25th ABCM International Congress of Mechanical Engineering
October 20-25, 2019, Uberlândia, MG, Brazil

COB-2019-0356
EFFECTS OF THE ZONAL HARMONICS J_2, J_3 AND J_4 ON OPTIMAL LOW-THRUST TRAJECTORIES

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Abstract. *In this work, a numerical-analytical procedure based on infinitesimal canonical transformation is developed for computing optimal time-fixed low-thrust limited power transfers between arbitrary orbits in Earth´s gravitational field which includes the effects of the zonal harmonics J_2, J_3 and J_4. The proposed procedure involves the development of a two-stage algorithm: in the first step, a neighboring extremals algorithm is applied to solve the two-point boundary-value problem governed by an average canonical system describing the secular behavior of the optimal trajectories; in the second step, a Newton-Raphson algorithm is applied to adjust the initial values of the adjoint variables when the first order periodic terms are included. The maximum Hamiltonian governing the average canonical system is computed by applying the concept of 'mean Hamiltonian', and, the first order periodic terms due the second zonal harmonic J_2 and the optimal thrust acceleration are recovered by means of Hori method. Some numerical results show the effects on the optimal trajectories due to the zonal harmonics considered in this study.*

Keywords: *optimal low-thrust trajectories, transfers between arbitrary orbits, zonal harmonics*

1. INTRODUCTION

In last decades, optimal space trajectories problems involving low-thrust systems (electric propulsion) have been renewed interest because of recent technological advances that allow the use these systems in space exploration. The two pioneer missions which employed such propulsion systems were the NASA-JPL Deep Space 1 and the ESA SMART1. Deep Space 1 was the first interplanetary spacecraft to use solar electric propulsion. It was developed by the NASA new Millennium program to test new technologies for future space programs. It was launched on 24 October 1998. The mission of Deep Space 1 ended on 18 December 2001, when its fuel supply exhausted. Smart-1 was the first of a series of missions for advanced research in technology of ESA. It was used to test the solar electric propulsion and other deep-space technologies. It was launched on 27 September 2003. Its mission ended on 3 September 2006, when the spacecraft, in a planned maneuver, impacted the lunar surface. Interesting details about these space missions can be found in Rayman et al (2000), Racca et al (2002) and Camino et al (2005). The Brazil has also demonstrated interest in the use of such new technology in exploring of asteroids with the ASTER mission (Sukhanov et al., 2010), scheduled for launching in 2021. ASTER mission involves the exploration of a triple asteroid system called 2001 SN263, and, it is being designed to use low-thrust propulsion system.

Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability – the ratio between the maximum thrust acceleration and the gravity acceleration on the ground is small, between 10^{-4} and 10^{-2}. The greatest benefits with these propulsion systems occur in interplanetary missions of high energy - rendez-vous with the asteroid belt, rendez-vous with comets, fly-by missions to Pluto, ... etc – and in geocentric missions involving communication systems and GPS satellites. Since the sixties, several researchers have obtained numerical and analytical solutions for several maneuvers involving specific initial and final orbits and specific thrust profiles (Gobetz, 1965; Edelbaum, 1965, 1966; Marec and Vinh, 1980; Haissig et al, 1993; Geffroy and Epenoy, 1997; Bonnard et al, 2006; Huang, 2012; Da Silva Fernandes et al, 2018). In the analytical studies, averaging techniques are applied and solutions of the averaged equations are obtained such that only secular behavior of the optimal solutions is discussed. Few works discuss the inclusion of periodic terms which are, in general, considered only for transfers between close orbits.

Considering the geocentric missions, this work describes a study of the problem of optimal low-thrust and limited-power trajectories in a non-central gravitational field which includes the main zonal harmonics – J_2, J_3 and J_4 – of the development of Earth´s gravitational potential. In this way, a numerical-analytical procedure based on infinitesimal canonical transformation is developed for computing optimal time-fixed low-thrust limited power transfers between arbitrary orbits in such gravitational field. The proposed procedure involves the development of a two-stage algorithm: in the first step, a neighboring extremals algorithm is applied to solve the two-point boundary-value problem governed by an average canonical system describing the secular behavior of the optimal trajectories; in the second step, a Newton-Raphson algorithm is applied to adjust the initial values of the adjoint variables when the first order periodic terms are included. The maximum Hamiltonian governing the average canonical system is computed by applying the concept of 'mean Hamiltonian'. Hori method (Hori, 1966) – a perturbation technique based on Lie series – is applied to recover the short periodic terms related to the second zonal harmonic and to the optimal thrust acceleration. The main advantage of this procedure is that the average canonical system is much simpler than the complete canonical system with the short periodic terms. This technique has been applied recently with good results in a study of optimal time-fixed low-thrust limited power transfers between coplanar orbits with small eccentricities in an inverse-square force field (Da Silva Fernandes et al, 2018). Numerical results for some transfers show the effects on the optimal trajectories due to the inclusion of the zonal harmonics in the development of Earth's gravitational potential.

