to handle large displacement conditions. It is shown that the present strategy based on the Cosserat theory successfully overcomes the known limitations of other implementations, such as the Modified Cosserat Rod Element (MCRE).

**Keywords:** Cosserat Rods, Large Displacements, Geometrical non-linearity

## 1. INTRODUCTION

Oil production has become the most important energy source for a foreseeable future (Kapitaniak, 2015). As drilling is essential to oil and gas extraction, the study of the dynamics of drill-strings is of utmost importance for oil-companies in order to improve their performance, gain insight into the processes that affect their expected life, and reduce costs associated to oil production. In particular, premature failure of the drill-string and drill-bit may occur due to undesired vibrations of the drill-column, with associated reduced performance, i.e. rate of penetration. For these reasons, companies invest on the research of new methods and technologies to predict and suppress unwanted vibrations (Ghasemloonia *et al.*, 2015). Thus, the development of dynamic models for drill-strings is essential to understand and control their behaviour.

The drilling assembly of a drill-column is composed of the following main components: a drill-string (DS) which is in charge of driving the necessary torque to the cutting tool; a bottom hole assembly (BHA) that, along with the DS, provide the required weight to the bit; and a drill-bit which performs the actual cutting of the rock.

Over the last decades, various strategies have been employed to study drill-string dynamics. A comprehensive review of the many models available for drill-string dynamics is presented in (Ghasemloonia *et al.*, 2015). In general, the approaches found on the literature can be classified in two groups, based upon the hypothesis involved with regards to the bending effects. The so-called soft-string models include those approaches that neglect the bending stiffness of the drill-string, such as in (Sheppard *et al.*, 2007; Aarsnes and Shor, 2018; Sampaio *et al.*, 2007; Trindade and Sampaio, 2005). These models justify disregarding the dynamics in the lateral planes upon the fact that drill-strings are slender structure. In contrast, the stiff-string models do not neglect the bending stiffness and, as a consequence, they take into account lateral displacements, as in (Ring *et al.*, 2014; Ritto *et al.*, 2009).

Next, the models from the previous classification can also be sorted into either lumped (discrete) or continuous models. The choice of either a discrete or a continuous representation is usually based on the complexities involved in the resolution of each model and, the required time for their computations.

As already stated, many different approaches have been used in the analysis of drill-string dynamics, based upon different theoretical frameworks. For example, in (Ghasemloonia *et al.*, 2013) a continuous Euler beam theory is employed to deal with a vertical straight borehole geometry. Then in (Tucker and Wang, 2003; Silveira, 2011), the special theory of Cosserat rods is used to analyse the dynamics of a drill-string in a straight vertical borehole, with a method that should, theoretically, be capable of dealing with arbitrary configurations.

Lastly, (Kapitaniak, 2015) reviews the Cosserat rod theory implementation through the Modified Cosserat Rod Element (MCRE), and studies the dynamics of drill-strings by means of the elasticity theory via the finite elements method.

The author disregards the MCRE approach as a useful strategy as it is shown in (Kapitaniak, 2015) that the model suffers instabilities in situations that involves large displacements. Then, (Kapitaniak, 2015) concludes that the MCRE approach is not suitable for arbitrary drill-string configurations.

In the present work it is shown that the Cosserat rod theory implementation employed in (Goicoechea *et al.*, 2019) can successfully overcome the limitations observed with the MCRE-based model for drill-string dynamics. The case of a cantilever beam subjected to an eccentric force load is herein analysed and the results are compared against those presented in (Kapitaniak, 2015). Other problems are presented in (Goicoechea *et al.*, 2019) to test the quality of the non-linear response of the present formulation, as well as the case of a drill-string moving within a curved borehole geometry.

## 2. Cosserat Rod Formulation

A brief introduction into the formulation for the special theory of Cosserat rods is herein presented. A Cosserat rod can be defined as a 1D representation of a 3D slender body, where the description of the 3D is made by means of the position of the rod centreline and the orientation of the cross-sections attached to it. As usual to most of beam theories, the cross-sections are considered to behave as rigid bodies and do not deform in the current configuration. Then, the orientation of the cross-section can be obtained as a rotation applied to the reference configuration. For this work, from the many different mathematical formulations that can be adopted in the representation of rotations, the quaternion approach is chosen.

