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DESIGN OPTIMIZATION OF A STOCHASTIC DYNAMIC PLATE BASED ON FATIGUE LIFE

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Abstract. *This work presents an optimization strategy to increase and evaluate fatigue life of a dynamic structure subjected to uncertainties in its geometry and loads. The main objective is to reduce thickness of an aluminum plate, which is directly in conflict with enhancing its durability when subjected to cyclic loads. Solving this multiobjective problem requires the application of metaheuristic methods, searching not only one point in the search space, but a whole group of solutions where all are treated as optimal. The chosen technique which was applied in the numerical simulations is the NSGA (Non-Dominated Sorting Genetic Algorithm). As the system is stochastic, this approach presents, in general, high sensitivity to small changes or fluctuations in the variables of interest. This can abruptly affect the objective functions in the region close to a global optimum. A robust method is applied herein and is compared with previous results, highlighting that, despite the robust optimum may not always be the global optimum, structural randomness makes this type of analysis necessary to ensure reliability. Numerical results are presented in terms of FRF, stress responses in the frequency domain and Sines fatigue index for each finite element composing the plate.*

Keywords: *stochastic finite elements method, fatigue, robust optimization, random vibrations*

1. INTRODUCTION

The development of computational methods for engineering structural design must always aim for fidelity to what happens in real applications. One way to obtain more accurate approximations for a model is to include, in its calculations, the effects caused by aleatory variables and uncertainties that may be present in geometry, material properties and loads, propagating through computational models affecting its responses and sensibility (Stefanou, 2009). Nevertheless, the classical theory presented in literature does not consider the uncertainties inherent in the structure.

Because it is a relatively new research area, it is believed that uncertainty evaluation can provide an even greater gain than classical deterministic approaches in terms of durability, reliability and safety of mechanical components. In addition, practical engineering applications usually require machines and equipment to operate under conditions where they are constantly subjected to dynamic disturbances and cyclic loads which can lead to undesirable levels of vibrations and noise and, therefore, cause fatigue cracks, leading to definitive fractures or total collapse of the structure.

In the context of fatigue analysis of dynamic systems, Lambert (2007) emphasizes that fatigue cracks usually appear at critical areas such as geometry changes, load application points and boundary conditions. Classic prediction methods involve statistical data of loads and stresses in the time domain. This approach is applicable, however, it presents a high computational cost, requiring great acquisition and post-treatment time. Besides that, external vibrations may excite natural frequencies, drastically increasing stress levels. Therefore, a frequency domain technique is adopted.

In literature, there are some ways of inserting uncertainties into a finite element model. The one applied herein is the method of expansion by Karhunen-Loève (Ghanem and Spanos, 2003), which perform its parametric inclusion through the development and modification of mass stochastic matrices and stiffness (De Lima *et al.*, 2010). The aleatory effects inserted via this method are not only present in random variables but also affect the integration process of the elementary mass and stiffness matrices, allowing its propagation throughout the entire structural domain (Rosa and de Lima, 2016).

The design of a dynamic system usually runs into an optimization problem, where two or more conflicting characteristics need to be analyzed, design constraints must be established and a final compromise adopted (Rao, 2009). The optimization problem associated with the analysis of a stochastic structure whose objective is the study of fatigue phenomena can be treated as a multiobjective problem, since the desired global solution, be it a maximum or a minimum, is not obtained for a single point in the search space but for a set of optimal solutions. To solve problems of this type, what is recommended in the literature is to use heuristic methods or the so-called metaheuristic methods (Lobato *et al.*, 2008).

2. STOCHASTIC FINITE ELEMENT FORMULATION FOR FATIGUE ANALYSIS

2.1 Stress Response in the Frequency Domain

Theory described herein follow the hypothesis proposed by Kirchhoff (Zienkiewicz and Taylor, 2005), so the system is considered as in a plane stress state. The transverse displacement is then independent of the coordinate along the thickness, meaning zero normal strain in this direction ($\varepsilon_z = 0$). Also both shear strains are zero ($\gamma_{xz} = 0, \gamma_{yz} = 0$). The adopted thin plate finite element is rectangular and composed by four nodes and five degrees of freedom per node, being two in-plane displacements (u_0, v_0), one transverse displacement (w_0) and two cross-section rotations (θ_x, θ_y), as shown in Fig. 1.