The paper is organized as follows. In Section 2, the optimization problem for time-fixed low-thrust limited power transfers is formulated as a Mayer problem of optimal control with position and velocity vectors and a consumption variable as state variables, and, the maximum Hamiltonian governing the optimal trajectories is derived. In Section 3, a canonical transformation is performed and classical orbits elements are introduced as state variables. In Section 4, a numerical-analytical procedure for solving the two-point boundary value problem of going forma an initial orbit to a final orbit is described. Finally, numerical results are presented for some transfers in Section 5, and, the concluding remarks are presented in Section 6.

2. OPTIMAL SPACE TRAJECTORIES

A low-thrust limited-power propulsion system, or, LP system, is characterized by low-thrust acceleration level and high specific impulse (Marec, 1979). The ratio between the maximum thrust acceleration and the gravity acceleration on the ground, γ_{max}/g_0, is between 10^{-4} and 10^{-2}. For such system, the fuel consumption is described by the variable J defined as

$$J = \frac{1}{2}\int_{t_0}^{t} \gamma^2 dt \ , \tag{1}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\gamma}$, used as control variable. The minimization of the final value of the fuel consumption J_f is equivalent to the maximization of m_f (final mass of the vehicle).

The optimization problem concerning with low-thrust limited-power transfers is formulated as follows: it is proposed to transfer the space vehicle M from the initial state $(\boldsymbol{r}_0, \boldsymbol{v}_0, 0)$ at the initial time $t_0 = 0$ to the final state $(\boldsymbol{r}_f, \boldsymbol{v}_f, J_f)$ at the specified final time t_f, such that the final consumption variable J_f is a minimum. The state equations are

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \qquad \frac{d\boldsymbol{v}}{dt} = -\frac{\mu}{r^3}\boldsymbol{r} + \nabla\left(\sum_{n=2}^{4} U_n\right) + \boldsymbol{\gamma} \qquad \frac{dJ}{dt} = \frac{1}{2}\gamma^2 \ , \tag{2}$$

where μ is the gravitational parameter and U_n, n = 2,3,4, denotes the terms of the disturbing force function associated to Earth´s gravitational potential related to the main zonal harmonics – J_2, J_3 and J_4 – respectively. These terms are given by

$$U_n = -\frac{\mu}{r} J_n \left(\frac{a_e}{r}\right)^n P_n(\sin\varphi), n = 2,3,4 \ , \tag{3}$$

where P_n is Legendre polynomial of n-th degree, and φ denotes the latitude.

It is assumed that the control $\boldsymbol{\gamma}$ is unconstrained, that is, the thrust direction is free and the thrust magnitude is unbounded.

Following the Pontryagin Maximum Principle (Pontryagin et al, 1962), the optimal thrust acceleration $\boldsymbol{\gamma}^*$ must be selected from the admissible controls such that the Hamiltonian function H reaches its maximum.

The Hamiltonian function is formed using Eq. (2),

$$H = \boldsymbol{p}_r \bullet \boldsymbol{v} + \boldsymbol{p}_v \bullet \left(-\frac{\mu}{r^3}\boldsymbol{r} + \nabla\left(\sum_{n=2}^{4} U_n\right) + \boldsymbol{\gamma}\right) + \frac{1}{2} p_J \gamma^2 , \tag{4}$$

where $\boldsymbol{p}_r$, $\boldsymbol{p}_v$ and p_J are the adjoint variables and dot denotes the dot product.

Since the optimization problem is unconstrained, the optimal thrust acceleration $\boldsymbol{\gamma}^*$ is given by

$$\boldsymbol{\gamma}^* = -\frac{\boldsymbol{p}_v}{p_J}. \tag{5}$$

the optimal trajectories are governed by the maximum Hamiltonian function H^*, obtained from Eqns. (4) and (5),

$$H^* = \boldsymbol{p}_r \bullet \boldsymbol{v} + \boldsymbol{p}_v \bullet \left(-\frac{\mu}{r^3}\boldsymbol{r} + \nabla\left(\sum_{n=2}^{4} U_n \right) \right) - \frac{{p_v}^2}{2 p_J}. \tag{6}$$

The consumption variable J is ignorable and p_J is a first integral whose value is obtained from the transversality conditions, $p_J(t) = -1$. So, Eq. (5) reduces to

$$\boldsymbol{\gamma}^* = \boldsymbol{p}_v.$$

The optimal thrust acceleration $\boldsymbol{\gamma}^*$ is modulated (Marec, 1979). Equation (6) simplifies and maximum Hamiltonian function H^* can be put in the form

$$H^* = H_0 + H_{\gamma^*} + H_U^*, \tag{7}$$

where

$$H_0 = \boldsymbol{p}_r \bullet \boldsymbol{v} - \boldsymbol{p}_v \bullet \frac{\mu}{r^3}\boldsymbol{r}, \tag{8}$$

denotes the undisturbed Hamiltonian function, and,

$$H_{\gamma^*} = \frac{{p_v}^2}{2}, \tag{9}$$

$$H_U^* = \mathbf{p}_v \bullet \nabla\left(\sum_{n=2}^{4} U_n \right), \tag{10}$$

denotes the disturbing function associated with the optimal thrust acceleration and the disturbing function associated with the perturbations due to the zonal harmonics of the geopotential, respectively.