Next, the rod section is analysed following the sketch in Fig. 1. A detailed derivation of the equations involved can be found in in (Goicoechea *et al.*, 2019).

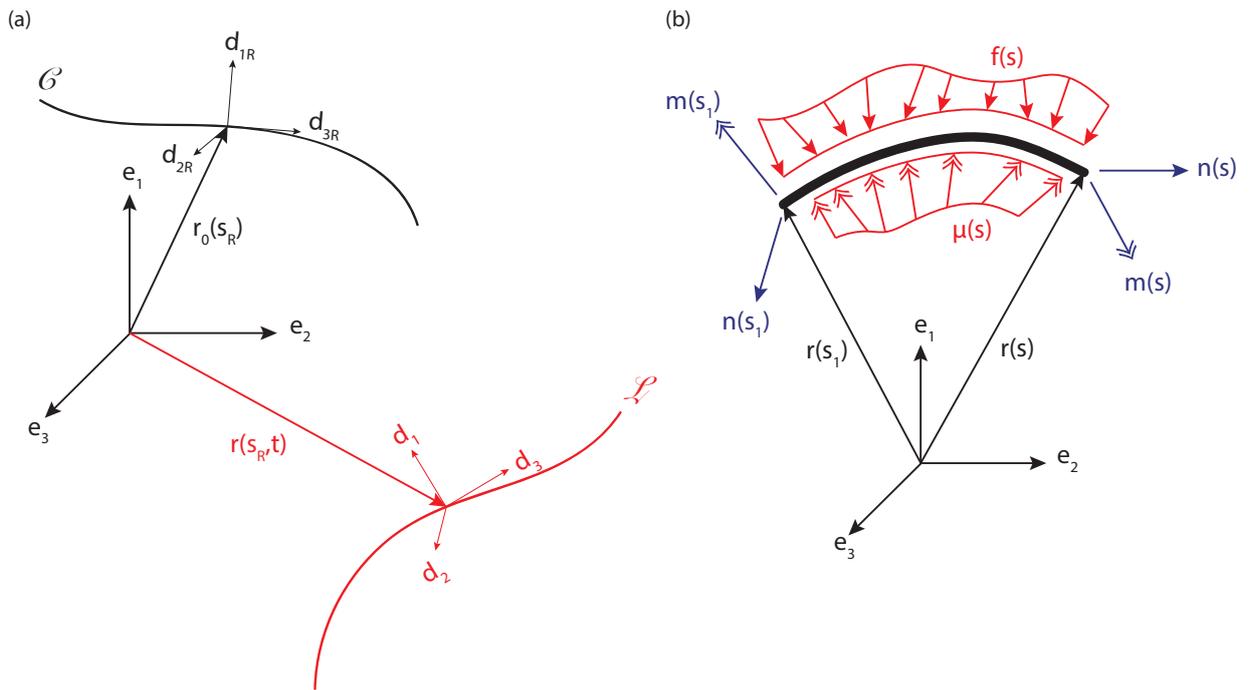


Figure 1. (a) The curves  $\mathcal{C}$  and  $\mathcal{L}$  refer to the reference and current centreline configurations, respectively; vectors  $d_i$  to the current moving frame;  $d_{iR}$  to the reference moving frame;  $e_i$  to an arbitrary inertial frame. (b) Equilibrium of a rod segment. Vectors  $\mathbf{n}$ ,  $\mathbf{m}$  represent applied loads and moments, respectively;  $\boldsymbol{\mu}$ ,  $\mathbf{f}$  are distributed forces and moments, respectively.