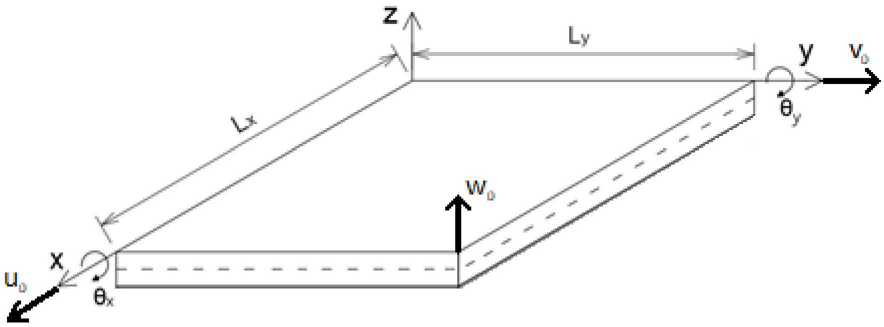

Figure 1: Four-nodes rectangular finite element

The notation adopted for displacement fields is shown in Eq. (1).

$$\mathbf{U}(x,y,z,t) = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} = \begin{Bmatrix} u_0 + z\theta_x \\ v_0 + z\theta_y \\ w_0 \end{Bmatrix} \tag{1}$$

As one can note, elements on the right side of Eq. (1) represent the structure's degrees of freedom, being $\theta_x = \partial w_0/\partial x$ and $\theta_y = \partial w_0/\partial y$. In finite elements method formulation the continuous system is approximated by shape functions (Reddy, 1997), as shown in Eq. (2) .

$$\mathbf{U}(x,y,z,t) = \mathbf{A}(z)\mathbf{N}(x,y)\mathbf{u_e}(t) \tag{2}$$

where $\mathbf{A}(z)$ contains the parameter z shown in Eq. (1), $\mathbf{N}(x,y)$ are the shape functions and $\mathbf{u_e}(t)$ is the nodal displacements vector.

Shape functions are also used in the calculation of mass and stiffness matrices of the system. These matrices are square and symmetric and their dimension is the total number of degrees of freedom. They are classically calculated for each element with elementary volume V_e based on energy variational calculus. After, they are expanded for the global system by means of node connectivity (Moaveni, 1999). The following equations show how they are formulated:

$$\mathbf{M}^{(e)} = \int_{V_e} \rho \mathbf{N}(x,y)^T \mathbf{N}(x,y) dV_e \tag{3}$$

$$\mathbf{K}^{(e)} = \frac{1}{2}\int_{V_e} \mathbf{N}'(x,y)^T \mathbf{H} \mathbf{N}'(x,y) dV_e \tag{4}$$

where ρ is the mass density, $\mathbf{N}'(x,y)$ is the space derivative of shape functions relative to (x,y) and $\mathbf{H}$ is the elasticity tensor.

The system's movement equation in the time domain is written as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t) \tag{5}$$

where $\mathbf{M} = \bigcup\limits_{Ndof} \mathbf{M}^{(e)}$, $\mathbf{K} = \bigcup\limits_{Ndof} \mathbf{K}^{(e)}$, $\mathbf{C} = \beta\mathbf{K}$ represents a stiffness-proportional structural damping, $\mathbf{u}(t)$ is the displacement vector and $\mathbf{f}(t)$ is the external forces vector.

