EVALUATION OF THE DUAL-TIME STEP TECHNIQUE COUPLED WITH MINIMAL GAIN MARCHING SCHEME

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Abstract: *Reference solutions are quite important in many engineering applications. Stability analysis, initial conditions for unsteady problems and reference states for numerical sponge zones are some examples. Recently, the minimal gain marching scheme was developed to generate reference solutions for any type of unstable problem. This method consists to modify the original numerical scheme coefficients such a way that absolute numerical gain is smaller than one for the required frequency spectra. This approach allows the respective disturbance group amplitude decay given enough time. However, these ideas were applied and obtained for implicit schemes. An alternative is the dual-time step technique. This method modifies the original transient evolution of the governing equations. It creates a pseudo-time which any numerical scheme can be used without affecting the physical-time. This is done by introducing in the governing equations another physical-time derivative that minimal gain scheme can be used while the original scheme becomes a pseudo-time derivative. A physical-time accurate solution is then generated upon convergence towards pseudo-time steady-state in each physical-time step. The present paper will discuss the efficient implementation of the dual-time step coupled with minimal gain marching scheme. Different problems will be investigated to evaluate the convergence process of the coupled method and compared with damping methods available in the literature.*
Keywords: *Minimal gain marching scheme, Dual-time step technique, Numerical stability analysis, Unstable problems and Select frequency damping technique*

1. INTRODUCTION

Reference solutions have an important role at the studies of several fluid dynamic research area. One such area is computational fluid dynamics, where reference solutions are necessary to minimize unwanted waves (Teixeira, 2010). Undesirable disturbances were developed when both approximate and similar solution were employed as initial condition in absolutely mixing layer (Lardjane *et al.*, 2004). Even though high order schemes are employed in mixing layer numerical investigations, the relative error in the calculations of the temporal disturbances growth rate are higher than 18% due approximate initial conditions (Germanos *et al.*, 2009). Numerical investigations of flows around two-dimensional bodies produce large disturbances at the beginning of the simulation due approximate initial conditions (Bijl *et al.*, 2002; Wang and Mavriplis, 2007). Absorbing layers boundary conditions requires reference solutions near the artificial boundaries (Blaschak and Kriegsmann, 1988; Bodony, 2006). These boundary conditions are quite useful in aeroacoustic and receptivity research (Collis and Lele, 1999; Barone and Lele, 2002; Saric *et al.*, 2003; Colonius and Lele, 2004). Linear stability analysis of complex flows are another research area which accurate reference solutions are quite important. These solutions are representative base flow for disturbances analysis (Michalke, 1984). Nevertheless, inappropriate results might develop, since numerical errors in the reference solution become forcing terms in the disturbance governing equations. The Method of Manufactured Solution overcome this problem reallocating the force term into the governing equations as a source term (Steinberg and Roache, 1985; Roache, 2002). Hence, the approximate solution of the original equation becomes an exact solution of the modified equation. Recently, the stability analysis and the method of manufactured solution have been blended, which direct numerical simulations of the modified governing equations is used to perform a linear analysis of a time-averaged solution (Jones *et al.*, 2010). This approach have a problem that numerical residue becomes an integral part of model, which reduces its capability to predict the correct flow physics. Therefore, the most used base states are the similarity solutions calculated by simplified models (Theofilis, 2003). Although this type of solutions may not be enough to obtain accuracy analysis of complex flow fields. Opposite trends for the range of unstable frequencies in the linear stability analysis of transverse jets using approximate (de B. Alves *et al.*, 2008) and similar (Kelly and de B. Alves, 2008) profiles as base flow are compelling evidence that an inaccurate base flow will lead to qualitatively incorrect predictions (Bagheri *et al.*, 2009a).