In the following mathematical expressions, the following notation is used: let  $\mathbf{r}(s_R, t)$  be a vector field that defines the current position of the centreline of a rod,  $\mathbf{d}_i(q_a(s_R, t), q_b(s_R, t), q_c(s_R, t), q_d(s_R, t))$  the directors of a moving frame that is fixed to the cross-sections,  $\mathbf{v}(s_R, t)$  a tangent vector to the centreline, and let  $\mathbf{r}_R, \mathbf{d}_{iR}, \mathbf{v}_R$  be analogous vector fields defined for the reference configuration.

Then, the weak formulation reads as follows

$$\begin{aligned}
& \int \left( -\phi'_{rx} n_x + \phi_{rx} f_x - \phi_{rx} \rho_0 A \ddot{r}_x \right) ds_R + \int \left( -\phi'_{ry} n_y + \phi_{ry} f_y - \phi_{ry} \rho_0 A \ddot{r}_y \right) ds_R \\
& \int \left( -\phi'_{rz} n_z + \phi_{rz} f_z - \phi_{rz} \rho_0 A \ddot{r}_z \right) ds_R + \int \left( -\phi'_{qb} m_x + \phi_{qb} (\eta_x + \mu_x) - \phi_{qb} \dot{\kappa}_x \right) ds_R \\
& \int \left( -\phi'_{qc} m_y + \phi_{qc} (\eta_y + \mu_y) - \phi_{qc} \dot{\kappa}_y \right) ds_R + \int \left( -\phi'_{qd} m_z + \phi_{qd} (\eta_z + \mu_z) - \phi_{qd} \dot{\kappa}_z \right) ds_R \\
& \int \left( \phi_{qa} (q_a^2 + q_b^2 + q_c^2 + q_d^2 - 1) \right) ds_R + \phi_{rx} n_x \Big|_0^L + \phi_{ry} n_y \Big|_0^L + \phi_{rz} n_z \Big|_0^L \\
& + \phi_{qb} m_x \Big|_0^L + \phi_{qc} m_y \Big|_0^L + \phi_{qd} m_z \Big|_0^L = 0
\end{aligned} \tag{1}$$

where  $\rho_0$  is the reference mass density,  $A$  the cross-section area,  $\phi_{(\cdot)}$  are the test functions,  $\kappa_{(\cdot)}$ , represent the components of the angular momentum,  $\eta_{(\cdot)}$  the vector components of the product  $\mathbf{r}^{*\prime} \times \mathbf{n}^*$ , and  $n_{(\cdot)}$ ,  $m_{(\cdot)}$ ,  $f_{(\cdot)}$ ,  $\mu_{(\cdot)}$  are the components of an applied force, applied moment, distributed force, distributed moment, respectively, with all the components expressed in an inertial frame.

Finally, the constitutive relations take the form

$$\mathbf{n} = \tilde{K}(\mathbf{v} - \mathbf{v}_R) \tag{2}$$

$$\mathbf{m} = \tilde{J}(\mathbf{u} - \mathbf{u}_R) \tag{3}$$

with

$$\mathbf{u} = \frac{1}{2} \sum_{i=1}^3 \mathbf{d}_i \times \frac{\partial \mathbf{d}_i}{\partial s_R} \tag{4}$$

$$\mathbf{u}_0 = \frac{1}{2} \sum_{i=1}^3 \mathbf{d}_{iR} \times \frac{\partial \mathbf{d}_{iR}}{\partial s_R} \tag{5}$$

As already stated, the dynamics of the Cosserat rod are defined by means of a translation of the points from the centreline and a rotation of the cross-sections, for which the quaternion approach is employed. Then, if the reference configuration is chosen so that the directors  $d_{iR}$  coincide with the directors of the inertial frame, a relationship between the directors in the current space and quaternion components can be established.

$$\begin{aligned}
d_{1x} &= q_a^2 + q_b^2 - q_c^2 - q_d^2 & d_{2x} &= 2(q_b q_c - q_a q_d) & d_{3x} &= 2(q_a q_c + q_b q_d) \\
d_{1y} &= 2(q_a q_d + q_b q_c) & d_{2y} &= q_a^2 - q_b^2 + q_c^2 - q_d^2 & d_{3y} &= 2(q_c q_d - q_a q_b) \\
d_{1z} &= 2(q_b q_d - q_a q_c) & d_{2z} &= 2(q_a q_b + q_c q_d) & d_{3z} &= q_a^2 - q_b^2 - q_c^2 + q_d^2
\end{aligned} \tag{6}$$