Considering a harmonic external excitation of the form $\mathbf{f}(t) = \mathbf{F}e^{j\omega t}$ that also generates a harmonic response $\mathbf{u}(t) = \mathbf{U}e^{j\omega t}$,it is possible to calculate the displacement frequency response function (FRF) by Eq. (6):

$$\mathbf{G}(\omega) = \left|\frac{\mathbf{U}}{\mathbf{F}}\right| = \frac{1}{[\mathbf{K} + j\omega\mathbf{C} - \omega^2\mathbf{M}]} \tag{6}$$

Performing fatigue analysis in the time domain requires a large stress data historic (Budynas and Nisbett, 2011). This method is applicable, but the calculation of convolution integrals for numeric analysis generates a high computational cost and it is even more critical when random loads are present. As an alternative, it is desirable to obtain stress responses in the frequency domain via power spectral density (PSD) (Lambert, 2007). Bendat and Piersol (2010) present a complete background for its calculation, starting from Fourier Transforms and involving several statistical concepts of data processing. Finally, the displacement PSD is formulated as follows:

$$\boldsymbol{\phi}_u(\omega) = \mathbf{G}(\omega)\phi_f(\omega)\mathbf{G}^H \tag{7}$$

where $\phi_u(\omega)$ is the displacament PSD matrix and $\phi_f(\omega)$ is the external forces PSD.
Applying Hooke's Law to Eq. (7) it is possible to calculate the stress response, as show in Eq. (8).

$$\boldsymbol{\phi}_S(\omega) = \mathbf{HBG}(\omega)\phi_f(\omega)\mathbf{G}^H(\omega)\mathbf{B}^T\mathbf{H}^T \tag{8}$$

where $\boldsymbol{\phi}_S$ is the stress PSD matrix and $\mathbf{B}$ is the derivative therms matrix.

2.2 Stochastic Finite Elements Method Via Karhunen-Loève Expansion

In general, uncertainties are included in computational models following non-parametric or parametric approaches. The first consists in introducing random variables directly in the model's global matrices (Soize, 2000; Ritto *et al.*, 2008). The second is mainly represented by the Stochastic Finite Elements Method (SFEM), which allows a combination of the classic FEM and statistic analysis (Schuëller, 2001; Ghanem and Spanos, 2003). In this approach, the inclusion of random effects is extended to the integration process of elementary matrices, generating Stochastic Matrices. The solution of the equation movement holds and propagates uncertainties along the simulations, affecting the responses.

A random field $X(x,\theta)$ is a group of random variables composed by a set of continuous parameters x representing the structure's physical geometry and θ representing variables belonging to the space of aleatory events. Adding another spatial variable y, the random field for the bidimensional case of a plate is rewritten as $X(x,y,\theta)$. The Karhunen-Loève Expansion (KL) is a random field discretization method which is part of the called *Series Expansion Mehotds* (Ghanem and Spanos, 2003). By applying this method, the random field $X(x,y,\theta)$ is approximated by $\hat{X}(x,y,\theta)$ as a Fourier-type series truncated at n therms:

$$X(x,y,\theta) \approx \hat{X}(x,y,\theta) = \mu(x,y) + \sum_{r=1}^{n} \sqrt{\lambda_r} f_r(x,y)\xi_r(\theta) \tag{9}$$

where $\mu(x,y)$ is the expected value for the field, $\{\xi_r, r = 1, ..., n\}$ is a set of random variables, λ_r is a constant and $f_r(x,y)$ is an orthonormal set of deterministic functions. $(\lambda_r, f_r(x,y))$ are obtained by the solution of a eigenproblem associated to the covariance function. Ghanem and Spanos (2003) present the complete procedure fot obtaining the eigenvalues and eigenfunctions by solving an integral equation. This methodology was also previously applied by the autors (De Lima *et al.*, 2010; Rosa and de Lima, 2016).

