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LEAST SQUARES FINITE ELEMENT METHOD - LSFEM FOR RESOLUTION OF EQUATION CONVECTION DIFFUSION TWO DIMENSIONAL

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Abstract*. This paper aims to validate the code implemented in C language to solve the equation convection diffusion two dimensional using the the least squares finite element method. The code is compared with the analytical solution of the equation using the method of separating the variables. In this way it is observed that the obtained error was very small 0.5 % which validates the developed code.*

Keywords: *Least Squares Finite Element Method, Equation Convection Diffusion Two Dimensional, Numerical Solution, Language C.*

1. INTRODUCTION

Proper description of the physical phenomena makes it possible to obtain the exact or approximate solution to the problem at hand, according to Reddy (1993).

The Finite Element Method - MEF was developed by aeronautical engineers in 1950 to analyze structural problems in aircraft. In his article Turner (1956) established the procedures for the formulation and assembly of the element matrix. Clough (1960) wrote an article where the term finite element was first used. Argyris and Kelsey (1963) developed the theory of the matrix of elements. structures for discrete elements establishing the concepts of flexibility and stiffness standard formulations of structural mechanics. Zienkiewicz and Cheung (1965) applied the finite element method to nonstructural problems such as fluid flow and electromagnetism. Oden and Wellford Junior (1972) applied the method to a wide range of nonlinear mechanics problems.

The partial differential equations - EDP are very important because they model several problems in the areas of electromagnetism, biomedical engineering, heat transfer, fluid mechanics, geotechnics, meteorology, econometrics, chemical engineering, food, among others. Currently the Finite Element Method - MEF is widely used to solve this kind of equation, in general terms is a mathematical method in which a is discretized into elements that maintain the properties of the originator, the least squares finite element method - LSFEM is one of the variants of the finite elements. According Bon (1998) the least squares finite element method offers many advantages in its use since it can be used for all types of partial differential equations.

2. LITERATURE REVIEW

Camprubi *et al.* (2000) makes a similar analysis comparing the Galerkin finite element method (GFEM) with the least squares finite element method (LSFEM) for the Peclet high number diffusion convection problem. The results showed oscillations in the solution by applying the Galerkin finite element method as expected because it is known from the literature that this method presents good results only for problems with dominant diffusion and the least squares finite element method showed good results.

Romão (2004) studied the finite element method in the Galerkin, Petrov-Galerkin and least squares variants for the convection equation - two-dimensional diffusion and compared their results with those of Camprubi *et al.* (2000) and concluded that the method presents in its formulation a matrix of coefficients that is symmetrical and positive definite for both diffusive and convective-diffusive problems.

Using the velocity-pressure-vorticity and velocity-pressure-stress formulations, named u - p - ω and u - p - τ , made incompressible flow simulations. Perrreira (2005) used the Least Square Finite Element (LSFEM) method. For the validation of the results, an element with nine nodal points was used, and for the resolution of the algebraic system, the frontal method was used instead of the most appropriate gradient method for a defined symmetric and positive matrix.

Bergstrom (2000) conducted a study on the application of LSFEM and obtained good results since the Maxwell equations are of first order being convenient to implement the LSFEM according to Perrreira (2005).

In the study by Chen *et al.* (2010) an approach was developed to solve problems related to non-Newtonian fluid flow using LSFEM. These surveys are conducted by the chemical, food and oil industries. Numerical solutions to this type of problem are difficult to obtain due to the hyperbolic behavior of partial differential equations.

3. FORMULATION MATHEMATICAL USING FINITE ELEMENT METHOD OF LEAST SQUARES - LSFEM

The central idea of the method of least squares is to determine u $\in \Omega^e$ for the minimization of the integral of the residue. Defining the quadratic functional as equation (1).

$$I(u) = |[R(u)]|^2 = \int_\Omega R(u)^2 d\Omega \tag{1}$$

with $u \in \Omega^e$={ $u \in H^2$} at where H^2 is the Hilbert space of order 2. To the solution u be a minimizer of the functional given by equation (1) the first variation δI equation (2) it should be zero.

$$\delta I(u) = 2\int_\Omega (\delta R)R(x)d\Omega = 0 \quad \text{ou} \quad \delta I(u) = \int_\Omega (\delta R)R(x)d\Omega = 0 \tag{2}$$

