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DIFFERENTIAL TRANSFORM METHOD APPLIED TO EULER-BERNOULLI BEAM THEORY

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Abstract.
The present work study on the application of the differential transform method, in the solution of differential equations. The objective of the work is to use the differential transform method to solve the proposed vibration problem through the Euler-Bernoulli continuous beam theory, thus determining the natural frequencies and modes of vibration for classical support conditions, comparing the results obtained by the difunctional transformation method with the results of the exact analytical solution. The differential transform method proved to be very effective in solving the differential equations proposed by the Euler-Bernoulli beam theory and a high degree of precision compared to the exact analytical solutions.

Keywords: *Mechanical vibrations, Natural frequencies, Vibration modes, Numerical method.*

1. INTRODUCTION

The differential transform method is applied to solve discrete, continuous, linear and non-linear problems. In this work, the differential transform method is applied to solve mechanical vibration problems by means of the dynamic equation of the Euler-Bernoulli beam theory Catal (2012) and Zhifeng *et al.* (2013). The differential transform method is applied in all boundary conditions and in the beam equation, making it possible to represent all classical support conditions together with the natural frequencies and the relevant vibration modes Yesilce (2010), Najafgholipour and Soodbakhshb (2016) and Giriraj (2016). The result of the differential transform method is compared with the analytical solution of the Euler-Bernoulli beam theory and other literature results Inman (2001).

2. Analysis Vibration Free Beam Euler-Bernoulli

The vibration in the Beam of Euler-Bernoulli is generally called transverse vibration or vibrations of flexion Malik .M (1998).

2.1 The beam governing differential equation

Whose solution $y(x,t)$ is obtained by separating variables. A is the cross section of the area, E is the Young's modulus, I is the moment of inertia of the beam, ρ is the mass per unit volume, and L is the length. The vertical displacement of the beam is $y(x,t)$.
The free-vibration differential equation of the Euler-Bernoulli beam is represented in "Eq. (1)".

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (1)$$

$$y(x,t) = Y(x)T(t) \quad (2)$$

Being, $Y(x)$ the beam deflection mode, $T(t)$ the harmonic function of time, ω being the angular frequency of $T(t)$, then "Eq. (3)".

$$\frac{EI}{Y(x)}\frac{d^4Y(x)}{dx^4} = -\frac{\rho A}{T(t)}\frac{d^2T(t)}{dt^2} = \omega^2 \quad (3)$$

Soon:

$$EI\frac{d^4Y(x)}{dx^4} - \rho A\omega^2 Y(x) = 0 \tag{4}$$

Making φ the dimensionless distance of the left end of the beam, $y(\varphi)$ the dimensionless transverse mode, according to "Eq. (5)".

$$\varphi = \frac{x}{L}; \quad Y(\varphi) = \frac{Y(x)}{L}; \quad \lambda^2 = \frac{\rho A\omega^2 L^4}{EI} \tag{5}$$

"Eq. (4)" in dimensionless form.

$$\frac{\partial^4 Y(\varphi)}{\partial \varphi^4} - \lambda^2 Y(\varphi) = 0 \tag{6}$$

Choosing the boundary condition for the left side of the beam with $x = 0$, according to "Eq. (7)".

$$y(x,t)|_{x=0} = 0 \quad \textbf{e} \quad \frac{\partial y(x,t)}{\partial x}|_{x=0} = 0 \tag{7}$$

Choosing the boundary conditions for the right side of the beam with $x = L$, according to "Eq. (8)".

$$\frac{\partial^2(x,t)}{\partial x^2}|_{x=0} = 0 \quad \textbf{e} \quad \frac{\partial^3(x,t)}{\partial x^3}|_{x=0} = 0 \tag{8}$$

And the initial conditions for $t = 0$ are, like "Eq. (9)".

$$y(x,t)|_{t=0} = y(x,0) = y_0(x) \quad \textbf{e} \quad \frac{\partial y(x,t)}{\partial t}|_{t=0}\frac{\partial y(x,0)}{\partial t} = \dot{y_0}(x) \tag{9}$$

