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USING RECURRENCE PLOTS TO CHARACTERIZE A NONLINEAR PENDULUM

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Abstract. Recurrence plots and their diagnostics are shown to be very useful in the study of chaotic systems. In this work we use recurrence plots to characterize a nonlinear pendulum. First, we study the system using usual tools, such as phase space, Poincaré sections and bifurcation diagrams. After that, we use the recurrence plots and compare them with the standard tools.

Keywords: Pendulum, Dynamical Systems, Classical Mechanics, Nonlinear Systems, Recurrence Plots.

1. INTRODUCTION

The Pendulum is probably the most famous system in classical mechanics. We can find its features in a large variety of systems such as spring motion (Goldstein and Safko, 2001), molecular interactions (Goldstein and Safko, 2001) and so on. With the discovery of chaos (Barrow-Green, 2005), classical mechanics has taken a new breath and various scientists and engineers started to study this new exciting phenomenon of nature. Although this branch of physics is so beautiful, it shows us that nature is really complex even in situations that seem simple. With this motivation, several tools were created to analyze chaotic systems, like Poincaré sections and phase spaces (Alligood and Yorke, 2000).

Physical phenomena have the property to go back to initial states with time. This was first stated by Poincaré in his famous recurrence theorem (Poincaré, 1890). Periodic systems have a definite pattern of recurrence while the chaotic ones don't. Examples of famous systems that recur periodically are the spring-mass system (Feynman, 2013) and the relative distance between Sun and Earth. The solutions for these two systems are periodic functions.

The tool used to analyze physical systems based on their recurrences is called Recurrence plot. Recurrence plots (RP) are shown to be very useful in dynamical systems. They can show properties of the systems that the classical tools can't. It can be applied in financial crisis (Addo and guegan, 2013), earthquake analysis (Garcia and Figueroa-Nazuno, 2013) and so on. For example, RP's can show not only if a trajectory is periodic, but also its period (Marwar *et al.*, 2007). Also, with quantitative diagnostics, RP's provide statistical data such as rate of recurrence.

In this work, we study the behavior of a nonlinear pendulum, identifying regularity and chaos. We use recurrence plots to analyze and characterize different behaviours of the system.

2. NONLINEAR PENDULUM

Figure 1 shows a schematic representation of the experimental apparatus. It consists of an aluminum disc (1) with a lumped mass (2) that is connected to a rotary motion sensor (4). This assembly is driven by a string-spring device (6) that is attached to an electric motor (7) and also provides torsional stiffness with the anchor mass (5). A magnetic device (3) provides an adjustable dissipation of energy.

The motion equation describing the system is the following (de Paula and Pereira-Pinto, 2006):

$$\ddot{\theta} + \frac{\xi}{I}\dot{\theta} + \frac{kd^2}{2I}\theta + \frac{2\mu}{\pi I}\arctan\left(\alpha\dot{\theta}\right) + \frac{mgD\sin\theta}{2I} = \frac{kd}{2I}\Delta L, \tag{1}$$

where,

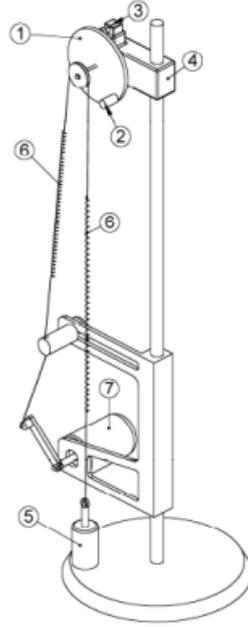$$\Delta L = \sqrt{a^2 + b^2 - 2ab\cos\Omega t} - (a - b). \tag{2}$$

Figure 1. Schematic figure representing the nonlinear pendulum. (de Paula and Pereira-Pinto, 2006).