The general solution of the undisturbed Hamiltonian plays an important role in the numerical-analytical procedure described in this work, because it defines a canonical transformation between the Cartesian and the orbital elements, including their respective adjoint variables.

3. CANONICAL TRANSFORMATION FROM CARTESIAN ELEMENTS TO ORBITAL ELEMENTS

Consider the canonical system of differential equations governed by the undisturbed Hamiltonian function H_0, defined by Eq. (8). The general solution of the state equations is well-known from the classical two-body problem (Battin, 1987), and, it is given by

$$\boldsymbol{r} = \frac{a\left(1 - e^2\right)}{1 + e\cos f}\boldsymbol{e}_r, \tag{11}$$

$$\boldsymbol{v} = \sqrt{\frac{\mu}{a(1-e^2)}}\left[(e\sin f)\boldsymbol{e_r} + (1+e\cos f)\boldsymbol{e_s}\right], \tag{12}$$

where a is the semi-major axis, e is the eccentricity, f is the true anomaly, $\boldsymbol{e_r}$ is the unit vector pointing radially outward of the moving frame of reference and $\boldsymbol{e_s}$ is the unit vector along circumferential direction. The unit vectors $\boldsymbol{e_r}$, $\boldsymbol{e_s}$ and $\boldsymbol{e_w} = \boldsymbol{e_r} \times \boldsymbol{e_s}$ are written in the fixed frame of reference as

$$\begin{aligned}\boldsymbol{e}_r &= (\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I)\boldsymbol{i} \\ &+ (\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I)\boldsymbol{j} + \sin(\omega+f)\sin I\,\boldsymbol{k},\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{e_s} &= -(\cos\Omega\sin(\omega+f) + \sin\Omega\cos(\omega+f)\cos I)\boldsymbol{i} \\ &+ (-\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos I)\boldsymbol{j} + \cos(\omega+f)\sin I\,\boldsymbol{k},\end{aligned} \tag{14}$$

$$\boldsymbol{e_w} = \sin\Omega\sin I\,\boldsymbol{i} - \cos\Omega\sin I\,\boldsymbol{j} + \cos I\,\boldsymbol{k}, \tag{15}$$

where I is the inclination of orbital plane, Ω is the longitude of the ascending node and ω is the argument of pericenter. Figure 1 shows the reference frames involved in the description of the transfer problem.

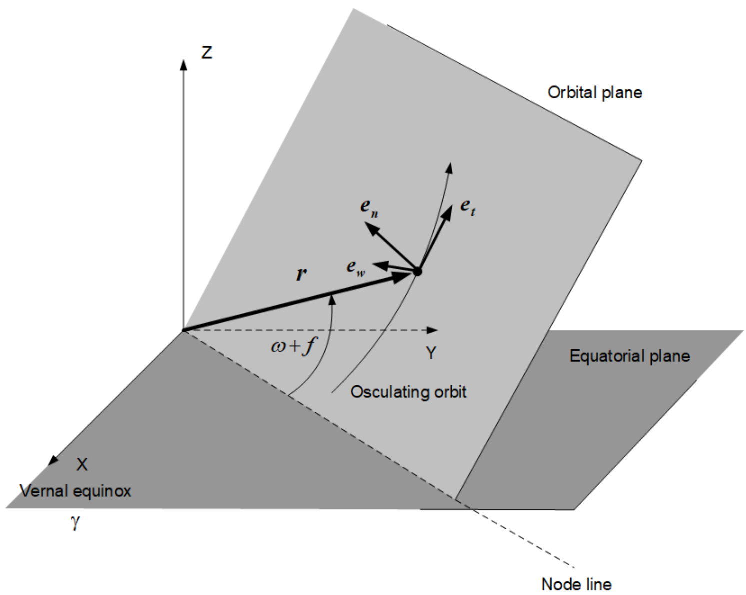

Figure 1. Reference frames.

The general solution of the differential equations for the adjoint variables $\boldsymbol{p}_r$ and $\boldsymbol{p}_v$ is obtained by computing the inverse of the Jacobian matrix of the point transformation between the Cartesian elements and the orbital ones and it is given by

$$\begin{aligned}\boldsymbol{p_r} &= \frac{a}{r^2}\left\{2ap_a + \left((1-e^2)\cos E\right)p_e + \left(\frac{r}{a}\right)\frac{\sin f}{e}\left(p_\omega - \frac{(1-e^3\cos E)}{\sqrt{1-e^2}}p_M\right)\right\}\boldsymbol{e_r} + \left\{\frac{\sin f}{a}p_e - \frac{(e+\cos f)}{ae(1-e^2)}p_\omega\right. \\ &\left.+ \frac{\sqrt{1-e^2}\cos f}{ae}p_M\right\}\boldsymbol{e_s} + \frac{1}{a\sqrt{1-e^2}}\left\{\left(\frac{a}{r}\right)\sin E\left[p_I\cos\omega + \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\sin\omega\right]\right. \\ &\left.+ \sqrt{1-e^2}\left(\frac{a}{r}\right)\cos E\left[p_I\sin\omega - \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\cos\omega\right]\right\}\boldsymbol{e_w},\end{aligned} \tag{16}$$