After determining KL method's parameters it is possible to calculate elementary stochastic matrices $\mathbf{M}^e(\theta)$ and $\mathbf{K}^e(\theta)$, composed by random variables θ:

$$\mathbf{M}^{(e)}(\theta) = \mathbf{M}^{(e)} + \sum_{r=1}^{n} \Delta\mathbf{M}_r^{(e)}\xi_r(\theta) \tag{10}$$

$$\mathbf{K}^{(e)}(\theta) = \mathbf{K}^{(e)} + \sum_{r=1}^{n} \Delta\mathbf{K}_r^{(e)}\xi_r(\theta) \tag{11}$$

where the part inside the sum is calculated by:

$$\Delta\mathbf{M}_r^{(e)} = \sqrt{\lambda_r}\int_{\Omega_x}\int_{\Omega_y} f_r(x,y)\rho\mathbf{N}(x,y)^T\mathbf{N}(x,y)d\Omega_y d\Omega_x \tag{12}$$

$$\Delta\mathbf{K}_r^{(e)} = \sqrt{\lambda_r}\int_{\Omega_x}\int_{\Omega_y} f_r(x,y)\mathbf{N}'(x,y)^T\mathbf{H}\mathbf{N}'(x,y)d\Omega_y d\Omega_x \tag{13}$$

After these changes, global stochastic matrices are concatenated in the same way as shown in section 2.1. Stochastic FRF and PSD equations, previously shown in Eq. (6) and Eq. (8) become:

$$\mathbf{G}(\omega,\theta) = \frac{1}{[\mathbf{K}(\theta) + j\omega\mathbf{C}(\theta) - \omega^2\mathbf{M}(\theta)]} \tag{14}$$

$$\phi_S(\omega,\theta) = \mathbf{HBG}(\omega,\theta)\phi_f(\omega,\theta)\mathbf{G}^H(\omega,\theta)\mathbf{B}^T\mathbf{H}^T \tag{15}$$

2.3 Sines' Criterion for Fatigue Analysis

The fatigue criterion proposed by Sines (1959) is highly suitable for the analysis performed in this study. As shown by Weber (1999), its formulation is simpler than other methods and requires only two material properties related to multiaxial fatigue: alternate torsinal (t_{-1}) and alternate traction (f_{-1}) fatigue resistances. It is based on the calculation of a coefficient D_{Sines} that must be lower than 1 to indicate that no crack initiation will occur in the structure. One may note that the formulation for random loadings presented in Eq. (16) imply in the calculation of a mathematical hope instead of direct evaluation.

$$D_{Sines} = \frac{E[\sqrt{J_{2,a}}] + \alpha E[p_{hid}(t)]}{t_{-1}} \tag{16}$$

$$\alpha = \frac{3t_{-1}(R_m + f_{-1})}{f_{-1}R_m} \tag{17}$$

where $\sqrt{J_{2,a}}$ is the square root of the stress deviatoric tensor second invariant, $p_{hid}(t)$ is the hydrostatic stress and R_m is the ultimate material strength.

From the stress PSD calculated by Eq. (15) it is possible to extract spectral moments (Bendat and Piersol, 2010) that are used to calculate $\sqrt{J_{2,a}}$. The whole procedure originally proposed by Li e de Fritas (2002), was later expanded for multiaxial and out of phase loads by Khalij *et al.* (2010) and applied in more recent works (Lambert *et al.*, 2010; Rosa, 2016; Rosa and de Lima, 2016).

3. ROBUST OPTIMIZATION PROCEDURE

The optimization problem associated with a stochastic structure whose objective is fatigue analysis is treated as a multiobjective problem, because the desired global solution, be it a maximum or a minimum, is not obtained directly for a unique point in the search space but for a group of optimal solutions, called Pareto front (Eschenauer *et al.*, 2012). This means that there is no single solution that can optimize both objectives at the same time. To solve this type of problem, heuristic or metaheuristic methods are recommended (Lobato *et al.*, 2008). A multiobjective optimization problem is formulated as follows:

$$\begin{cases} \min\limits_x \quad F(x) = (f_1(x), f_2(x), ..., f_n(x)) \\ \quad g_j(x) \le 0 \quad j = 1, ..., m) \\ \quad x_L \le x \le x_U \quad x \in C \end{cases} \tag{18}$$

where $F(x)$ is the vector of objective functions to be optimized, $n \geq 2$ is the number of objective functions, $x = (x_1, x_2, ..., x_k)$ is the project variables vector, $C \subset \mathbb{R}^k$ is the project space associated to the restrictions defined by $g_j(x)$ and its lower (x_L) and upper (x_U) boundaries.