Actually, there are few methods for the construction of accurate reference solutions. The most traditional method for the construction of accurate reference solutions is the Newton-type method coupled with continuation techniques (Tuckerman and Barkley, 2000). There are a vast literature of this subject. However, these schemes have several convergence difficulties for globally unstable problems, they are quite sensitive to initial guess and require heavy computational resources for large systems. An alternative method consists to introduce a source term in the governing equations that forces the unsteady solution to converge towards a filtered reference solution, where a filtered version of the unsteady solution is marched in time with an additional filter equation (Åkervik *et al.*, 2006). This source term vanishes at steady-state and the reference solution is obtained in this limit. The method is known by Selective Frequency Damping (SFD). The method is usually employed for globally unstable flows were investigated with selective frequency damping technique, i. e., flows with a single dominant self-excitation frequency (Benoît, 2008; Nichols and Schmid, 2008; Bagheri *et al.*, 2009b; Sipp *et al.*, 2010; Mack and Schmid, 2010; Chandler *et al.*, 2012; Ilak *et al.*, 2012). Furthermore, the technique has severe convergence difficulties when the problem is susceptible to stationary instabilities as well as instabilities with a broad frequency spectrum. Recently, the Minimal Gain Marching (MGM) scheme was developed as an alternative method to generate steady-states solutions for any unstable problem (de S. Teixeira and de B. Alves, 2013; Teixeira, 2014; de S. Teixeira and de B. Alves, 2014). This new class of schemes is based on the fundamental of numerical analysis, where the convergence of given dynamical system will occur if the numerical analysis gain of the marching scheme is sufficiently small. Therefore, the coefficients of any time marching scheme can be modified to achieve the optimal minimal gain for the convergence of unstable conditions. Nevertheless, the method requires an implicit solve of the system. The current paper presents an extension procedure of the MGM scheme (Teixeira, 2014). The dual-time approach developed by Merkle and co-workers (Venkateswaran and Merkle, 2000; Buelow *et al.*, 2001) are employed to extend the MGM for explicit schemes. It employs dual-time-stepping (Zeng *et al.*, 2003) to replace the original marching scheme by the implicit Euler method, removing the need to calculate implicit Jacobians and use special solvers for implicit matrix. Different explicit time marching schemes with dual-time MGM are tested for Lorenz equation case and a comparison with SFD techinique is investigated.

2. MATHEMATICAL MODEL

2.1 Minimal Gain Marching Scheme with Dual-Time Step

Numerical methods have been the main investigation source of partial differential equation (PDE) solutions. Finite-difference method were applied in present research (Tannehill *et al.*, 1997). The mathematical model present here is based on treatment of the temporal derivatives of a system of PDEs like as

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{f}(\mathbf{Q}) \tag{1}$$

where $\mathbf{Q}$ is the variable vector.

The steady-state solver consist in establish temporal numerical scheme as source term that provides minimal gain. Thus, it is easy to implement the methodology in any arbitrary code using a Dual-Time Step method (DTS), such as equation below

$$\frac{\partial \mathbf{Q}}{\partial \tau} = \mathbf{f}(\mathbf{Q}) - \frac{\partial \mathbf{Q}}{\partial \mathbf{t}} \tag{2}$$

where it has two temporal derivatives. This procedure was developed by Merkle and Choi (1988). The pseudo-time derivative ($\partial \mathbf{Q}/\partial \tau$) is measured with any numerical schemes. The temporal source term is calculated with steady-state solver. Once the steady-state solution is obtained in each physical time step the original government Eq. (1) is recovered, since $\partial \hat{\mathbf{Q}}/\partial \tau \simeq 0$ in this limit.

The temporal derivative were discretized with Minimal-Gain Marching scheme methodology (de S. Teixeira and de B. Alves, 2014). Hence, the temporal derivative ($\partial \mathbf{Q}/\partial t$) of Eq. (2) is written as

$$\frac{\partial \mathbf{Q}}{\partial t} = \theta_1 \frac{\mathbf{Q}^{n+1} - \mathbf{Q}^{n}}{\Delta t} + (1-\theta_1)\frac{\mathbf{Q}^{n+1} - \mathbf{Q}^{n-1}}{2\Delta t}, \tag{3}$$

where θ_1 is the dissipative control parameter, resulting in the following equation

$$\frac{\partial \mathbf{Q}}{\partial t} = \frac{(1+\theta)\mathbf{Q}^{n+1} - 2\theta \mathbf{Q}^{n} - (1-\theta)\mathbf{Q}^{n-1}}{2\Delta t}. \tag{4}$$

