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BOUNDS FOR THE PROPAGATION MODEL OF CRACK PARIS-ERDOGAN USING THE "FAST CRACK BOUNDS" METHODOLOGY

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Abstract: *The Linear Elastic Fracture Mechanics has as one of its studies the analysis of the microcracks's increase in the material due to aggravating factors, as fatigue. There are models to figure out the propagation of these cracks, and they are, in terms of amplitude and tension, the constant (CAC) and the variable (CAV), considering that both are formulated as an initial value problem (IVP) and obtained results from a numerical model. If complex geometries are taken into account, the calculation process becomes heavy and complex for computers. In this article is presented the methodology Fast Crack applied in the method that determines quota functions in the "Paris – Erdogan" model, which estimates superior and inferior quotas, that are applied in any number of cycles, estimating how the crack will behave in the extremes of the behavior. An example was a finite width plate with center crack. The work identifies computational time gains of the FCB methodology in the order of 144148.63 times smaller than that obtained by RK4. With a slight variation of relative deviations. Having maximum deviations of 2.26%. Validating the efficiency of the FCB methodology for the analysis of the "crack size" phenomenon.*

Keywords: *Microcrack; Paris-Erdogan; Fracture Mechanics; Initial value problem; tension.*

1. INTRODUCTION

In the scientific scope of the materials, Linear Elastic Fracture Mechanics (LEFM) is the responsible field for the study of the propagation of cracks phenomenon and consequently the fatigue process, defined as a localized degradation, progressive and permanent, which occurs in materials subject to dynamic tensions and deformations.

Cracks are discontinuities that occur in materials with applied tensions below the project's tension, generating flaws in the components. Representing a kind of fragile crack, stimulated by tension concentrators that restrain the plastic deformation. These microcracks may be disseminate from previous problems or by nucleation on duty.

Therefore, the propagation tax gains relevance depending on the intensity of work and the appliance of the component in engineering, making necessary to keep track of the cracks' evolution and recommendations of predictive or preventive maintenance. This way, the modelling of these phenomena became crucial, enabling the development of several representative methods through the years.

The Linear Elastic Fracture Mechanics (LEFM) encompass mathematical models that represents approximately the behavior and increase of cracks in order to simulate the cracks' evolution graphically, being classified as constant tension amplitude (CTA) and variable tension amplitude (VTA). These models are developed through an Initial Value Problem (IVP), using the Runge-kutta numerical method of order four (RK4) to find out the approximate numerical solution of the IVP, that demands long time of computational processing.

The goal of this work is the appliance of the "Fast Crack Bouns" (FCB) methodology for the evolution of the crack's size model, constant tension amplitude (CTA) type, proposed by Paris-Erdogan, this way obtaining the crack evolution's curve by the Runge-Kutta numerical method of order four.

For this purpose, the FCB methodology determines the superior and inferior quota functions for the Paris-Erdogan model obtained through the Taylor's series expansion, retaining the second order terms, with Lagrange's remains wrapping the approximate numerical solution. The performance of the proposal will be measured by the combination of the SMC with the Runge-Kutta method of order four. The intention is to quantify the performance of the proposal for a "classic" example of the Fracture Mechanics, called "Plate with finite width and central crack" where the function of correction of the tension intensity factor is known.

2. METHODOLOGY

The research was developed in two steps, they are:

1- "Fast Crack Bounds" (FCB) method applied to the Forman's model: In this step, the mathematical formulation for the superior and inferior quota functions for the crack's growth of the Formam's model was reached through the FCB methodology, the quotas were calculated by second order Taylor's series with Lagrange's remains.

2- Computational Implementation: For the accomplishment of the second step, the software MatLab was used purposing to implement the following methods: Monte Carlo (SMC), Runge-Kutta method of order four (RK4) and Fast Crack Bounds (FCB) of propagation model of crack Paris-Erdogan. In agreement with the traditional examples of "Plate with infinite width and central crack", "Plate with finite width and central crack" and "Plate with finite width and crack in the edge". Applying the iteration in the MatLab, it will result in an algorithm that will estimate the growth of the crack and it will be compared with the RK4 method.