$$\boldsymbol{p_v} = \frac{1}{na\sqrt{1-e^2}}\left\{\left\{2ae\sin f\, p_a + \left(\left(1-e^2\right)\sin f\right)p_e - \frac{\left(1-e^2\right)\cos f}{e}p_\omega + \frac{\left(1-e^2\right)^{3/2}}{e}\left(\cos f - \frac{2e}{1+e\cos f}\right)p_M\right\}\boldsymbol{e_r}\right.$$
$$+\left\{2a\left(1-e^2\right)\left(\frac{a}{r}\right)p_a + \left(1-e^2\right)\left(\cos f + \cos E\right)p_e + \frac{\left(1-e^2\right)\sin f}{e}\left(1+\frac{1}{1+e\cos f}\right)\left(p_\omega - \sqrt{1-e^2}\,p_M\right)\right\}\boldsymbol{e_s} \tag{17}$$
$$\left.+\left\{\left(\frac{r}{a}\right)\cos(\omega+f)p_I + \left(\frac{r}{a}\right)\sin(\omega+f)\left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\right\}\boldsymbol{e_w}\right.,$$

where $p_a,\ldots,\ p_M$ are adjoint variables to the orbital elements. Equations (11) through (17) define a Mathieu transformation between the Cartesian elements and the orbital ones, including the adjoint variables. The Hamiltonian function is invariant with respect to this canonical transformation. Thus, after some simplifications, one finds

$$H_0 = np_M\,, \tag{18}$$

$$H_\gamma^* = \frac{1}{2n^2a^2\left(1-e^2\right)}\left\{\frac{1}{2}(1-\cos 2f)\left[2aep_a + \left(1-e^2\right)p_e\right]^2 + 2\left(1-e^2\right)\sin 2f\left[-ap_a p_\omega - \frac{\left(1-e^2\right)}{2e}p_e p_\omega\right]\right.$$
$$+4\left(1-e^2\right)^{3/2}\sin f\left(\frac{-2e}{1+e\cos f}+\cos f\right)\left[ap_a p_M + \frac{\left(1-e^2\right)}{2e}p_e p_M\right] + \frac{\left(1-e^2\right)^2}{2e^2}(1+\cos 2f)p_\omega{}^2$$
$$-\frac{2\left(1-e^2\right)^{5/2}}{e^2}\left(\frac{-2e}{1+e\cos f}+\cos f\right)\cos f\, p_\omega p_M + \frac{\left(1-e^2\right)^3}{e^2}\left(\frac{-2e}{1+e\cos f}+\cos f\right)^2 p_M{}^2$$
$$+4a^2\left(1-e^2\right)^2\left(\frac{a}{r}\right)^2 p_a{}^2 + 4a\left(1-e^2\right)^2\left(\frac{a}{r}\right)(\cos E + \cos f)p_a p_e + \left(1-e^2\right)^2(\cos E+\cos f)^2 p_e{}^2$$
$$+\frac{4a\left(1-e^2\right)^2}{e}\left(\frac{a}{r}\right)\sin f\left(1+\frac{1}{1+e\cos f}\right)\left[p_a p_\omega - \left(1-e^2\right)^{1/2}p_a p_M\right]$$
$$+\frac{2\left(1-e^2\right)^2}{e}(\cos E+\cos f)\left(1+\frac{1}{1+e\cos f}\right)\sin f\left[p_e p_\omega - \sqrt{1-e^2}\,p_e p_M\right]$$
$$+\left[\frac{\left(1-e^2\right)}{e}\left(1+\frac{1}{1+e\cos f}\right)\sin f\left[p_\omega - \sqrt{1-e^2}\,p_M\right]\right]^2 + \frac{1}{2}\left(\frac{r}{a}\right)^2\left[p_I{}^2 + \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)^2\right]$$
$$\left.+\frac{1}{2}\left(\frac{r}{a}\right)^2\cos 2(\omega+f)\left[p_I{}^2 - \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)^2\right] + \left(\frac{r}{a}\right)^2\sin 2(\omega+f)p_I\left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\right\}. \tag{19}$$

and,

$$H_U^* = \frac{2}{na}\frac{\partial U}{\partial M}p_a + \frac{\sqrt{1-e^2}}{na^2e}\left[-\frac{\partial U}{\partial\omega} + \sqrt{1-e^2}\frac{\partial U}{\partial M}\right]p_e + \frac{1}{na^2\sqrt{1-e^2}\sin I}\left[-\frac{\partial U}{\partial\Omega} + \cos I\frac{\partial U}{\partial\omega}\right]p_I$$
$$\left.+\frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial U}{\partial I}p_\Omega + \frac{\sqrt{1-e^2}}{na^2e}\left[\frac{\partial U}{\partial e} - \frac{e\cot I}{(1-e^2)}\frac{\partial U}{\partial I}\right]p_\omega + \frac{1}{na}\left[-2\frac{\partial U}{\partial a} - \frac{(1-e^2)}{ae}\frac{\partial U}{\partial e}\right]p_M\right\}, \tag{20}$$