Due to the usual high sensibility of the stochastic problem and function discontinuities, slight parameters fluctuation may affect abruptly the regions close to a global optimum (Moreira *et al.*, 2015). Thus, the applied optimization routine should have the lowest sensibility as possible, requiring robustness. The designer must also by aware that the chosen optimum may not always be the multiobjective global optimum. Figure 2 illustrate two different optimal solutions. Solution A is called deterministic optimum and solution B is called robust optimum (Lee and Park, 2001). It is noted that a slight perturbation Δx_1 around point A results in a greater variation of $f(x_1)$ than around point B.

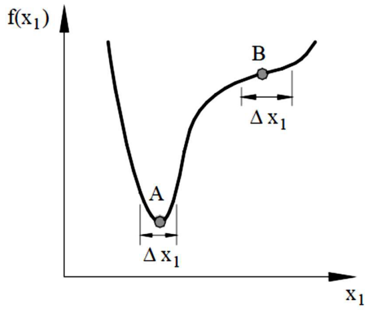

Figure 2: Representation of optimal solutions (adapted from Lee and Park (2001))

Robust optimization is a strong tool that allows to obtain less sensitive solutions to small design variations when project and cost function are subjected to uncertainties. Thus, it is possible to obtain a solution that despite being considered "sub-optimal", is stable concerning parametric uncertainties.

A robustness function evaluates the impacts of parameters variation. Robust optimal solutions are those that simultaneously optimize the initial cost functions and at the same time maximize its robustness or minimize its vulnerability. The classic form of a robust multiobjective optimization problem is presented in Eq. (19).

$$\begin{cases} \min_{x} \quad F(x) = (f_1(x), f_1^v(x), f_2(x), f_2^v(x), ..., f_n(x), f_n^v(x)) \\ \qquad\qquad g_j(x) \leq 0 \quad j = 1, ..., m) \\ \qquad\qquad x_L \leq x \leq x_U \quad x \in C \end{cases} \tag{19}$$

where $f_i^v(x)$ is the vulnerability function of the objective $f_i(x)$, calculated by the inverse of the dispersion:

$$f^v = (\sigma_f / \mu_f)^{-1} \tag{20}$$

In the study described herein the multiobjective optimization problem is solved by using the NSGA (Non-Dominated Sorting Genetic Algorithm) by a MatLab^{TM}toolbox (Lobato *et al.*, 2015). The main goal is to minimize the structure thickness and consequently its mass and to maximize fatigue life.

4. NUMERICAL APPLICATION

Numerical simulations were performed over a thin rectangular aluminum plate with mechanical properties given in Tab. 1. Its dimensions are $654mm \times 527mm$, thickness is $1.6mm$, double-clamped along y axes, load applied and measured at the central node. It was discretized in 8×8 rectangular four-nodes finite elements with 5 degrees of freedom per node, as shown in section 2.1

Table 1: Mechanical properties of the thin plate.

Young's Module (E)	Poisson Ratio (ν)	Mass Density (ρ)	Ultimate Strength Limit (R_m)	Alternate Torsional Limit$^{(1)}$ (t_{-1})	Alternate Traction Limit$^{(1)}$ (f_{-1})
[GPa]		[kg/m^3]	[MPa]	[MPa]	[MPa]
70	0.33	2700	343	92	132

(1) obtained for $2.0 \times 10^6 cycles$

In the present study thickness is the interest random variable. A parametrization was performed in mass and stiffness matrices, allowing the inclusion of stochasticity in the system:

$$\mathbf{K}(\theta) = Eh\mathbf{K}_m(\theta) + Eh^3\mathbf{K}_b(\theta) \tag{21}$$

$$\mathbf{M}(\theta) = \rho h\mathbf{M}_m(\theta) + \rho h\mathbf{M}_b(\theta) \tag{22}$$

where E is the Young modulus and subscripts m and b designate membrane and bending effects.