2.1. Fast Crack Bounds method applied to the Paris-Erdogan model

By the study of Linear Elastic Fracture Mechanics, we may predict how a crack will evolve in a determined number of cycles. This way we can know in advance when a material may break because of the fatigue. The mathematical model generally used is defined by the following IVP:

$$\begin{cases} \text{Encontrar a } \in C^1[N_0, N_1] \text{ tal que:} \\ \frac{da}{dN}(N) = h(\boldsymbol{a}, \Delta K), \forall N \in (N_0, N_1); \\ a(N_0) = a_0; \end{cases} \tag{1}$$

Which are ordinary diferencial equations of first order, nonlinear and autonomous. Appling the expansion in Taylor's series with Lagrange's remains considering the coming hypotheses (SANTOS, 2015):

$$\begin{aligned} &H_1: \Delta\sigma(\mathrm{N}) = \Delta\sigma_0, \forall \mathrm{N} \in [\mathrm{N0}, \mathrm{N1}]; \\ &H_2: \begin{cases} \mathrm{f} \in \mathrm{C}^1([a_{0,} a_1];\ \mathrm{R} + \backslash\{0\}); \\ 0 < \mathrm{f}(a_0) \le \mathrm{f}(\mathrm{x}) \le \mathrm{f}(\mathrm{y}), \mathrm{x} \le \mathrm{y}, \forall\, \mathrm{x}, \mathrm{y} \in [a_{0,} a_1]; \\ f'(a_0) \le f'(\mathrm{x}) \le f'(\mathrm{y}), \mathrm{x} \le \mathrm{y}, \forall\, \mathrm{x}, \mathrm{y} \in [a_{0,} a_1]; \end{cases} \\ &H_3: \mathrm{m} \ge 2; \end{aligned} \tag{2}$$

The first hypothesis H1 reveals that the loading must always be constant. The second hypothesis says that the geometric function of the crack cannot be a descending function as its derivative. These hypotheses are attended since, generally, a crack never decreases in size. These two propositions are the basis of the quotas method that limit the crack function in both superior and inferior parts. Answering the hypotheses above, we obtain the following equations applied to the Paris-Erdogan model:

Theorem (FCB Method applied to Paris-Erdogan model): being $f(\cdot)$ e $\Delta(\cdot)$ functions that satisfy the hypotheses (H_1), (H_2) e (H_3) and $a* \in [a_0, a_1]$. So the following superior and inferior quotas are valid

$$\begin{cases} a(N) - a_0 \leq C \left\{ \begin{matrix} \left(\Delta K(a_0)\right)^m + \left(\frac{mC}{2}\right)\left(\Delta K(a^*)\right)^{2m} \\ \times \left[\frac{1}{2a^*} + \left(\frac{f'}{f}\right)(a^*)\right](N - N_0) \end{matrix} \right\} (N - N_0); \\ a(N) - a_0 \geq C\left(\Delta K(a_0)\right)^m \left\{ \begin{matrix} 1 + \left(\frac{mC}{2}\right)\left(\Delta K(a_0)\right)^m \\ \times \left[\frac{1}{2a_0} + \left(\frac{f'}{f}\right)(a_0)\right](N - N_0) \end{matrix} \right\} (N - N_0), \forall \mathrm{N} \in [\mathrm{N0}, \mathrm{N1}]. \end{cases} \quad (3)$$

Proof: Appling the second order Taylor's series with Lagrange's remains for the crack growth model, we have:

$$a(N) = a_0 + \left(\frac{da}{dN}(N_0)\right)(N - N_0) + \frac{1}{2}\left(\frac{d^2a}{dN^2}(n)\right)(N - N_0)^2 \ \eta \in [\mathrm{N0}, \mathrm{N1}] \quad (6)$$

Using the hypotheses and appling it to th second order Taylor's series, we obtain the following inequations:

$$a(s) \leq a(t), s \leq t \ com \ s, t \in [N_0, N] \Rightarrow (\mathrm{a(s)})^n \leq (\mathrm{a(t)})^m.$$

In hypothesis H2, we have:

$$(\mathrm{f(s)})^m \leq (\mathrm{f(t)})^m \left(\mathrm{a}^{\frac{1}{2}}.f\right)^m (\mathrm{s}) \leq \left(\mathrm{a}^{\frac{1}{2}}.f\right)^m (t). s \leq t \ com \ s, t \in [N_0, N]$$