Equation (1) is a nonlinear, non autonomous and second order differential equation in θ. The term ΔL represents the external force, I is the moment of inertia, ξ and μ are terms concerning the energy dissipation of the system, viscous damping and dry friction respectively, k is the spring constant, m is the anchor's mass (5), g the gravitational acceleration. d, D, a and b are geometric parameters, Ω is the external forcing frequency. Ω is in $\frac{rad}{s}$. One needs that the function $\arctan(\alpha\dot\theta)$ behaves like a signal function. To have it, α must be very large.

Making the variable change $\omega = \dot\theta$ one has the following system:

$$\dot\theta = \omega, \tag{3}$$

$$\dot\omega = \frac{kd}{2I}\Delta L - \frac{\xi}{I}\omega - \frac{kd^2}{2I}\theta - \frac{2\mu}{I\pi}\arctan(\alpha\omega) - \frac{mgD}{2I}\sin(\theta). \tag{4}$$

3. DEFINITIONS

3.1 Recurrence plots

Suppose $\mathbf{X}(t)$ is a trajectory on a $2N$ dimensional phase space, where $\mathbf{X}$ is the tuple $(q_1, p_1, q_2, p_2, \ldots, q_N, p_N)$ and the canonical coordinates all depend on time. Consider now $\mathbf{X}$ at some time t_a. The system is then at state $\mathbf{X}(t_a)$. Constructing now an open ball with radius ϵ and $\mathbf{X}(t_a)$ at its center we have the following set of states:

$$\mathbb{P}_{t_a} = \{t \in [0, \infty), \mathbf{X}(t) \in \mathbb{R}^{2N} : |\mathbf{X}(t) - \mathbf{X}(t_a)| < \epsilon\}. \tag{5}$$

If at some time t_b, $\mathbf{X}(t_b) \in \mathbb{P}$, we call this event a recurrence, otherwise it will be a non-recurrence. With the definition of a recurrence in hands, we can now define a recurrence plot (RP).

A **RP** is a N dimensional matrix in which each pair of coordinates i, j verify if the state $\mathbf{X}(t_i)$ recurs with the state $\mathbf{X}(t_j)$, where $t_i = i\Delta t$, Δt being the time step. In other words, if $\mathbf{X}(t_j) \in \mathbb{P}_{t_i}$, we have a recurrence and the element RP_{ij} will be assigned to the value 1, otherwise it will be 0.

Recurrence plots are classified with respect to their typology. A RP can be homogeneous presenting no pattern of recurrence, periodic with well defined recurrent structures, drift presenting splotches on both sides of the main diagonal and disrupted with isolated recurrences.

3.2 Quantification of recurrences plots

The quantification of recurrence plots is made by using statistical quantities. These are functions that depend on the neighborhood used and on the form of the RP's. In this work two measures are considered: the recurrence rate $\mathcal{RR}$ and

the average diagonal length $\mathcal{L}$. The $\mathcal{RR}$ is the density of recurrence points. It is defined as follows (Marwar *et al.*, 2007):

$$\mathcal{RR} = \frac{1}{N^2} \sum_{i,j=1}^{N} R_{ij}, \tag{6}$$

where N is the matrix dimension. When $N \to \infty$, $\mathcal{RR}$ represents the probability of a state to visit again its neighborhood. $\mathcal{L}$ is the average diagonal length on the RP. It is defined as follows (Marwar *et al.*, 2007):

$$\mathcal{L} = \frac{\sum_{l_{min}}^{N} lP(l, \epsilon)}{\sum_{l_{min}}^{N} P(l, \epsilon)}, \tag{7}$$

where l_{min} is the smallest diagonal considered and $\mathcal{P}(l, \epsilon)$ is the frequency of each diagonal length given by (Marwar *et al.*, 2007):

$$\mathcal{P}(l, \epsilon) = \sum_{i,j=1}^{N} (1 - RP_{i-1,j-1}(\epsilon))(1 - RP_{i+l,j+l}(\epsilon)) \prod_{k=0}^{l-1} RP_{i+k,j+k}(\epsilon). \tag{8}$$

$\mathcal{L}$ represents how many well defined structures exist on the RP. Greater is $\mathcal{L}$, more periodic the system is. Further measures are found in (Marwar *et al.*, 2007).