Using the "Eq. (7)" and "Eq. (8)", the boundary boundary conditions are:

$$Y(\varphi)|_{\varphi=0}\frac{\partial Y(\varphi)}{\partial \varphi}|_{\varphi=0} = 0 \quad \textbf{e} \qquad Y(\varphi)|_{\varphi=1}\frac{\partial Y(\varphi)}{\partial \varphi}|_{\varphi=1} \tag{10}$$

3. Differential Transform Method (DTM)

The differential transformation method was first proposed by Zhou (1986). It is one of the numerical methods used to solve ordinary and partial differential equations, with fast convergence rate and small calculation error. The differential transform method uses the form of polynomials as approximations of the exact solutions of ordinary and partial differential equations. The method is based on the expansion of the Taylor series.

The main difference between the Taylor series method and the differential transform method is that the former requires higher order derivative calculations, whereas the latter involves iterative procedures.

The application of the differential transform method in the solution of differential equations usually involves two transformations, that is, the differential transformation and the inverse differential transformation. Let $y(x)$ be a continuous function belonging to domain D and $x = x_i$ being any point in D. The Taylor function through the series of $y(x)$ is of the form presented in "Eq. (11)" and "Eq. (12)".

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x - x_i)^k \left[\frac{d^k y(x)}{dx^k}\right]_{x=x_i} \tag{11}$$

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!}x^k \left[\frac{d^k y(x)}{dx^k}\right]_{x_i=0} \tag{12}$$

From the "Eq. (11)" and "Eq. (12), the Taylor series order differential equation n of the expansion function $y(x)$ is defined at the point $x = x_i$, as shown in "Eq. (11)" and "Eq. (13).

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_i} \tag{13}$$

Note that $Y(k)$ denotes the differential transform of the function $y(x)$. In the case of $x_i = 0$ the function $y(x)$ can be represented in terms of the differential transform $Y(k)$ in the form shown by “Eq. (14).

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k Y(k) \tag{14}$$

In real applications, the function $y(x)$ is expressed by a finite series of “Eq. (14)" that can be written by “Eq. (15)".

$$y(x) = \sum_{k=0}^{N} \frac{1}{k!} x^k Y(k) \tag{15}$$

Being, N the number of elements of the series. From the definitions and properties of the differential transform method we have in Tab. 1.

Table 1. Original function x Transform DTM

Original function	Transform DTM
$y(t) = x(t) + z(t)$	$Y(k) = X(k) + Z(k)$
$y(t) = ax(t)$	$Y(k) = aX(k)$
$y(t) = dx(t)/dt$	$Y(k) = (k+1)X(k+1)$
$y(t) = d^2x(t)/dt^2$	$Y(k) = (k+1)(k+2)X(k+2)$

4. Application of the Differential Transformation Method in the Euler-Berboulli equation

The differential transform method was applied to “Eq. (6)” which results in.

$$(k+4)(k+3)(k+2)(k+1)k!Y(k+4) - \lambda^2 k!Y(k) = 0 \tag{16}$$

Rewriting “Eq. (16)”, as follows:

$$Y(k+4) = \frac{\lambda^2 Y(k)}{(k+4)(k+3)(k+2)(k+1)} \tag{17}$$

combining “Eq. (16)”, with the properties of the boundary conditions of the differential transform method and considering the boundary conditions for the spherical beam of “Figure 1”, with the left end being $\varphi = 0$.

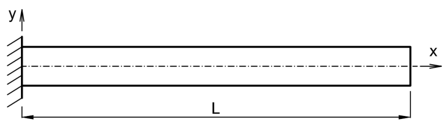

Figure 1. Balancing beam.