Therefore we conclude that the function w_i will be equal to the first variation of the residue δ being the least squares method. The equation of convection diffusion two dimensional with physical properties constants with temperature using finite element method of least squares in a generic domain $\Omega \in R^2$ it is limited and closed, with Dirichlet boundary conditions and initial condition is given by equation (3).

$$\rho(T)c_p(T)\left(\frac{\partial T(x,y,t)}{\partial t} + u\frac{\partial T(x,y,t)}{\partial x} + v\frac{\partial T(x,y,t)}{\partial y}\right) = k(T)\frac{\partial T^2(x,y,t)}{\partial x^2} + k(T)\frac{\partial T^2(x,y,t)}{\partial y^2} \tag{3}$$

where $\rho(T)$ is the density with unit of measure in kg/m^3, $c_p(T)$ it is the specific heat in J/kg .K e $k(T)$ thermal conductivity W/m.K. Isolating the transient term of the equation (3) obtains the equation (4).

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha(T)\frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha(T)\frac{\partial^2 T(x,y,t)}{\partial y^2} - u\frac{\partial T(x,y,t)}{\partial x} - v\frac{\partial T(x,y,t)}{\partial y} \tag{4}$$

The term $\alpha(T) = k(T)/\rho(T)c_p(T)$ it is called thermal diffusivity in m^2/s. The approximate solution $\widetilde{T}(x,y,t)$ is given by equation (5).

$$\widetilde{T}(x,y,t) = \sum_{i=1}^{Nnos} T_i(t)N_i(x,y) \tag{5}$$

From the approach of family θ can approach the transient term adopting $\theta = 0.5$ called a scheme of Cranck - Nicolson.

$$\begin{aligned}\frac{\widetilde{T}_n^{s+1} - \widetilde{T}_n^s}{\Delta t} &= \theta\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2} - u\frac{\partial\widetilde{T}}{\partial x} - v\frac{\partial\widetilde{T}}{\partial y}\right\}^{s+1} + (1-\theta)\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2} - u\frac{\partial\widetilde{T}}{\partial x}\right. \\ &- \left. v\frac{\partial\widetilde{T}}{\partial y}\right\}^s \\ &= 0.5\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2} - u\frac{\partial\widetilde{T}}{\partial x} - v\frac{\partial\widetilde{T}}{\partial y}\right\}^{s+1} + 0.5\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2} - u\frac{\partial\widetilde{T}}{\partial x}\right. \\ &- \left. v\frac{\partial\widetilde{T}}{\partial y}\right\}^s\end{aligned} \tag{6}$$

The residue $R(x,y)$ is given by equation (7).

$$R(x,y) = \frac{\widetilde{T}^{s+1}}{\Delta t} - \frac{\widetilde{T}^s}{\Delta t} - 0.5\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2} - u\frac{\partial\widetilde{T}}{\partial x} - v\frac{\partial\widetilde{T}}{\partial y}\right\}^{s+1} - 0.5\left\{\alpha(T)\frac{\partial^2\widetilde{T}}{\partial x^2} + \alpha(T)\frac{\partial^2\widetilde{T}}{\partial y^2}\right.$$

$$- \quad u\frac{\partial \widetilde{T}}{\partial x} - v\frac{\partial \widetilde{T}}{\partial y}\Bigg\}^{s} \tag{7}$$

Grouping the terms in time s and time $s+1$ it has equation (8).

$$\begin{aligned} R(x,y) &= \left\{\frac{\widetilde{T}}{\Delta t} - 0.5\alpha(T^s)\frac{\partial^2 \widetilde{T^s}}{\partial x^2} - 0.5\alpha(T^s)\frac{\partial^2 \widetilde{T^s}}{\partial y^2} + 0.5u\frac{\partial \widetilde{T}}{\partial x} + 0.5v\frac{\partial \widetilde{T}}{\partial y}\right\}^{s+1} + \left\{-\frac{\widetilde{T}}{\Delta t} - 0.5\alpha(T^{s-1})\frac{\partial^2 \widetilde{T}}{\partial x^2}\right. \\ &- \left. 0.5\alpha(T^{s-1})\frac{\partial^2 \widetilde{T}}{\partial y^2} + 0.5u\frac{\partial \widetilde{T}}{\partial x} + 0.5v\frac{\partial \widetilde{T}}{\partial y}\right\}^{s} \end{aligned} \tag{8}$$