The first simulation regarded the thin plate subjected to a 10% uncertainty level over thickness nominal value. As the mean value is known $(1.6mm)$ and standard deviation is defined by this uncertainty level, a Gaussian density function was chosen. Furthermore, Ghanem and Spanos (2003) mentioned that KL method converges using variables with this distribution. Uncertainties in the equivalent endurance limit and mass density will not be covered because this analysis was already been performed by the authors in previous studies (Rosa, 2016).

Figure 3 depicts a convergence analysis performed via root-mean-square deviation (RMSD) of FRF, normalized by its mean. This analysis is also used to estimate an optimal number of samples n_s. Equation (23) shows how it is calculated. A satisfactory convergence is noted around $n_s = 300$ samples.

$$RMSD(n_s) = \sqrt{\frac{1}{n_s}\sum_{i=1}^{n_s} |\mathbf{G}_i(\omega,\theta) - \overline{\mathbf{G}}(\omega,\theta)|^2} \tag{23}$$

Figure 4 shows the FRF envelope of the displacement due to a unitary load applied and measured on the central node, calculated by Eq. (14). As one may note, the deterministic FRF is inside the envelope, showing that results are coherent. The deterministic calculation of Sines' fatigue index provides a critical value of $E[D_{Sines}] = 0.9484$. It is less than 1 meaning that the plate does not fail in this case. Nevertheless, when subjected to stochasticity the maximum value is $E[D_{Sines}] = 1.1668$, indicating crack nucleation before 2 million cycles, as shown in Fig. 5.