In the rigth-hand side of Eq. (1) we applied the generalized Crank-Nicholson method (Anderson, 2000) with the a control parameter θ_2 can be observed in Eq. (5)

$$\mathbf{f}(\mathbf{Q}) = \theta_2 \mathbf{f}(\mathbf{Q}^{n+1}) + (1-\theta_2)\mathbf{f}(\mathbf{Q}^{n}). \tag{5}$$

Furthermore, temporal methodology consist in choose a right combination parameters between θ_1 and θ_2 that provides a steady-state solution. Hence, the scheme proposed is

$$\frac{(1+\theta_1)\mathbf{Q}^{n+1} - 2\theta_1 \mathbf{Q}^{n} - (1-\theta_1)\mathbf{Q}^{n-1}}{2\Delta t} = \theta_2 f(\mathbf{Q}^{n+1}) + (1-\theta_2)\mathbf{f}(\mathbf{Q}^{n}), \tag{6}$$

where setting $\theta_1 = 1$ is obtained a implicit scheme or configuring $\theta_2 = 0$ the explicit method is generated.

Now we present the Eq. (2) with steady-state methodology in temporal source term. Initially, the implicit scheme is applied in physical-time derivative. The Eq. (2) becomes

$$\frac{\partial \mathbf{Q}}{\partial \tau} = \theta_2 \mathbf{f}(\mathbf{Q}^{p}) + (1-\theta_2)\mathbf{f}(\mathbf{Q}^{n}) - \frac{(1+\theta_1)\mathbf{Q}^{p} - 2\theta_1 \mathbf{Q}^{n} - (1-\theta_1)\mathbf{Q}^{n-1}}{2\Delta t} \tag{7}$$

where the original explicit scheme is employed in pseudo-time derivative. This methodology can be used adding a temporal derivative source term in governing equations with MGM discretization while the original marching scheme becomes a pseudo-time.

2.2 Selective Frequency Damping Method

Usually, globally unstable flows poses several challenges to provide a steady-state solution for Navier-Stokes equations. Selective Frequency Damping (SFD) was developed as an alternative method to generate these solutions (Åkervik *et al.*, 2006). This technique consists to add a source term in governing equations (1) as

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{f}(\mathbf{Q}) - \chi(\mathbf{Q} - \bar{\mathbf{Q}}) \tag{8}$$

where χ is the coefficient control. This procedure results in a converged solution $\bar{\mathbf{Q}}$ time-filtered by a non-linear equation.

For a continues function vector $\mathbf{Q}$, a causal low-pass time filter is defined as

$$\bar{\mathbf{Q}}(t) = \int_{-\infty}^{t} T(\tau - t; \Delta)\mathbf{Q}(\tau)d\tau, \tag{9}$$

where $\bar{\mathbf{Q}}$ is the temporally filtered quantity, T is the parameterized filter kernel, and Δ is its associated filter width. The integral formulation (9) is impractical for engineering problems, since it requires the storage of the complete time history of the signal $\mathbf{Q}$. Consequently, equivalent differential form is adopted,

$$\frac{\partial \bar{\mathbf{Q}}}{\partial t} = (\mathbf{Q} - \bar{\mathbf{Q}})/\Delta. \tag{10}$$

The filter coefficients have to be chosen to sufficiently damp the lowest unstable frequencies; information about these frequencies can straightforwardly be extracted from the simulations (Bagheri *et al.*, 2009b).

2.3 Test Case

A system of ordinary differential equations (ODE) will be employed to certificate the method behavior. This problem was implemented to improve the readers comprehension. On the other hand, the ordinary differential equation Eq. (11) represent a good nonlinear problem to test our methodology, due to their chaotic characteristic. This system of equations was first formulated in 1963 by E. N. Lorenz and possesses what has come to known as a “strange attractor"(Hirsch *et al.*, 2004). The Lorenz equation is defined as

$$\begin{cases} \dfrac{dx}{dt} & = & \sigma\,(y - x) \\ \dfrac{dy}{dt} & = & \rho x - y - xz \\ \dfrac{dz}{dt} & = & -\beta z + xy \end{cases} \tag{11}$$

where x, y and z are the independent variables, σ is the Prandtl number, ρ is the Rayleigh number and β is related to the physical size of the system.