The result may be written as:

$$(\Delta K)^m(a(s)) \leq (\Delta K)^m(a(t))$$

As C >0, we obtain:

$$\frac{da}{dN}(s) \leq \frac{da}{dN}(t), s \leq t \ com \ s, t \in [N_0, N]$$

Soon, in the second derivative of the crack growth function:

$$\frac{d^2a}{dN^2}(a(N)) = \frac{d}{dN}\left(\frac{da}{dN}(a(N)\right) = \frac{d}{da}\left(\frac{da}{dN}(a)\right)\frac{da}{dN}\big(a(N)\big)$$

$$= C^2(\Delta\sigma\sqrt{\pi})^{2m}\frac{d}{da}\left(a^{\frac{m}{2}}\big(f(a)\big)^m\right)\left(a^{\frac{m}{2}}\big(f(a)\big)^m\right)$$

$$= \mathrm{mC}^2\left(\Delta\sigma\sqrt{\pi}\right)^{2m}\left[\frac{1}{2}a^{\frac{m}{2}-1}(\mathrm{f(a)})^m + a^{\frac{m}{2}}(\mathrm{f(a)})^{m-1}\mathrm{f'(a)}\right]\left(a^{\frac{m}{2}}(\mathrm{f(a)})^m\right)$$

$$= \mathrm{mC}^2\left(\Delta\sigma\sqrt{\pi}\right)^{2m}\left[f(a)\sqrt{a}\right]^{2m}\left[\frac{1}{2a} + \left(\frac{f'}{f}\right)(a)\right]$$

$$= \mathrm{mC}^2\left(\Delta \mathrm{K(a)}\right)^{2m}\left[\frac{1}{2a} + \left(\frac{f'}{f}\right)(a)\right] \quad (7)$$

Appling the Eq. (7) in the Eq. (6), we have the following formulation:

$$a(N) - a_0 = C\left(\Delta K(a_0)\right)^m(N - N_0) + \frac{mC^2}{2}\Big(\Delta K\big(a(\eta)\big)\Big)^{2m}\left[\frac{1}{2\mathrm{a}(\eta)} + \left(\frac{f'}{f}\right)\big(\mathrm{a}(\eta)\big)\right](\mathrm{N} - \mathrm{N}_0)^2, com \ \eta \in [\mathrm{N0}, \mathrm{N}]. \quad (8)$$

The superior and inferior limits are obtained through a change in the Lagrange's remains of the Eq. (8). Through the hypothesis H2 in function of the crack's geometry, we obtain:

$$\left[f(a)\sqrt{a}\right]^{2m}\left[\frac{1}{2}f(a) + af'(a)\right] \leq \left[\sqrt{a^*}f(a^*)\right]^{2m}\left[\frac{1}{2a^*} + \left(\frac{f'}{f}\right)(a^*)\right], \forall a \in [a_0, a_1]. \tag{9}$$

So:

$$\left[f(a)\sqrt{a}\right]^{2m}\left[\frac{1}{2}f(a) + af'(a)\right] \geq \left[\sqrt{a_0}f(a_0)\right]^{2m}\left[\frac{1}{2a_0} + \left(\frac{f'}{f}\right)(a_0)\right], \forall a \in [a_0, a_1]. \tag{10}$$

Replacing the Eq. (9) in Eq. (7) it is obtained the superior function of quota:

$$\frac{d^2a}{dN^2}(\eta) \leq mC^2\left(\Delta K(a^*)\right)^{2m}\left[\frac{1}{2a^*} + \left(\frac{f'}{f}\right)(a)\right], \forall\eta \in [a_0, a_1] \tag{11}$$

And just inserting the Eq. (10) in Eq. (7), we have the inferior function of quota:

$$\frac{d^2a}{dN^2}(\eta) \leq mC^2\left(\Delta K(a_0)\right)^{2m}\left[\frac{1}{2a_0} + \left(\frac{f'}{f}\right)(a_0)\right], \forall\eta \in [a_0, a_1] \tag{12}$$

Appling the Eqs. (11) and (12) in Eq. (8), the superior and inferior functions of quota abovementioned are obtained. It can be seen that the calculations depend on the geometric function of the crack and on the derivative in two points a0 and a*. The result comes from the second order Taylor's series with Lagrange's remains.