4. RESULTS

4.1 Simulation Details

Numerical results are obtained by integrating the system 3, 4 using the **Runge-Kutta** routine **ode45** in **MATLAB**. The integration step is given by $h = 2\pi/120\Omega$, where Ω is the forcing frequency, Ω in $\frac{rad}{s}$. In all simulations the parameters are the same as used by (de Paula and Pereira-Pinto, 2006): $d = 4, 8 \cdot 10^{-2}m$; $D = 9, 5 \cdot 10^{-2}m$; $a = 1, 6 \cdot 10^{-1}m$; $b = 6, 0 \cdot 10^{-2}m$; $m = 1, 47 \cdot 10^{-2}Kg$; $I = 1, 738 \cdot 10^{-4}Kgm^2$; $\xi = 2, 368 \cdot 10^{-5}Kgm^2s^{-1}$; $k = 2, 47N/m$; $\mu = 1, 272 \cdot 10^{-4}Nm$; $g = 9, 81m/s^2$. To obtain the bifurcation diagram, the system is integrated over 2400 periods $\tau = \frac{2\pi}{\Omega}$. To compute the Lyapunov exponents, we used the Wolf's algorithm (Wolf *et al.*, 1984) considering four trajectories with slightly different initial conditions. To construct the recurrence plots we used the software available on the website (Marwar *et al.*, 2000). The measures are obtained from time series with 20000 points each, divided in blocks of 500 points each. The neighborhood used is $\epsilon = 0.12$.

4.2 Lyapunov exponents and Poincaré sections

Figure 3 shows the Lyapunov exponents for Ω between 2 and 7. Observing the Lyapunov graph, we can see that the chaotic behaviour begins when Ω is greater than 5. In the transition to chaos, the system presents orbits with different periods (see fig. 2). It is important to note that when the system is in the chaos zone, it still presents periodic trajectories for some values of Ω.

4.3 Recurrence plots

Figure 4 shows the phase space and the recurrence plot for $\Omega = 3.59$. The recurrence plot shows only one recurrent structure in its typology because the phase space has period 1 (you can see that because the Poincaré map is just one point). One of the advantages of RP's is that we can calculate the period of the trajectory just by taking the horizontal (or vertical) length of the recurrent structure in the RP. In this case, the period is $\tau = 1.75s$. Figure 5 shows a zoom of the recurrence plot.

Figure 6 shows the phase space and recurrence plot for $\Omega = 5.22$. In this case we see that the trajectory has period 2 (see the Poincaré map in red). It means that the RP has to two recurrent structures in its typology. Figure 7 shows a zoom of the recurrence plot.

To calculate the period of one trajectory using the recurrence plots, we used the following relation:

$$\tau = \frac{2\pi D_t}{120\Omega}, \tag{9}$$

where τ is the period of motion, D_t is the structure length (horizontal or vertical) and Ω is the forcing frequency. Figure 8 shows the phase space and recurrence plot for $\Omega = 5.43$. As we can see, this case presents a trajectory of period 4 and, because of that, the recurrence plot has to show four structures well defined. In this case it is difficult to see them, hence we can't calculate the periods of the four characteristic motions.
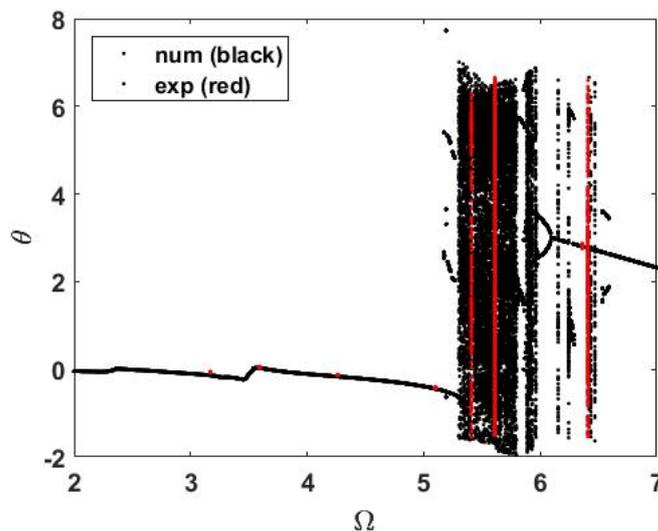
Figure 2. Bifurcation diagram for the parameter Ω. Black points are numerical data and red points experimental data (de Paula and Pereira-Pinto, 2006). In the beginning of the evolution of Ω the system presents only period-1 trajectories. In the transition to chaos, we can see orbits with different periods.
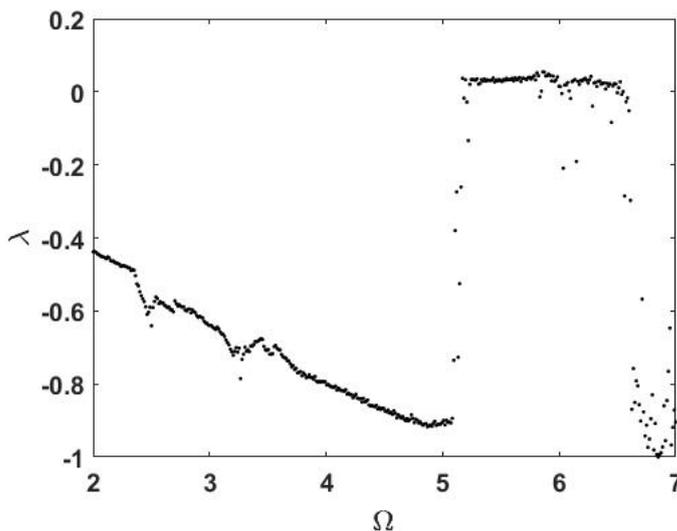

Figure 3. Lyapunov exponents with respect to Ω, forcing frequency. For Ω between 2 and approximately 5, the system presents periodic behavior ($\lambda < 0$). When Ω is greater than 5 the system is chaotic, but still presents some periodic trajectories ($\lambda > 0$). It is important to notice that increasing Ω doesn't mean that the system will be forever chaotic.

Figure 9 shows the phase space and sequential recurrence plots for $\Omega = 5.40$. This is a chaotic case (see Poincaré map in red). It is important to notice that all the sequential RP's are different of each other. This happens because the trajectory is chaotic. If we continue with the time series, the next points in the Poincaré map would not follow a pattern. Hence, the next RP's would be different.