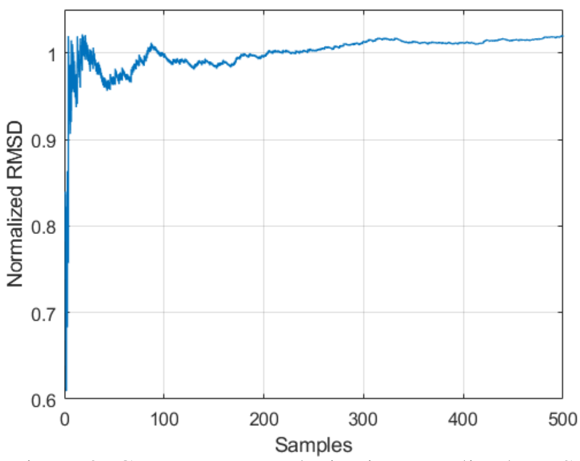

Figure 3: Convergence analysis via normalized RMSD

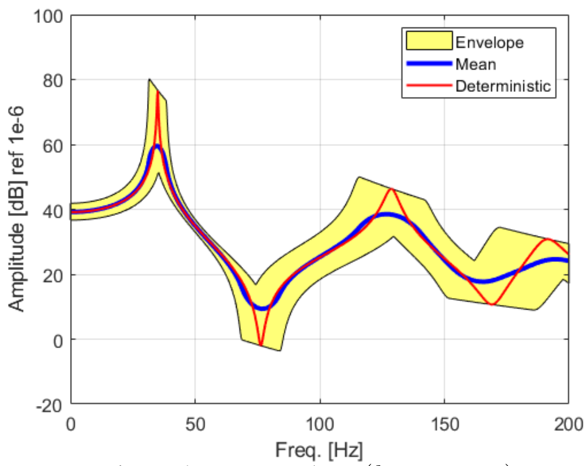

Figure 4: FRF envelope $(h = 1.6mm)$

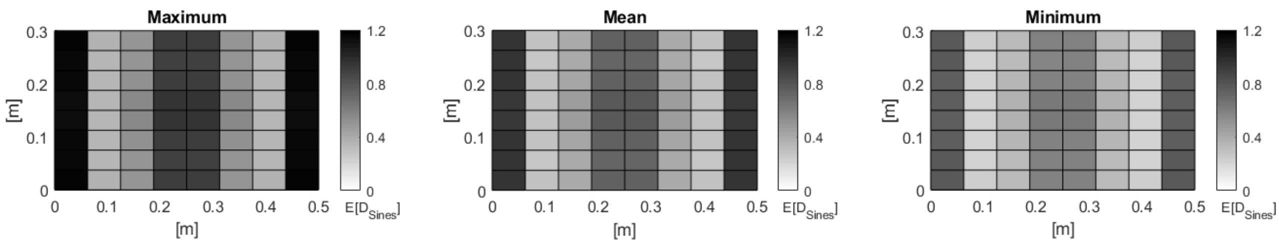

Figure 5: Sines coefficient distribution $(h = 1.6mm)$

After presenting the original results for a plate with fatigue failure occuring before 2 million cycles, the optimization procedure is applied. The simulation consisted in minimizing critical Sines' index values and plate thickness (and consequently its mass) as well as vulnerability functions, defined in Eq. (20). With two objective functions, the trade-off curve can be visualized in two dimensions. Figure 6 presents Pareto front obtained for a robust optimization routine.

All Pareto optimal solutions presented in the fronts are considered equally good. The choice becomes subjective and it is up to the designer to analyze the structure, the environment where it will work and other restrictions and decide which optimal solution will be adopted. In this study, the chosen thickness must not provide a fatigue index above 1. It is noted in Fig. 6 that a thickness $h = 1.8mm$ is below the limit, so it was chosen as the optimal value.

The deterministic value for the critical element's fatigue index is $E[D_{Sines}] = 0.7457$. A simulation similar to the previous one is performed, with the thin plate subjected to a 10% uncertainty level over thickness nominal value $h = 1.8mm$.

Figure 7 shows the convergence analysis for this case. Again, an excellent convergence is noted above $n_s = 300$. Comparing Fig. 8 with Fig. 4 one can easily notice that FRF levels are lower in both envelopes and deterministic curves. Fatigue index is directly proportional to stress response and FRF amplitude, so values shown in Fig. 9 are lower than those in Fig. 5. The higher critical value obtained in the stochastic simulation is $E[D_{Sines}] = 0.9248$, meaning that the structure won't fail before 2 million cycles.