where H_U^* denotes the disturbing function associated to the zonal harmonics, that is, $U = \sum_{n=2}^{4} U_n$. The terms of the disturbing force function associated to Earth´s gravitational potential related to the main zonal harmonics – J_2, J_3 and J_4 – are respectively expressed in classical orbital elements as follows:

$$U_2 = \frac{\mu}{a}J_2\left(\frac{a_e}{a}\right)^2\left[\left(\frac{1}{2} - \frac{3}{4}\sin^2 I\right)\left(\frac{a}{r}\right)^3 + \frac{3}{4}\sin^2 I\left(\frac{a}{r}\right)^3\cos 2(\omega+f)\right], \tag{21}$$

$$U_3=-\frac{\mu}{a}J_3\left(\frac{a_e}{a}\right)^3\sin I\left\{\left(\frac{15}{8}\sin^2 I-\frac{3}{2}\right)\left(\frac{a}{r}\right)^4\sin(\omega+f)-\frac{5}{8}\sin^2 I\left(\frac{a}{r}\right)^4\sin 3(\omega+f)\right\}, \quad (22)$$

$$U_4=-\frac{\mu}{a}J_4\left(\frac{a_e}{a}\right)^4\left\{\left(\frac{3}{8}-\frac{15}{8}\sin^2 I+\frac{105}{64}\sin^4 I\right)\left(\frac{a}{r}\right)^5 + \left(\frac{15}{8}\sin^2 I-\frac{35}{16}\sin^4 I\right)\left(\frac{a}{r}\right)^5\cos 2(\omega+f)+\frac{35}{64}\sin^4 I\left(\frac{a}{r}\right)^5\cos 4(\omega+f)\right\}. \quad (23)$$

Parameter a_e is the mean equatorial radius

4. A NUMERICAL-ANALYTICAL APPROXIMATE SOLUTION

After building the canonical transformation defined in the previous section, an average maximum Hamiltonian is computed by applying the concept of 'mean Hamiltonian'. Then, Hori method (Hori, 1966) – a perturbation technique based on Lie series – is applied to recover the short periodic terms related to the second zonal harmonic J_2 and to the optimal thrust acceleration.

According to the procedure described in the previous paragraph, one finds that the mean maximum Hamiltonian is computed as follows

$$\left\langle H^*\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}H^*dM .$$

So, taking into account that terms factored by p_M can be ignored for simple transfers, one finds after some calculations that the mean maximum Hamiltonian concerning to the optimal thrust acceleration is given by

$$\left\langle H_\gamma^*\right\rangle=\frac{a}{2\mu}\left\{4a^2p_a^2+\frac{5}{2}(1-e^2)p_e^2+\frac{p_I^2}{2(1-e^2)}\left[\left(1+\frac{3}{2}e^2\right)+\frac{5}{2}e^2\cos 2\omega\right]+\frac{5e^2\sin 2\omega}{2(1-e^2)}p_I\left(\frac{p_\Omega}{\sin I}-\cot I p_\omega\right) + \frac{1}{2(1-e^2)}\left(\frac{p_\Omega}{\sin I}-\cot I p_\omega\right)^2\left[\left(1+\frac{3}{2}e^2\right)-\frac{5}{2}e^2\cos 2\omega\right]+\frac{(5-4e^2)}{2e^2}p_\omega^2\right\}. \quad (24)$$

The mean part of the maximum Hamiltonian concerning with the disturbing function associated to the zonal harmonics, $\left\langle H_U^*\right\rangle$, can be computed from Eqn. (20) with $\langle U\rangle$ replacing U. So, the mean maximum Hamiltonian involves the mean part of the disturbing function, which is expressed as

$$\langle U_2\rangle=U_{2\sec}=\frac{\mu}{a}J_2\left(\frac{a_e}{a}\right)^2(1-e^2)^{-\frac{3}{2}}\left(\frac{1}{2}-\frac{3}{4}\sin^2 I\right), \quad (25)$$

$$\langle U_3\rangle=U_{3\sec}=-\frac{3}{8}\frac{\mu}{a}J_3\left(\frac{a_e}{a}\right)^3 e(1-e^2)^{-\frac{5}{2}}(1-5\cos^2 I)\sin I\sin\omega, \quad (26)$$

$$\langle U_4\rangle=U_{4\sec}+U_{4lp}=-\frac{\mu}{a}J_4\left(\frac{a_e}{a}\right)^4(1-e^2)^{-\frac{7}{2}}\left(1+\frac{3}{2}e^2\right)\left(\frac{3}{8}-\frac{15}{8}\sin^2 I+\frac{105}{64}\sin^4 I\right) -\frac{3}{4}\frac{\mu}{a}J_4\left(\frac{a_e}{a}\right)^4(1-e^2)^{-\frac{7}{2}}e^2\left(\frac{15}{8}\sin^2 I-\frac{35}{16}\sin^4 I\right)\cos 2\omega. \quad (27)$$

Note that the mean part of the disturbing function U_4 has secular and long periodic terms.