Finally, they must also satisfy the unitary quaternion constraint

$$q_a^2 + q_b^2 + q_c^2 + q_d^2 = 1 \tag{7}$$

### 3. BEAM UNDERGOING LARGE DISPLACEMENTS

The displacements of a cantilever beam subjected to an eccentric force are studied. The problem has already been presented in (Kapitaniak, 2015) as a means to test the capabilities of the MCRE to handle large displacements. Here, it is shown that the implementation of the Cosserat rod theory as in (Goicoechea *et al.*, 2018) is free of the aforementioned instabilities that are observed for the MCRE approach under load-states that involve large displacements. A sketch of the beam and its geometrical properties is presented in Fig. 2.

The beam has a length  $L = 20$  m, a cross section with diameter  $d = 0.2$  m, Young elastic modulus  $E = 210$  GPa. The applied load and the bending moment are modelled by means of a force with an equivalent eccentricity  $e = L/100$ . For this case, the classical Euler buckling theory predicts a maximum critical load  $P_{cr} = \pi^2 EI / (2L)^2$ . The eccentric force is varied by considering a parameter  $\eta$ , with  $F = \eta \cdot P_{cr}$ , and  $M = \eta \cdot P_{cr} \cdot e$ .

An analytical solution to this problem is reported by (Kapitaniak, 2015), based on the solution presented in (Gere and Goodno, 2013). The expression is obtained by considering second-order effects in Bernoulli-Euler beam theory, and reads as

$$\delta = \frac{M}{F} \left( \sec \left( L \sqrt{\frac{F}{EI}} \right) - 1 \right) \tag{8}$$

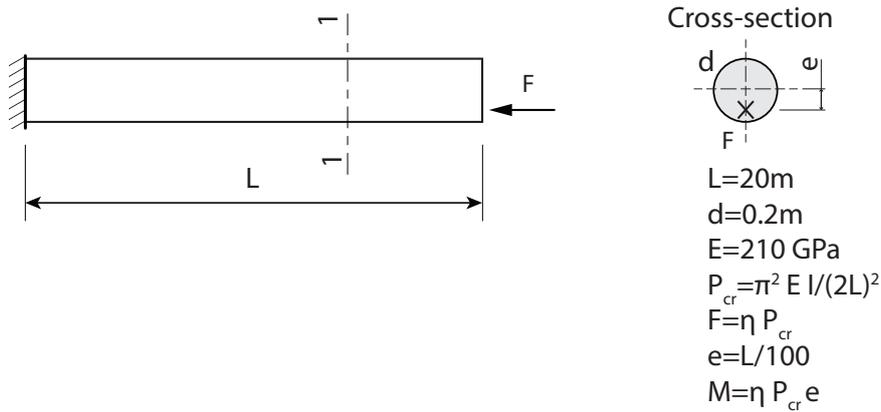


Figure 2. Sketch of a cantilever beam with length  $L$ , diameter  $d$ , eccentricity  $e$ , Young elastic module  $E$ , under the effects of an eccentric compressive force  $F$ .  $M$  is the equivalent bending moment that results from the application of the eccentric load  $F$ .

#### 4. RESULTS

The displacements obtained by means of the Cosserat rod (long-dash short-dash black line) are illustrated in Figure 3. The solution is compared with the results obtained in a commercial software (COMSOL AB, 2018), where the Timoshenko theory for large displacements is used (solid red line). Also results published in (Kapitaniak, 2015) for a 3D model based on the elasticity theory are presented (solid black line), as well as an analytical solution (dashed black line) given by (8).