Therefore:

$$\begin{aligned} R(x,y) &= \left\{\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s+1}N_i - 0.5\alpha(T^s)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T^s)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial x}\right. \\ &+ \left. 0.5v\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial y}\right\} + \left\{-\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s}N_i - 0.5\alpha(T^{s-1})\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T^{s-1})\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial y^2}\right. \\ &+ \left. 0.5u\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial y}\right\} \end{aligned} \tag{9}$$

The first variation δR the residue equation (9) is given by:

$$\begin{aligned} \delta R &= \frac{\partial R}{\partial T_i}\delta T_i = \left(\frac{1}{\Delta t}\sum_{i=1}^{Nnos} N_i - 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial x}\right. \\ &+ \left. 0.5v\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial y}\right)\delta T_i \end{aligned} \tag{10}$$

Applying the method of least squares equation (11).

$$\begin{aligned} \int_\Omega R(x,y)\delta R d\Omega &= \int_\Omega\Bigg\{\Bigg(\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s+1}N_i - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial y^2} \\ &+ 0.5u\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial y}\Bigg) + \Bigg(-\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s}N_i - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial x^2} \\ &- 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial y}\Bigg)\Bigg\}\Bigg\{\frac{1}{\Delta t}\sum_{i=1}^{Nnos} N_i \\ &- 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial y}\Bigg\}\delta T_i d\Omega = 0. \end{aligned} \tag{11}$$

as $\delta T_i \neq 0$.

$$\begin{aligned} \int_\Omega R(x,y)\delta R d\Omega &= \int_\Omega\Bigg\{\Bigg(\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s+1}N_i - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial^2 N_i}{\partial y^2} \\ &+ 0.5u\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos} T_i^{s+1}\frac{\partial N_i}{\partial y}\Bigg) + \Bigg(-\frac{1}{\Delta t}\sum_{i=1}^{Nnos} T_i^{s}N_i - 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial x^2} \\ &- 0.5\alpha(T)\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos} T_i^{s}\frac{\partial N_i}{\partial y}\Bigg)\Bigg\}\Bigg\{\frac{1}{\Delta t}\sum_{i=1}^{Nnos} N_i \\ &- 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T)\sum_{i=1}^{Nnos}\frac{\partial^2 N_i}{\partial y^2} + 0.5u\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial x} + 0.5v\sum_{i=1}^{Nnos}\frac{\partial N_i}{\partial y}\Bigg\}d\Omega = 0. \end{aligned}$$

$$
\begin{aligned}
= & \int_\Omega \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} T_i^{s+1} N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial^2 N_i}{\partial y^2} \\
+ & 0.5u \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial N_i}{\partial x} + 0.5v \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial N_i}{\partial y} \Bigg) \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial x^2} \\
- & 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial y^2} + 0.5u \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial x} + 0.5v \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial y} \Bigg) d\Omega + \int_\Omega \Bigg(-\frac{1}{\Delta t} \sum_{i=1}^{Nnos} T_i^s N_i \\
- & 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^s \frac{\partial^2 N_i}{\partial x^2} + 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^s \frac{\partial^2 N_i}{\partial y^2} + 0.5u \sum_{i=1}^{Nnos} T_i^s \frac{\partial N_i}{\partial x} \\
+ & 0.5v \sum_{i=1}^{Nnos} T_i^s \frac{\partial N_i}{\partial y} \Bigg) \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial y^2} \\
+ & 0.5u \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial x} + 0.5v \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial y} \Bigg) d\Omega = 0.
\end{aligned} \tag{12}
$$

Because the values of T_i^s are known:

$$
\begin{aligned}
\int_\Omega \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} T_i^{s+1} N_i & - 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial^2 N_i}{\partial y^2} + 0.5u \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial N_i}{\partial x} \\
& + 0.5v \sum_{i=1}^{Nnos} T_i^{s+1} \frac{\partial N_i}{\partial y} \Bigg) \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial y^2} \\
& + 0.5u \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial x} + 0.5v \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial y} \Bigg) d\Omega
\end{aligned}
$$