For showing the Lorenz solution the Fig. 1 is presented with different time step Δt. Figure 1 (right) show the difference between Lorenz solution with more precision than sixteenth digits and numerical solutions for temporal step (Δt) variation. The verification has a good agreement with deterministic chaos theory.

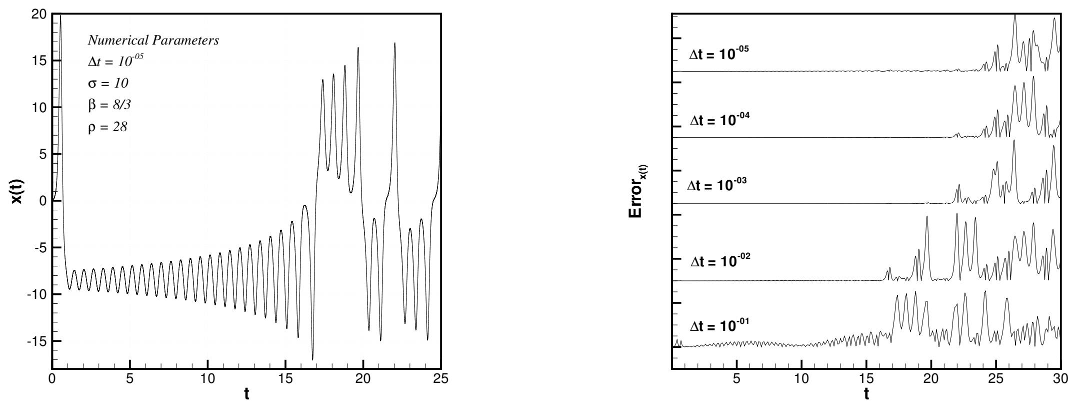

Figura 1: Numerical Solution of Lorenz problem for different time step Δt.

3. RESULTS AND DISCUSSIONS

Firstly, the current presents numerical verifications for different schemes employed. The Figure 3. shows the error behave for studied explicit method. Each line color is associated to numerical scheme. The methods are Explicit Euler (EE), second-order Adams-Bashforth (AB2), second and forth-order explicit Runge-Kutta (ERK2 and ERK4). The solid black lines represent the error slope for each precision order. As can be seen, the curves slope is consistent with numerical precision. Then, the original explicit marching code was adapted with MGM scheme coupled dual-time marching. Therefore, the explicit scheme became a pseudo-time which MGM scheme were employed as physical-time marching. Figure 3 presents the numerical verification for different parameters combination of MGM approach. Furthermore, solid black lines are the order precision slope. Hence, for $\theta_1 = 1.0$ & $\theta_2 = 0.0$ is first-order explicit Euler and $\theta_1 = 1.0$ & $\theta_2 = 1.0$ is implicit Euler. As can be observed, the error behaviour is according with numerical truncation error analysis. The Crank-Nicolson scheme is studied for $\theta_1 = 1.0$ & $\theta_2 = 0.5$ and numerical verification is consistent. An investigation

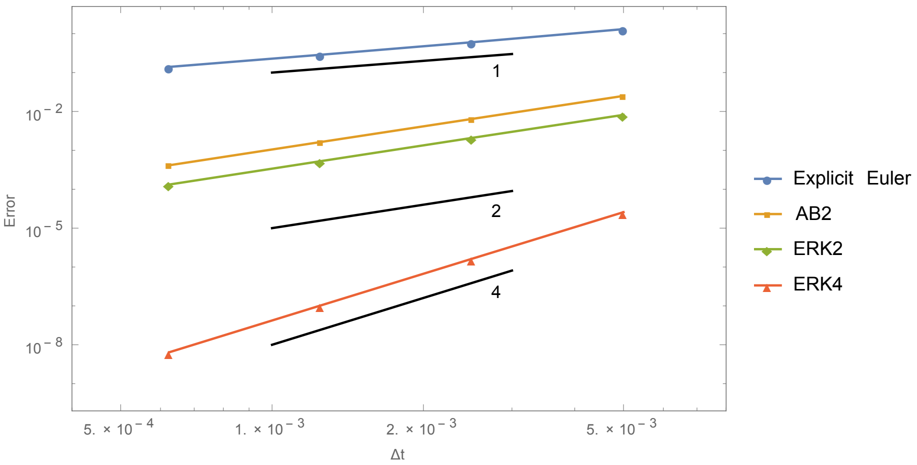

Figura 2: Numerical verification for explicit schemes.