4.4 Quantification of the RP's

Figures 10 and 11 show the evolution of the measures $\mathcal{RR}$ and $\mathcal{L}$ as a function of Ω, respectively. In the evolution of $\mathcal{RR}$, we see that for small values of Ω, $\mathcal{RR}$ undergoes a big variation and then stabilize when Ω is between 4 and 5. Comparing with fig. 2, the abrupt variation of $\mathcal{RR}$ for Ω between 5 and 6 represents the transition to chaos. In terms of the measure $\mathcal{RR}$, it means that when the pendulum presents chaotic behaviour, given the state $\vec{X}$ at any time t, the system is less likely to recur in a future time. The same happens to $\mathcal{L}$. When the system is in the chaotic zone, it will show less diagonal structures on the RP's, hence it will be less periodic. It is important to notice that $\mathcal{RR}$ and $\mathcal{L}$ present similar behaviours. This is not occasional. Less periodic trajectories will be less likely to recur.
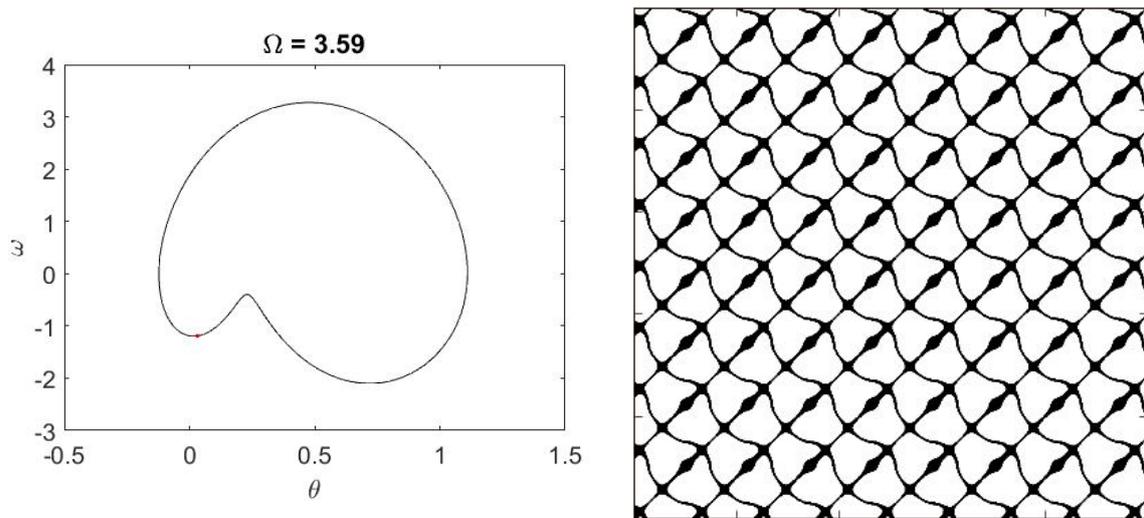
Figure 4. Phase space and recurrence plot for $\Omega = 3.59$. On phase space we see that the trajectory has period 1 (Poincaré map in red). The recurrence plot for this case has only one recurrent structure. The RP has dimension of a thousand.
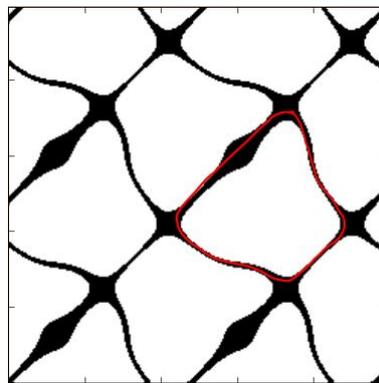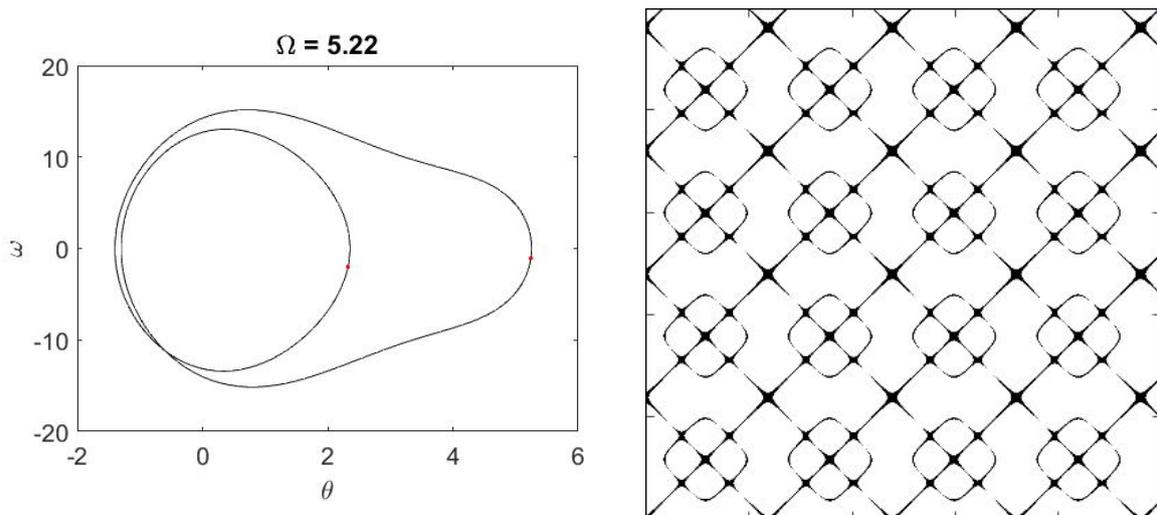