$$
\begin{aligned}
= - & \int_\Omega \Bigg(-\frac{1}{\Delta t} \sum_{i=1}^{Nnos} T_i^s N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^s \frac{\partial^2 N_i}{\partial x^2} + 0.5\alpha(T) \sum_{i=1}^{Nnos} T_i^s \frac{\partial^2 N_i}{\partial y^2} + 0.5u \sum_{i=1}^{Nnos} T_i^s \frac{\partial N_i}{\partial x} \\
+ & 0.5v \sum_{i=1}^{Nnos} T_i^s \frac{\partial N_i}{\partial y} \Bigg) \Bigg(\frac{1}{\Delta t} \sum_{i=1}^{Nnos} N_i - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \sum_{i=1}^{Nnos} \frac{\partial^2 N_i}{\partial y^2} + 0.5u \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial x} \\
+ & 0.5v \sum_{i=1}^{Nnos} \frac{\partial N_i}{\partial y} \Bigg) d\Omega
\end{aligned} \tag{13}
$$

Rearranging the equation (13) get the linear system equation (14).

$$[A]\{T^{s+1}\} = [b]\{T^s\} \tag{14}$$

where

$$
\begin{aligned}
A_{i,j} \;=\; & \int_\Omega \left(\frac{1}{\Delta t} N_i - 0.5\alpha(T) \frac{\partial^2 N_i}{\partial x^2} - 0.5\alpha(T) \frac{\partial^2 N_i}{\partial y^2} + 0.5u \frac{\partial N_i}{\partial x} + 0.5v \frac{\partial N_i}{\partial y} \right) \left(\frac{1}{\Delta t} N_j - 0.5\alpha(T) \frac{\partial^2 N_j}{\partial x^2} \right. \\
- & \left. 0.5\alpha(T) \frac{\partial^2 N_j}{\partial y^2} + 0.5u \frac{\partial N_j}{\partial x} + 0.5v \frac{\partial N_j}{\partial y} \right) d\Omega
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
b_i \;=\; & -\int_\Omega \left(-\frac{1}{\Delta t} N_i - 0.5\alpha(T) \frac{\partial^2 N_i}{\partial x^2} + 0.5\alpha(T) \frac{\partial^2 N_i}{\partial y^2} + 0.5u \frac{\partial N_i}{\partial x} + 0.5v \frac{\partial N_i}{\partial y} \right) \left(\frac{1}{\Delta t} N_j - 0.5\alpha(T) \frac{\partial^2 N_j}{\partial x^2} \right. \\
- & \left. 0.5\alpha(T) \frac{\partial^2 N_j}{\partial y^2} + 0.5u \frac{\partial N_j}{\partial x} + 0.5v \frac{\partial N_j}{\partial y} \right) d\Omega
\end{aligned} \tag{16}
$$

4. METHODOLOGY

For resolve the equation of convection - diffusion two dimensional using least squares finite element method - LSFEM a code in C language was implemented quadratic two - dimensional elements with 8 nodes were used for spatial discretization. The Cranck-Nicolson method was used to discretize the time, the method of Gaussian Quadrature - Legendre are used for the calculation of the integrals, the solving the linear system obtained will be used the method of conjugate gradients why the global matrix is symmetric and positive definite sparse result of using LSFEM.

The quadratic two - dimensional elements with 8 nodes are apresentado in figure 1.

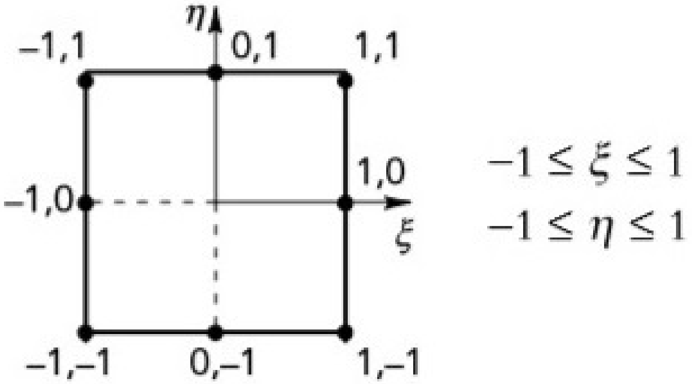


Figure 1. Quadratic two - dimensional elements with 8 nodes

The Cranck-Nicolson method for temporal discretization is a θ family method widely used for integrating first order differential equations. It is a one-step method where the value of T^{n+1} is unknown in time $n+1$, but can be calculated from the value of T^n in time n. In this case a weighted average was made between T^{n+1} e T^n at the endpoints of the integration range.

$$\frac{\partial T}{\partial t}\frac{T^{n+1}-T^n}{\Delta t} = (1-\theta)\frac{\partial T^n}{\partial t} + \theta\frac{\partial T^{n+1}}{\partial t} \tag{17}$$

For $\theta = \frac{1}{2}$ we have the precisely stable Crank Nicolson scheme $O(\Delta t)^2$.