The first-order short periodic terms related to the second zonal harmonic J_2 and to the optimal thrust acceleration are computed from the generating function obtained by applying Hori method (Hori, 1966),

$$S'_{J_2} = J_2\left(\frac{a_e}{a}\right)^2\left\{\left\{\left(1-\frac{3}{2}\sin^2 I\right)\left[\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right]+\frac{3}{2}\sin^2 I\left(\frac{a}{r}\right)^3\cos 2(\omega+f)\right\}ap_a\right.$$
$$+\left\{\frac{3}{4}\frac{\sin^2 I}{e(1-e^2)}\left[-\cos 2(\omega+f)-e\left(\cos(2\omega+f)+\frac{1}{3}\cos(2\omega+3f)\right)\right]\right.$$
$$\left.+\frac{(1-e^2)}{e}\left[\left(\frac{1}{2}-\frac{3}{4}\sin^2 I\right)\left[\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right]+\frac{3}{4}\sin^2 I\left(\frac{a}{r}\right)^3\cos 2(\omega+f)\right]\right\}p_e$$
$$+\frac{3}{8}\frac{\sin 2I}{(1-e^2)^2}\left[\cos 2(\omega+f)+e\left(\cos(2\omega+f)+\frac{1}{3}\cos(2\omega+3f)\right)\right]p_I$$
$$+\left\{\frac{3}{2}\frac{\cos I}{(1-e^2)^2}\left[-(f-M+e\sin f)+\frac{1}{2}\sin 2(\omega+f)+\frac{e}{2}\left(\sin(2\omega+f)+\frac{1}{3}\sin(2\omega+3f)\right)\right]\right\}p_\Omega$$
$$+\left\{\frac{3}{4}\frac{(5\cos^2 I-1)}{(1-e^2)^2}(f-M+e\sin f)+\frac{1}{4}\frac{(3\cos^2 I-1)}{e(1-e^2)}\left[\left(\frac{a}{r}\right)^2(1-e^2)+\left(\frac{a}{r}\right)+1\right]\sin f\right.$$
$$+\frac{3}{8}\frac{\sin^2 I}{e(1-e^2)}\left[\left(-\left(\frac{a}{r}\right)^2(1-e^2)-\left(\frac{a}{r}\right)+1\right)\sin(2\omega+f)+\left(\left(\frac{a}{r}\right)^2(1-e^2)+\left(\frac{a}{r}\right)+\frac{1}{3}\right)\sin(2\omega+3f)\right.$$
$$\left.\left.\left.+\frac{3}{8}\frac{(3-5\cos^2 I)}{(1-e^2)^2}\left[\sin 2(\omega+f)+e\left(\sin(2\omega+f)+\frac{1}{3}\sin(2\omega+3f)\right)\right]\right]\right\}p_\omega\right\}, \tag{28}$$

and,

$$S'_\gamma = \frac{1}{2}\sqrt{\frac{a^5}{\mu^3}}\left\{8e\sin E\, a^2 {p_a}^2+8(1-e^2)\sin E\, ap_a p_e-\frac{8\sqrt{1-e^2}}{e}\cos E\, p_a p_\omega\right.$$
$$+(1-e^2)\left[-\frac{5}{4}e\sin E+\frac{3}{4}\sin 2E-\frac{1}{12}e\sin 3E\right]p_e^2+\frac{\sqrt{1-e^2}}{e}\left[\frac{5}{2}e\cos E\right.$$
$$\left.-\frac{1}{2}(3-e^2)\cos 2E+\frac{1}{6}e\cos 3E\right]p_e p_\omega+\frac{1}{(1-e^2)}\left[\left(-e+\frac{3}{8}e^3\right)\sin E+\frac{3}{8}e^2\sin 2E\right.$$
$$\left.-\frac{1}{24}e^3\sin 3E\right]\left[{p_I}^2+\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\right]+\frac{1}{(1-e^2)}\left[{p_I}^2\cos 2\omega+2p_I\left(\frac{p_\Omega}{\sin I}\right.\right.$$
$$\left.\left.-p_\omega\cot I\right)\sin 2\omega-\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\cos 2\omega\right]\left[\frac{1+2e^2}{4}\sin 2E+\frac{-2e+e^3}{24}\sin 3E\right.$$
$$\left.+\frac{5}{8}\left(-2e+e^3\right)\sin E\right]+\frac{1}{\sqrt{1-e^2}}\left[\frac{5}{4}e\cos E-\frac{1+e^2}{4}\cos 2E+\frac{1}{12}e\cos 3E\right]\left[-p_I^2\sin 2\omega\right.$$
$$\left.+2p_I\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)\cos 2\omega+\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\sin 2\omega\right]$$
$$\left.+\frac{{p_\omega}^2}{e^2}\left[\left(\frac{5}{4}e-e^3\right)\sin E+\left(-\frac{3}{4}+\frac{1}{2}e^2\right)\sin 2E+\frac{1}{12}e\sin 3E\right]\right\}. \tag{29}$$

Variable E denotes the eccentric anomaly.