Figure 5. A zoom on the recurrence plot of fig. 4. The recurrent structure is outlined in red. It represents the trajectory on phase space of fig. 4. Its horizontal (or vertical) length is 120 units of time or $1.75s$. It means how long the system takes to make that trajectory once.

Figure 6. Phase space and recurrence plot for $\Omega = 5.22$. On phase space we see that the trajectory has period 2 (Poincaré map in red). The recurrence plot for this case shows two recurrent structures. The RP has dimension of a thousand.

5. CONCLUSIONS

Analyzing the bifurcation diagram and the Lyapunov exponent we see that the system presents periodic and chaotic behaviours. In the chaotic zone, the nonlinear pendulum still presents some periodic trajectories. The bifurcation diagram
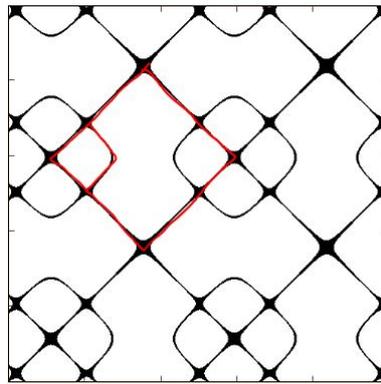
Figure 7. A zoom on the recurrence plot of fig. 6. The recurrent structures are outlined in red. We can see two structures, a small and a big lozenge. The small one has 80 units of time of length ($\tau_1 = 0.8s$) and the big one has 240 units of time of length ($\tau_2 = 2.4s$).
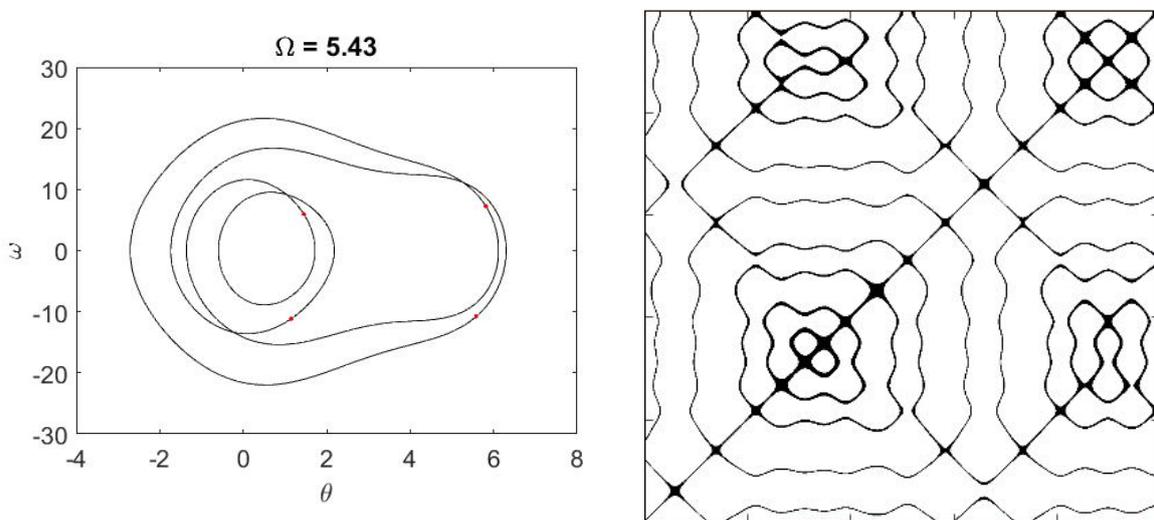