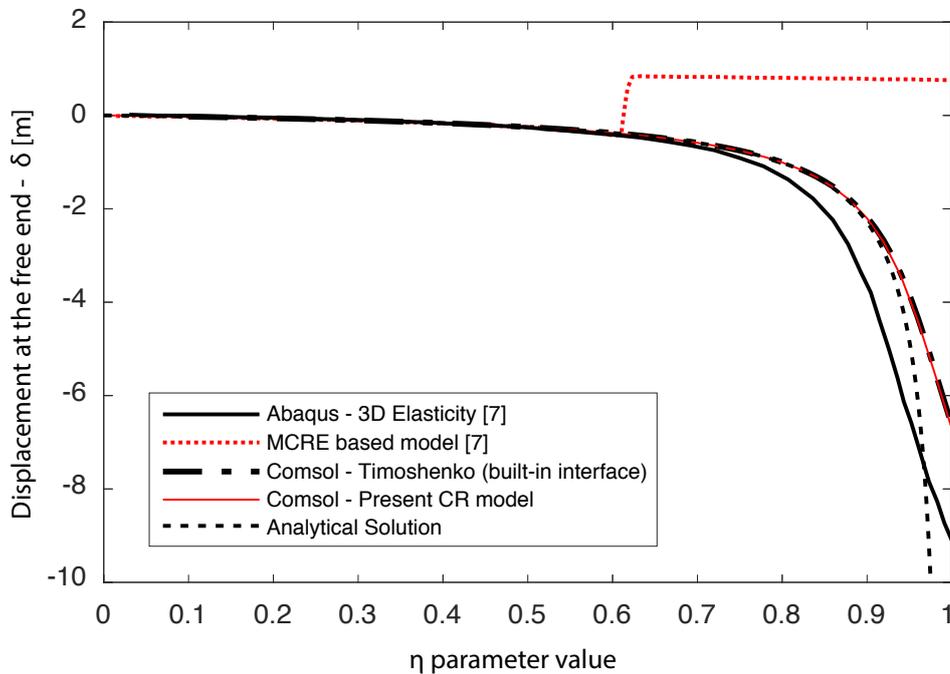


Figure 3. Results for the displacement of the cantilever beam at the free end. In long-dash short-dash black line, the solution for the built-in (COMSOL AB, 2018) Timoshenko theory that accounts for large displacements; in red line the present Cosserat theory solution; in black line the results for a 3D model based on the elasticity theory presented in (Kapitaniak, 2015); in dotted red line the results for the MCRE approach (Kapitaniak, 2015); and, in dashed black line the results from the analytical solution following (8).

The results obtained by the Cosserat rod model agrees with the solution obtained for the Timoshenko theory for large displacements. Moreover, they satisfy the analytical solution with no appreciable differences for  $\eta \leq 0.9$ , where the

validity of the analytical solution is lost due to the occurrence of large rotations.

## 5. CONCLUSIONS

In this work, the problem of a cantilever beam subjected to an eccentric compressive force is solved. The beam is modelled as a Cosserat rod. The present solution is compared with other approaches, such as a Timoshenko beam theory with geometric non-linearity present in a commercial software (COMSOL AB, 2018), as well as the solutions discussed in (Kapitaniak, 2015). It is observed that the CR model response qualitatively follows the same behaviour as the solution for the 3D elasticity theory reported in (Kapitaniak, 2015). Moreover, the present CR implementation overcomes the limitations and singularities analysed in (Kapitaniak, 2015), for the solution based on the Modified Cosserat Rod Element (MCRE), as no singularities are observed. Therefore, the authors find it feasible to employ the current CR implementation to deal with drill-string dynamics, even in situations that require accounting for large deformations (geometrical non-linearities), which actually is the great advantage of using Cosserat rod models.

## 6. ACKNOWLEDGEMENTS

This study was funded by Secretaría General de Ciencia y Tecnología, Universidad Nacional del Sur (Grant Number 24/J075); FONCyT, Agencia Nacional de Promoción Científica y Tecnológica (Grant Number PICT-2015-0220); CONICET (Grant Number 112 201301 00007 CO), all institutions from Argentina. The Brazilian authors thank CNPQ and FAPERJ for their support.

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