The method of Gaussian Quadrature choose the points for the calculation optimally rather than equally spaced as in Newton Cotes formulas. The points $x_1, x_2, \ldots, x_n$ between $[a, b]$ and the weights $w_1, w_2, \ldots, w_n$ are chosen so that the error made in the approximation equation 18.

$$\int_a^b f(x)dx \simeq \sum_{i=0}^{n} W(x)f(x)dx \simeq \sum_{i=0}^{n} w_i f(x_i). \tag{18}$$

In Gaussian - Legendre Quadrature the interpolator polynomial $p_n(x)$ is Legendre polynomial with weight function $W(x) = 1$. Legendre polynomials are defined by the recurrence formula given by the equation 19.

$$P_n(x) = \frac{(2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)}{n} \tag{19}$$

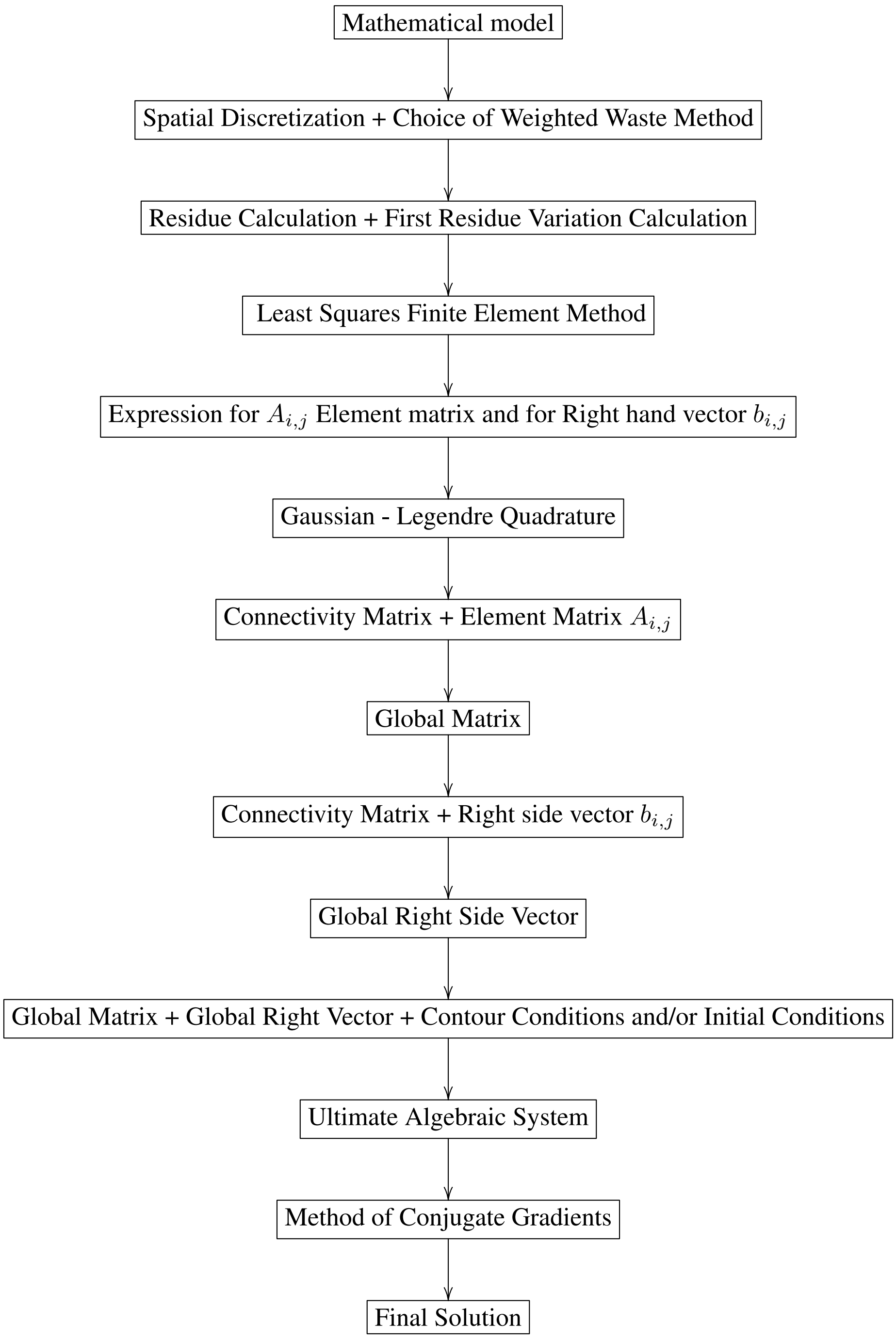
with $P_0(x) = 1$ e $P_1(x) = x$.