Figure 8. Phase space and recurrence plot for $\Omega = 5.43$. On phase space we see that the trajectory has period 4 (Poincaré map in red). In this case we can't see four recurrent structures because the RP is too complex. The recurrence plot has dimension of a thousand.

shows that the transition to chaos is constituted by trajectories of different periods. It means that this change of behaviour is not soft. Another important point is that increasing indefinitely Ω doesn't mean more chaotic behaviour. Lyapunov exponents show that when Ω is close to 7, λ assumes again values smaller than zero.

With the recurrence plots we can easily compute the period of motion of simple trajectories. But for trajectories of period greater than 2, it is very hard to compute them, because the RP's become more complex. In the chaotic case, sequential RP's will be different. Perhaps it is difficult to see chaotic behaviour only using RP's because periodic trajectories with high period will present complex RP's just like the chaotic ones.

The statistical of RP's shows clearly when chaotic behaviour happens by an abrupt change on their values. For $\mathcal{RR}$ and $\mathcal{L}$ it means that in the chaotic domain the system is less likely to recur and will present less diagonal structures. It is important to remember that this relation between $\mathcal{RR}$ and $\mathcal{L}$ is particular to this case because there can exist non periodic systems (low $\mathcal{L}$) which present high values of $\mathcal{RR}$.

6. REFERENCES

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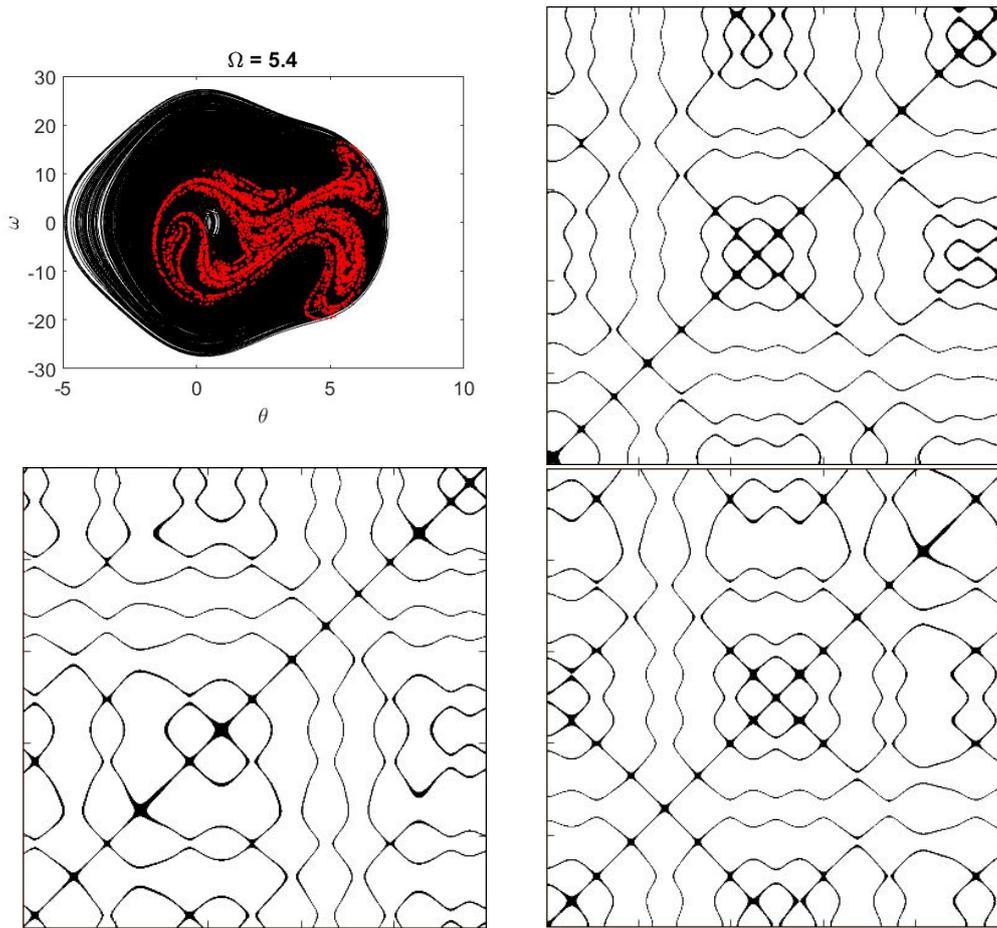Feynman, R., 2013. "The harmonic oscillator". http://www.feynmanlectures.caltech.edu/I₂1.