In order to solve the two-point boundary value problem, described by the maximum Hamiltonian H^*, of going from an initial orbit O_0 to a prescribed final orbit O_f, a two-stage algorithm is used (Da Silva Fernandes et al, 2018): in the first step, a neighboring extremals algorithm is applied to solve the two-point boundary-value problem described by the canonical system governed by the mean maximum Hamiltonian; in the second step, a Newton-Raphson algorithm is applied to adjust the initial values of the adjoint variables when the first order short periodic terms related to the second zonal harmonic J_2 and to the optimal thrust acceleration are included.

5. RESULTS

The numerical-analytical procedure briefly described in the previous section is applied to solve the transfer problem defined in Table 1. The Figures 2 and 3 depicts the time behavior of orbital elements of optimal trajectory considering different harmonics in the model of the Earth's gravitational potential; that is: solution 1 - optimal solution in central force field (classical model commonly used in the literature), solution 2 - optimal solution with the effects of the second zonal harmonic J_2, and, solution 3 - optimal solution with the effects of the three main zonal harmonics J_2, J_3 and J_4. The results show that effects of the zonal harmonics exhibited in solutions 2 and 3 cannot be ignored when these solutions are compared to the solution 1. On the other hand, the influence of the harmonics on the fuel consumption is too small.

Table 1 – Set of orbital elements of initial and final orbits.

	Semi major axis (km)	Eccentricity	Inclination (degrees)	Longitude of ascending node (degrees)	Argument of pericenter (degrees)
Initial orbit	7000	0.05	10.0	0.0	30.0
Final orbit	10500	0.25	30.0	0.0	30.0

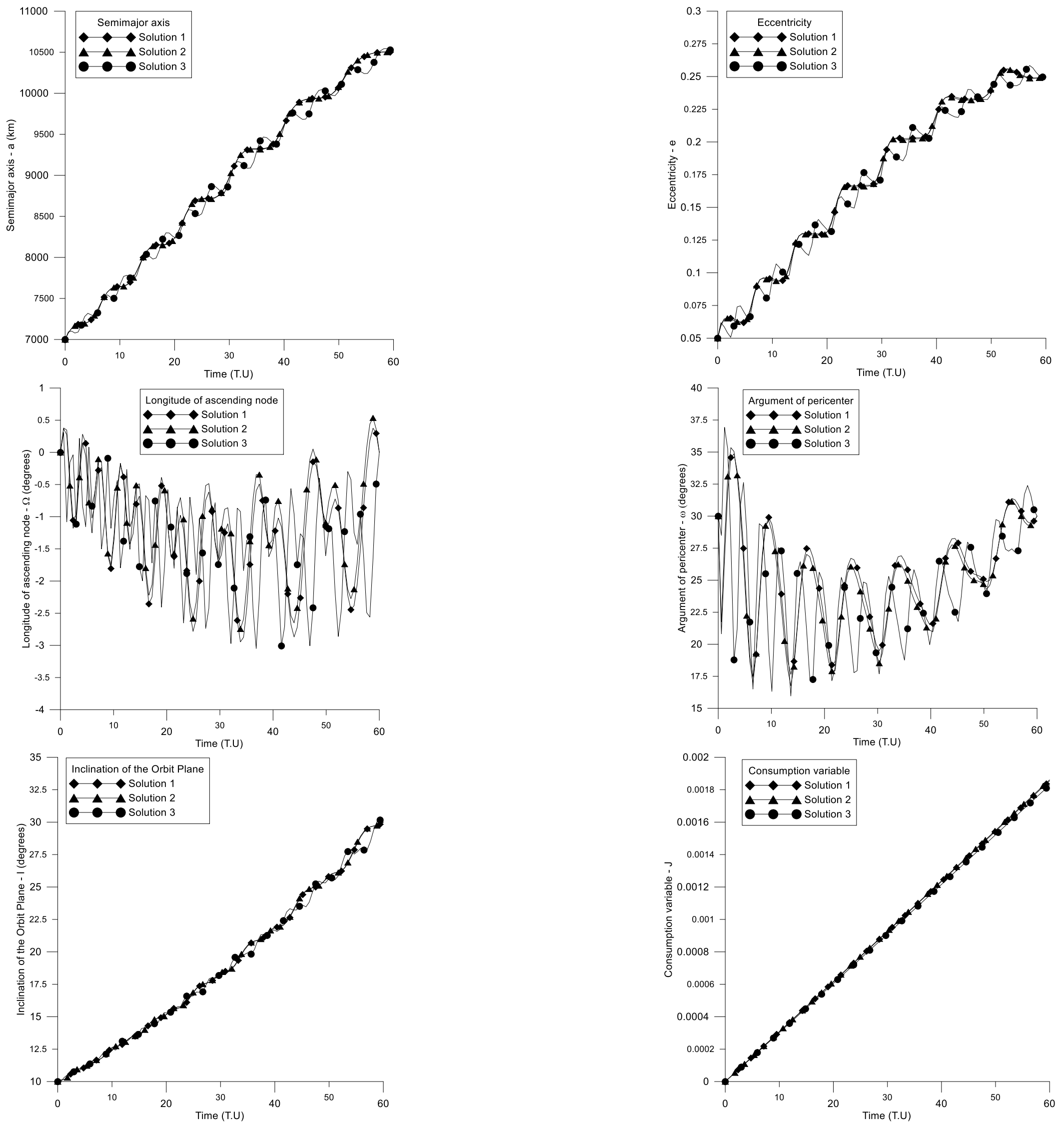

Figure 2 – Time behavior of orbital elements of optimal trajectory for time 60 T.U..