3. NUMERICAL RESULTS

The numerical results were generated by the MatLab software (version R2015a), using a computer with Intel Celeron B830 (1,8 GHz) processor and RAM memory of 4,00 GB to do the reading of the algorithm. From the Paris-Erdogan model of propagation of cracks, the quotas were examined based on the foundations presented on the works of Castro and Meggiolaro (2009), applied to a ferritic steeled material. The data used in the numerical simulations are described on Tab. 1.

Table 1. Data used in the numerical simulations.

Parameter	Numerical Value	Unidade
C	6,90e-12	m/cycle
m	3,0	Dimensionless
K_c	250	MPA√m
Width of the plate (b)	0.1	m
Number of Cycles (N)	900.000	Cycles

Add up to the data of table 1, the size of the star crack to the superior quota (a*). According to Santos (2015), the value of a* is determined by inspection, being responsible to ensure that the superior quota will not be trepassed by the numerical solution, limiting it. For this study, the methodology was applied in an classic example of the literature, adopting a a* = 1,3*a0.

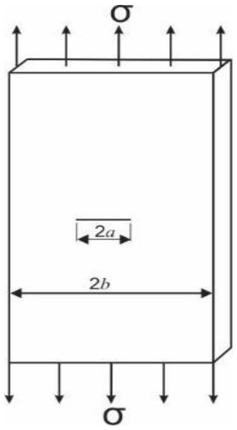
In order to mesure the performance of the presented proposal, functions of "relative deviation" of quotas were established, presented by the Eq. (13) and Eq. (14).

$$\delta_{superior}(N_k) = 100 * \left(\frac{a_{cs}-a_{RK4}}{a_{RK4}}\right)(N_k) \qquad [\%], \forall N_k\{0,1,\dots,N\} \tag{13}$$

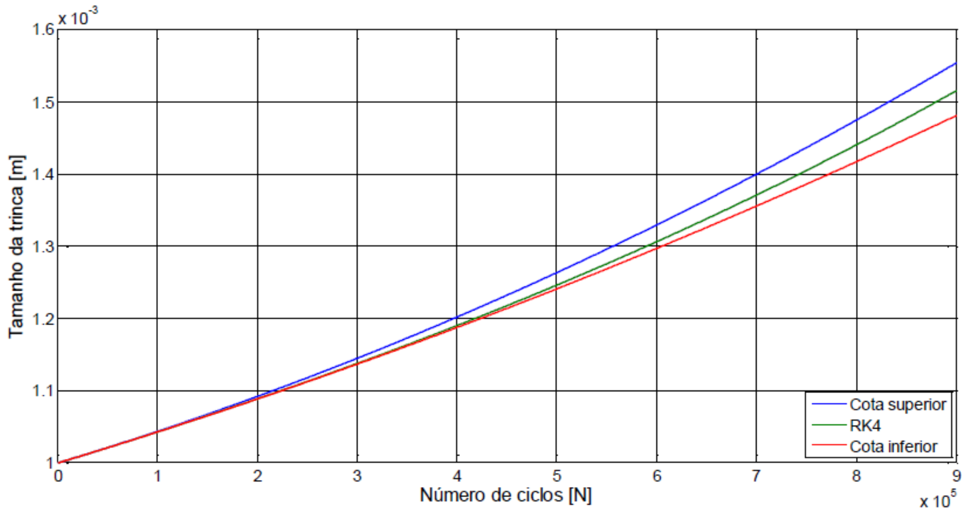
$$\delta_{inferior}(N_k) = 100 * \left(\frac{a_{ci}-a_{RK4}}{a_{RK4}}\right)(N_k) \qquad [\%], \forall N_k\{0,1,\dots,N\} \tag{14}$$

Next, there is na application of the methodology for a classic example used on the fracture mechanics, the "Plate with finite width and central crack" which facilitate the results and simplify the calculation. A correcting function of the tension intensity factor was used, described by Bannantine (1989).

$$f(a) = \sqrt{\sec\left(\frac{\pi a}{2b}\right)}. \tag{15}$$



Picture 1. Plate with finite width and central crack.
Source: Formulated by the authors.