Deflection	Inclination
$y(x,t)\|_{x=0} = 0$	$\frac{\partial y(x,t)}{\partial x}\|_{x=0}$
$Y(\varphi)\|_{\varphi=0} = 0$	$\frac{dy(x)}{dx}\|_{x=0} T(t) = 0$
$Y(0) = 0$	$\frac{dY(\varphi)}{d\varphi}\|_{\varphi=0} = 0$

Applying the differential transform method to the boundary conditions on the left side of the beam.

Deflection	Inclination
$Y(0) = 0$	$Y(1) = 0$

considering the boundary conditions of the free set beam for $\varphi = 1$, representing the free end of the right side.

After applying the differential transform method in the boundary conditions and in the equation of motion of the beam, we will find a system, where the solution of this system gives us the natural frequencies. $f^k(\lambda) = 0$.

$$f^k(\lambda) = \left(\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k)!} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k)!} \right) - \left(\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k-1)!} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+1)!} \right) \tag{18}$$

Bending Moment	Cutting Force
$\frac{\partial^2 y(x,t)}{\partial x^2}\|_{x=L}$	$\frac{d^3 y(x,t)}{dx^3}\|_{x=L}$
$\frac{d^2 Y(x)}{dx^2}\|_{x=L} T(t) = 0$	$\frac{d^3 Y(x)}{dx^3}\|_{x=L} T(t) = 0$
$\frac{d^2 Y(\varphi)}{d\varphi^2}\|_{\varphi=1} = 0$	$\frac{d^3 Y(\varphi)}{d\varphi^3}\|_{\varphi=1} = 0$

In the sequence the equation of the modal form is determined:

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+2)!} x^{(4k+2)} + 3c_1 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+3)!} x^{(4k+3)} \tag{19}$$

Thus we find the modal form of the beam.

$$y(x) = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+2)!} x^{(4k+2)} - \frac{\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k)!}}{\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+1)!}} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(4k+3)!} x^{(4k+3)} \tag{20}$$

5. Results and Discussion

A computational program was developed for continuous beam models, through the presentation of the Euler-Bernoulli beam theory and the differential transform method. For the simulations made here, the data from are used, where information is provided regarding 1020 steel and the geometric properties of the beam studied Oke (2017). Geometric and material properties of simulated continuous beam, modulus of elasticity E_1 =210,00 GPa, transverse modulus of elasticity E_2 =210,00 GPa, shear modulus in the plane 1-2 G_{12} = 81,0 GPa, Shear modulus in the plane 1-3 G_{13} = 81,0 GPa, Shear modulus in the plane 2-3 G_{23} = 81,0 GPa, especific mass ρ = 7.800 kg/m^3, poisson coefficient v_{12} = 0,3, coefficient of shear k_c = 5/6, base b = 0,075 m, thickness h = 0,075 m, length L = 1,5 m, number of blades 1, Orientation of blade fibers 0^0.

In Tab. 2 the first three dimensionless natural frequencies $\hat{\beta}=\sqrt{\lambda}$ are presented for a continuous beam as a function of different boundary conditions. The non-dimensional natural frequencies obtained here are compared with those found analytically by using the Euler-Bernoulli model.
From Tab. 2, it can be observed that the natural frequencies for the continuous beam obtained by the author present the same values found in the literature. This helps the reliability of the program developed.

Table 2. 1st, 2nd and 3rd. Dimensional natural frequencies for a clamped-clamped continuous beam.

	$\hat{\beta}$	Continuous beam $\frac{L}{h} = 20, L = 1,5\ m$
Freq	Inman(2000)	Author (DTM)
1st	4,730	4,730
2nd	7,853	7,853
3rd	10,996	10,996

6. Conclusion

In this work, the differential transform method was introduced to solve the vibration problem in continuous beams of the Euler-Bernoulli theory. The natural frequency and vibration mode results were successfully calculated by the differential transform method and compared with the exact analytical solutions. The solution of the difunctional transform is in the form of an infinite series. The calculated results clearly show that these series are of very rapid convergence, in addition, the calculated results of the different transform method have a very high accuracy over exact analytical solutions.

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