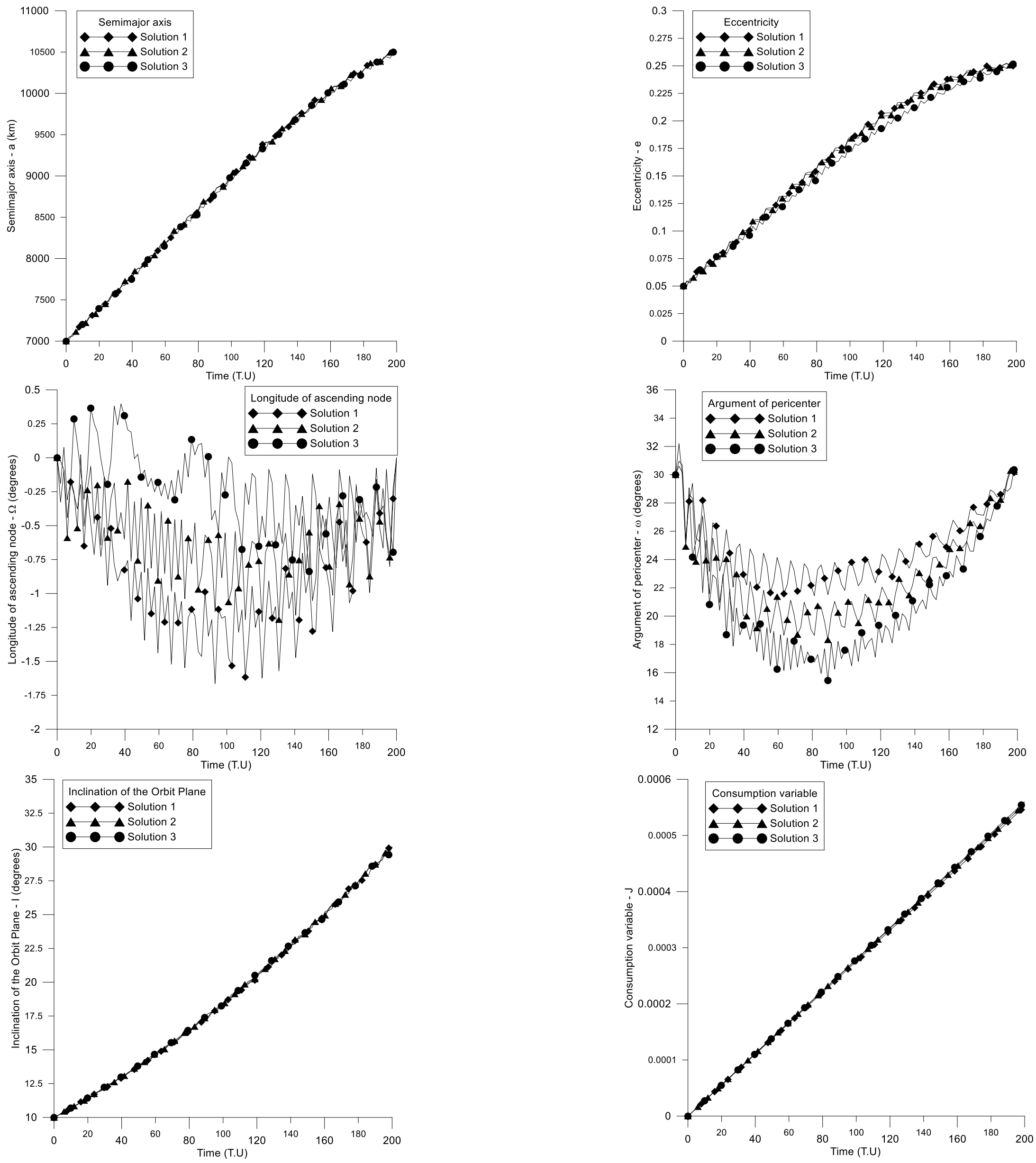

Figure 3 – Time behavior of orbital elements of optimal trajectory for time 200 T.U..

6. CONCLUDING REMARKS

In this paper, a numerical-analytical procedure for computing optimal time-fixed low-thrust limited power transfers between arbitrary orbits in Earth´s gravitational field which includes the effects of the main zonal harmonics is developed using the concept of mean Hamiltonian and Hori method. The proposed algorithm involves the development of a two-stage algorithm for solving the two-point boundary value problem of going from an initial orbit to final orbit and it is applied in the analysis of an example which involves changes in the three orbital elements: semi-major axis, eccentricity and inclination of the orbital plane. Numerical results show that effects of the zonal harmonics J_2, J_3 and J_4 cannot be ignored when compared with the results obtained using the classical model which considers the central force field hypothesis.

7. ACKNOWLEDGEMENTS

This research has been supported by CAPES and by CNPq under contract 301875/2017-0.

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