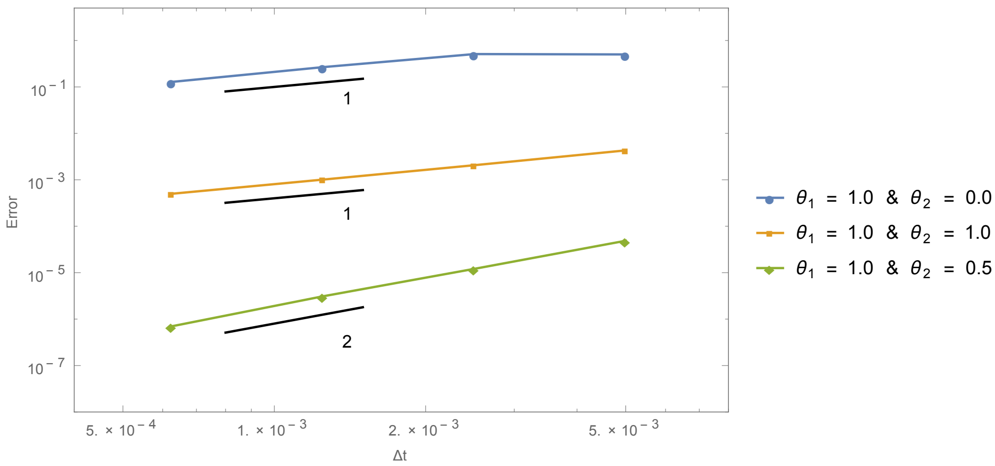

Figura 3: Numerical verification for dual-time step with MGM schemes.

about SFD with different temporal explicit scheme have been studied. The Figure 4 shows the behavior of the technique applied in explicit Euler, second-order Adams-Bashforth, second-order and forth-order Runge-Kutta temporal schemes. Each graph is composed by total iterations number for convergence towards steady-state solution with χ control parameter for different filter width Δ. As can be seen, the temporal discretization method has low influence on the convergence behavior. The number of iterations for convergence for all pseudo-time explicit schemes was about 500 iterations and the optimal parameter are approximately equal for all cases about $\chi \simeq 4.8$ and $\Delta \simeq 0.2$. As reveled in figure 4, the precision order of each temporal scheme has low impact for optimal parameters.