Picture 2. Superior and inferior quota functions compared to the numerical solution approximate for the example 1, according to Paris' model.
Source: Formulated by the authors.

Analyzing picture 2, it is evident that the deviation presented by superior and inferior quotas is not considerable in its numerical values, in terms of approximation, when compared with the the numerical solution method approximate of RK4, which can be proven by the function of relative deviation graphics. This way, there is a foothold to claim that in these examples, the Paris-Erdogan model for superior and inferior quotas wraps, in narrow way, the crack's propagation curve. The relative deviation of quotas function is established to measure the development of the FBC methodology in relation to RK4, equation 4.4. Picture 3 shows the relative deviation of quotas, for the proposed example.

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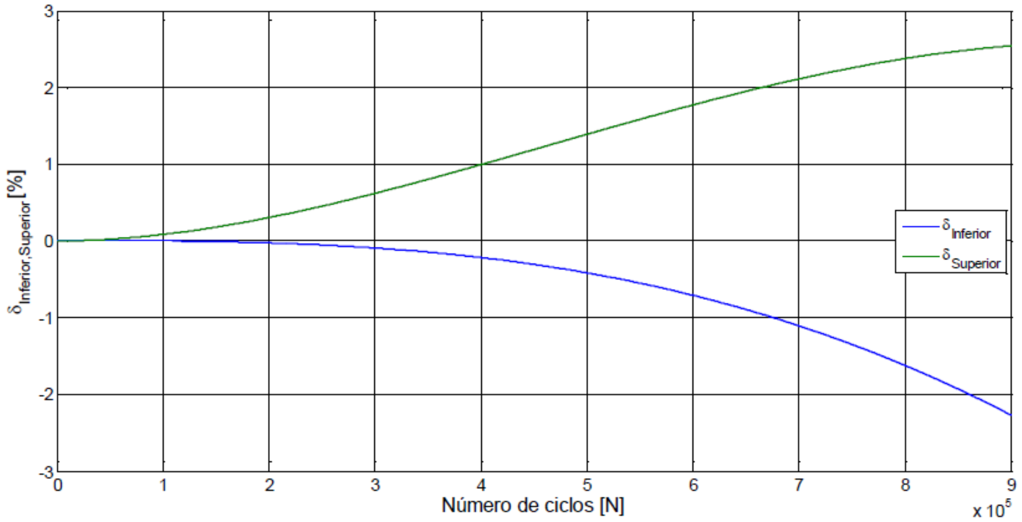

Picture 3. Relative deviation function between superior and inferior quotas for example 2, according to Paris's modelo.
Source: Formulated by the authors.

In picture 3 is observed that the relative deviation was 2,54% for the superior quota and -2,26% for the inferior quota, in other words, they got similar values and little variation. The Table 2, next presented, shows the computational runtime resulting of the software operation which determines the quotas via FCB-SMS and numerical solution via RK4-SMS for a 900.000 cycles condition. The comparison of processing runtimes derives from Eq. (16).

$$\rho = \left(\frac{RK4-FCB}{FCB}\right) \quad (16)$$

Table 2. Runtime (in seconds) for 900000 cycles for the Paris-Erdogan model.

Plate with finite width and central crack	Runtime (s)	ρ
RK4	1470,3263	144148,63
FCB	0,0102	

The ratio between the computational runtime is equal to 144148,63, therefor much faster computationally, if compared to the RK4 method's solution. This parameter (ρ), along with the relative deviation function, show that the proposed methodology has an accurate development and saves time.

4. CONCLUSION

Refering to the wrapping of the numerical solution deriving from RK4 method, the superior and inferior quotas had positive outcomes. Due to the usage of quotas, there are relative deviations, being the biggest deviation in the superior quota with a 2,54% maximum, and in the inferior quota, the maximum deviation was 2,26%. Concerning to the computational runtime, the quotas had satisfactory results, in some simulations the quotas were 144148,63% faster than the RK4 method, proving its proficiency.

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