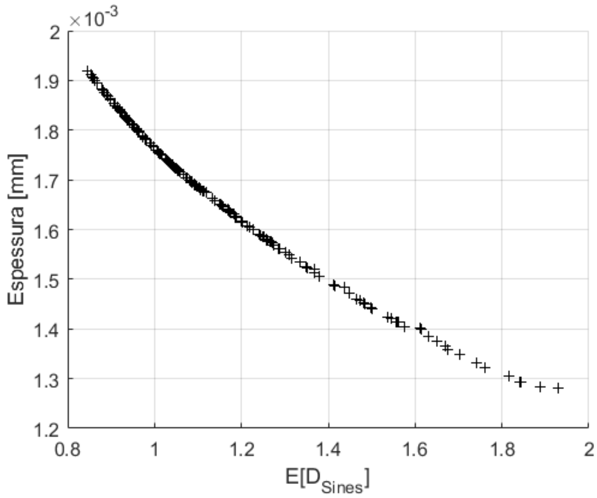

Figure 6: Robust optimization Pareto front

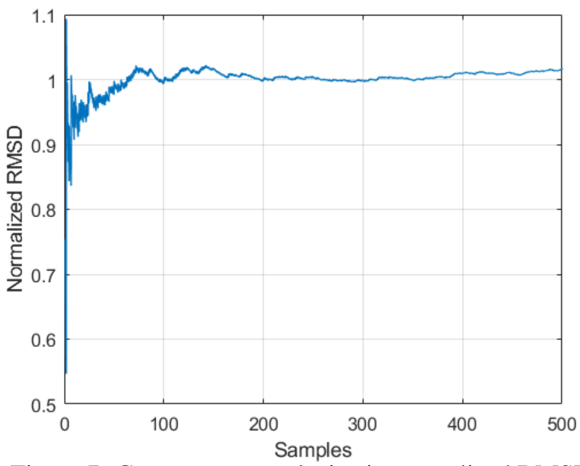

Figure 7: Convergence analysis via normalized RMSD

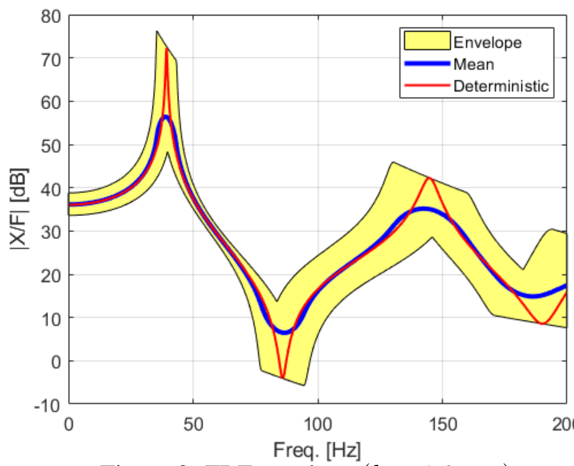

Figure 8: FRF envelope $(h = 1.8mm)$

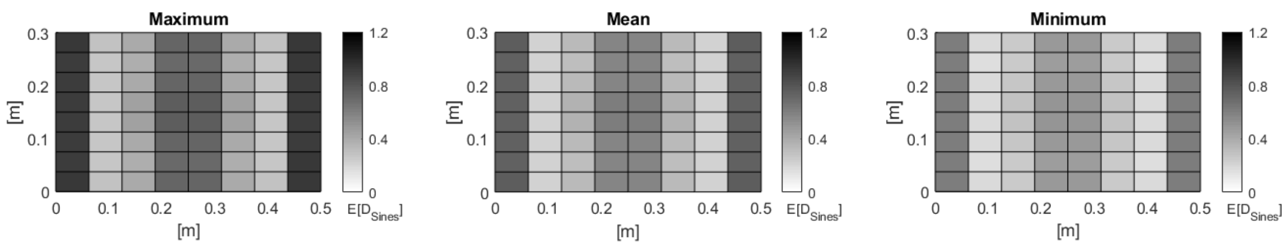

Figure 9: Sines coefficient distribution $(h = 1.8mm)$

5. CONCLUDING REMARKS

Fatigue strength limits such as those used in the present work are obtained experimentally. Thus, uncertainties present in the tests are propagated through results and posteriorly to computational simulations. By including these effects in geometry, material properties and loads, a more accurate and realistic approach than the classical application is obtained.

Considering random effects in the original system changed the subjective failure condition. Simulations over a thin plate whose thickness $h = 1.6mm$ is not subjected to parameter fluctuations did now show crack nucleation before 2 million cycles. Its Sines' fatigue index is always the same value and less than 1. Nevertheless, when thickness was considered as a random variable and a 10% uncertainty level was added, $E[D_{Sines}]$ became greater than 1, indicating possible fatigue failure. When a crack appears and is not controlled, it may grow and propagate towards the structure, causing a catastrophic fatigue break. This highlights the importance of including uncertainties in computational models.

The optimization procedure aimed at minimizing the thickness in a robust way, avoiding sensitive points and changes in the failure or non-failure condition. By changing the plate's thickness to $h = 1.8mm$, mass was increased by 12.5% but even when performing stochastic analysis, it does not fail before 2 million cycles. With the Pareto front in hand, it is up to the designer to choose the point and weigh its structural effects.

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