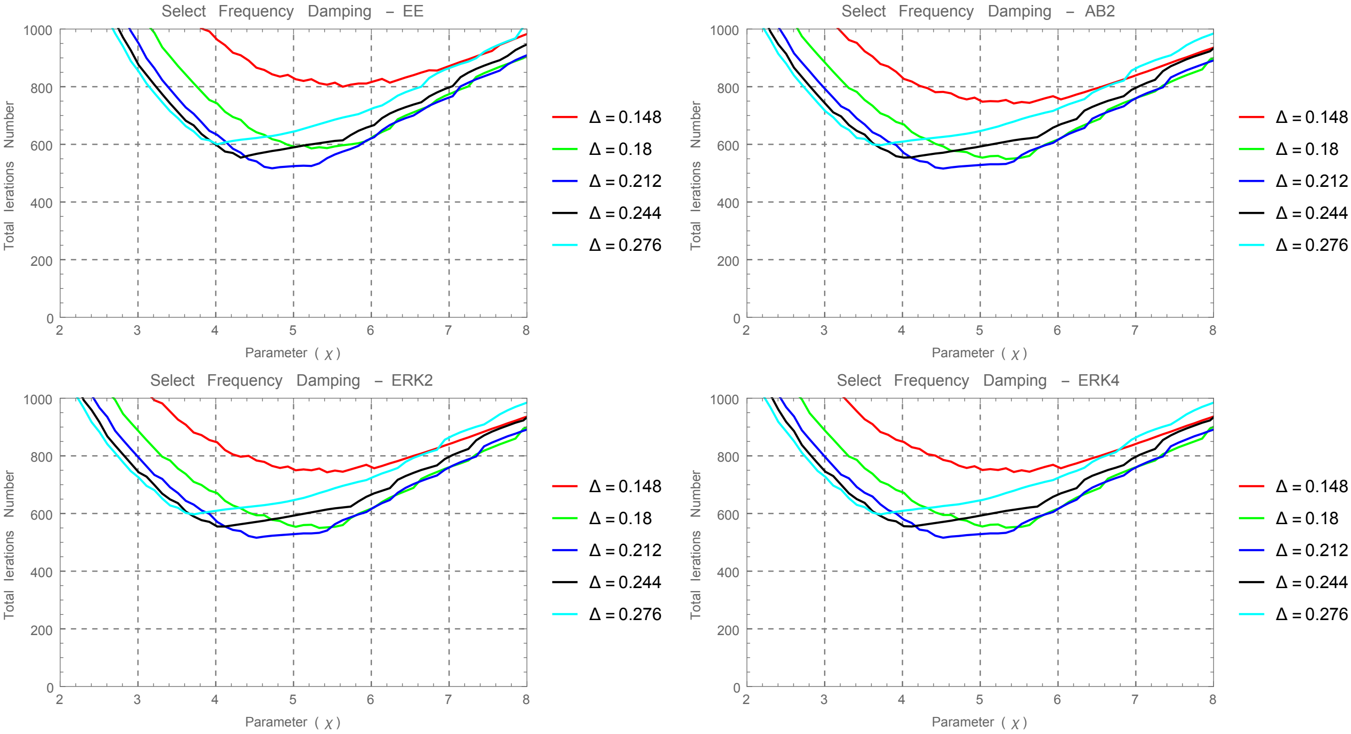

Figura 4: Comparison between SFD with different temporal schemes.

On the other hand, the above study was employed for minimal gain marching scheme with dual-time ste methodology. The figure 5 shows the total iterations number by physical-time step (Δt) variation for some combinations between θ_1 and θ_2 parameters form minimal gain marching scheme. Each graph was generated with different temporal numerical scheme employed in pseudo-time derivative, i.e., the original explicit temporal discretization became the pseudo-time and minimal gaing marching scheme was applied in the added temporal source term. As can be observed, the DTS-MGM with explicit multi-stage Runge-Kutta schemes in pseudo-time presents the optimal convergence with approximatelly 580 total iterations number for convergence towards steady-state solution. The multi-step second-order Adams-Bashforth and explicit Euler schemes was twice expensive cost. Finally, the optimal convegence case for DTS-MGM with ERK4 is 16% more expensive than SFD scheme.