Figure 9. Phase space and recurrence plots for $\Omega = 5.40$. On phase space we see that the trajectory is chaotic because the Poincaré map has an undefined period (in red). The three sequential recurrence plots show how a chaotic case is presented by RP's. All the recurrence plots have dimension of a thousand.
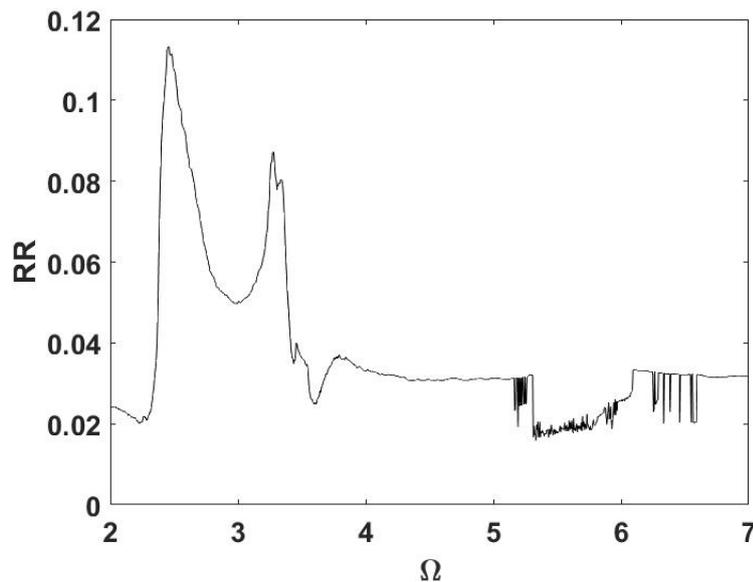

Figure 10. $\mathcal{RR}$ in function of Ω.

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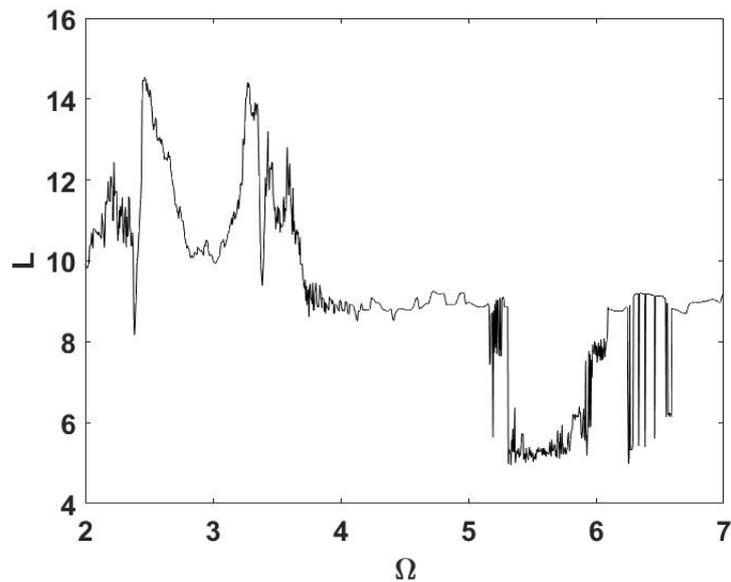
Figure 11. $\mathcal{L}$ in function of Ω.

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