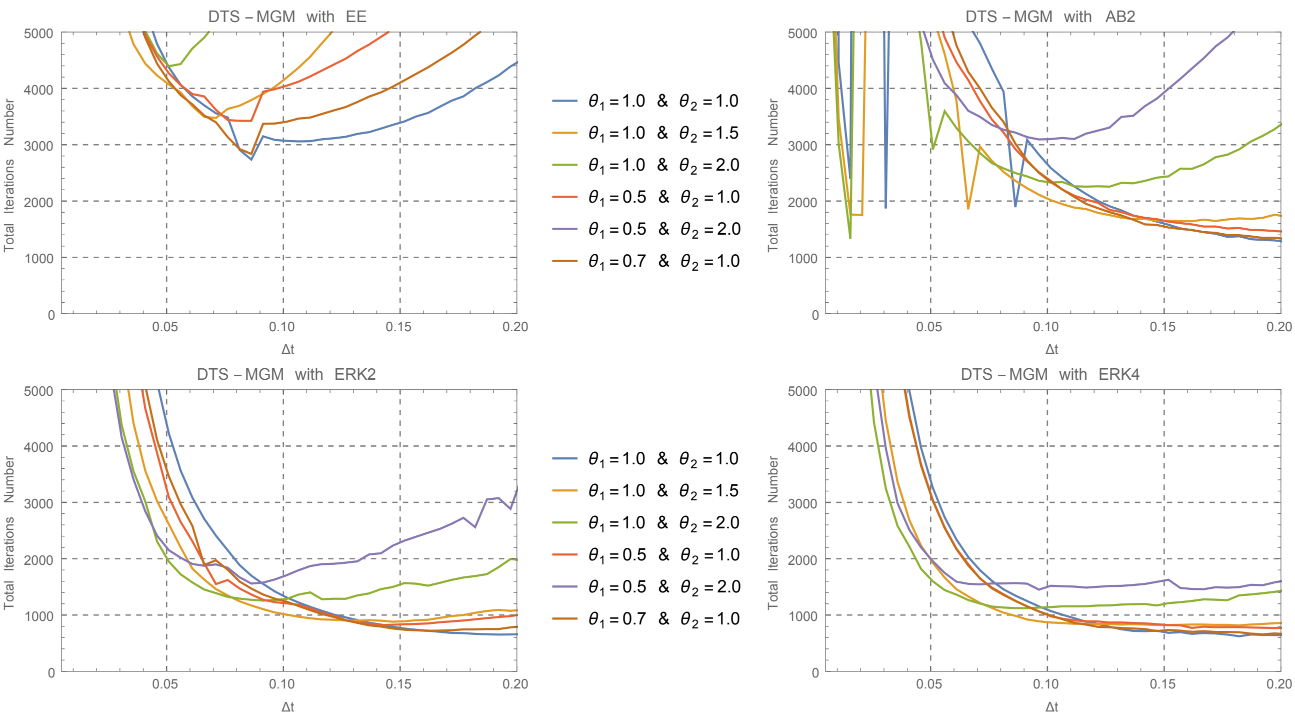

Figura 5: Comparison between MGM-DTS with different pseudo-time schemes.

The figure 6 presents a comparison between optimal convergence for MGM scheme and SFD technique. As can be seen, the DTS-MGM has approximately the same cost of SFD technique. This result show that DTS-MGM can be an alternative technique to use minimal gain marching scheme for explicit temporal codes.

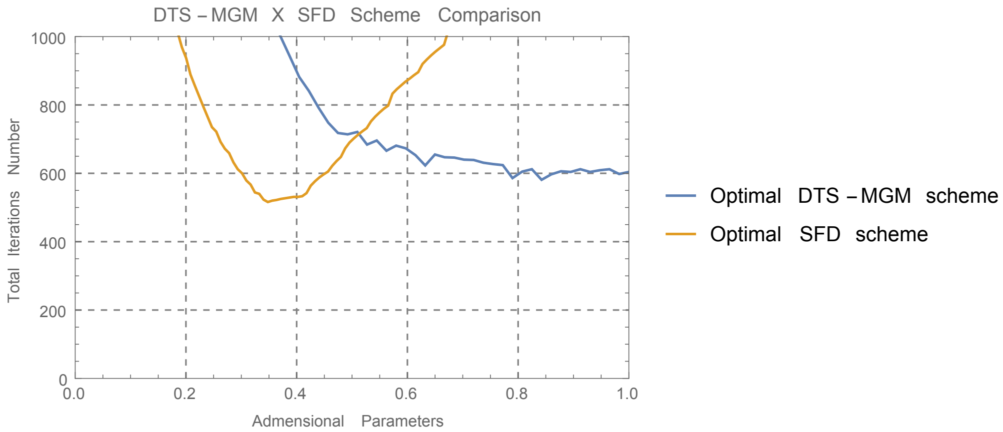

Figura 6: Comparison between MGM-DTS with different pseudo-time schemes.

4. CONCLUSION

The development of reference solutions for unstable flows has an important role for numerical problems. This research presents an investigation about dual-time step with Minimal Gain Marching scheme to provide steady-state solutions for unstable problems. The main objective was to evaluate the cost of this method compared with Selective Frequency Damping.

Firstly, the numerical schemes was verified. Furthermore, the chaotic behavior of Lorenz problem was shown. The study of SFD method shows the low impact of temporal scheme to accelerate convergence. On the other hand, the MGM scheme has a large impact of numerical scheme employed in original physical-time, i. e., the pseudo-time scheme has an important role in the process of convergence for MGM method. However, the MGM method is just 16% more expensive than SFD technique. Thus, the Minimal Gain Marching scheme possibly is a good alternative methodology for explicit temporal codes.

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