The method of conjugate gradients proposed is the most popular iterative method for solving large linear systems where the coefficient matrix is symmetric, sparse and positive, this method proposes a function to be optimized so that the minimum of this function is the solution of the. linear system in question. Given a linear system of the for (equation20).

$$Ax = b \tag{20}$$

with A $\in R^{nxn}$, x $\in R^n$ e b $\in R^n$ The basic idea of an optimization method is to create the optimization function. $F(x) : R^n \to R$, because minimizing this function is equivalent to solving the linear system 20.

The flow diagram below presents the methodology used in this article.



5. RESULTS

The differential equation for this problem is given by equation 21.

$$u\frac{\partial T(x,y,t)}{\partial x} + v\frac{\partial T(x,y,t)}{\partial y} - k\frac{\partial^2 T(x,y)}{\partial x^2} - k\frac{\partial^2 T(x,y)}{\partial y^2} = 0. \tag{21}$$

The boundary conditions of Direchlet type are equation (22).

$$\begin{aligned} T(x,0) &= T(L,y) = T(0,y) = 0^{\circ}\text{C} \\ T(x,L) &= 1^{\circ}\text{C} \qquad \text{with} \qquad 0 \leq (x,y) \leq 1 \quad . \end{aligned} \tag{22}$$

The thermal conductivity is equal to $k = 10^3$ W/m.K, $U(u,v)$ where $u = v = 20$ m/s.

The problem domain is given by Figure 2.

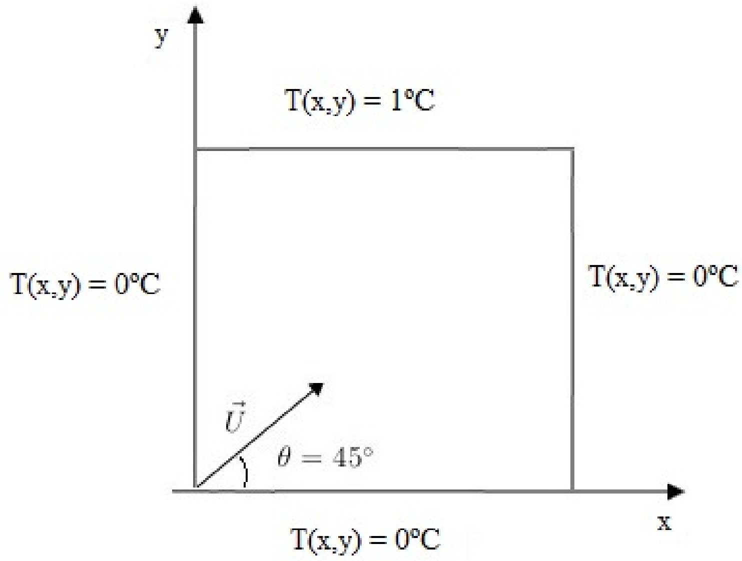


Figure 2. The problem domain

Applying variable separation method for obtaining analytic solution of the problem (21 - 22) given by equation (23).

$$T(x,y) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{(n\pi - n\pi e^{-2}\cos(n\pi) - e^{-2}\sin(n\pi))}{(1+n^2\pi^2)}\right](e^{\sqrt{2+n^2\pi^2}} - e^{-\sqrt{2+n^2\pi^2}})\sin(n\pi x)(e^{y\sqrt{2+n^2\pi^2}} - e^{-y\sqrt{2+n^2\pi^2}}) \quad (23)$$

The analytic solution to the permanent convection diffusion problem are shown below Figure 3

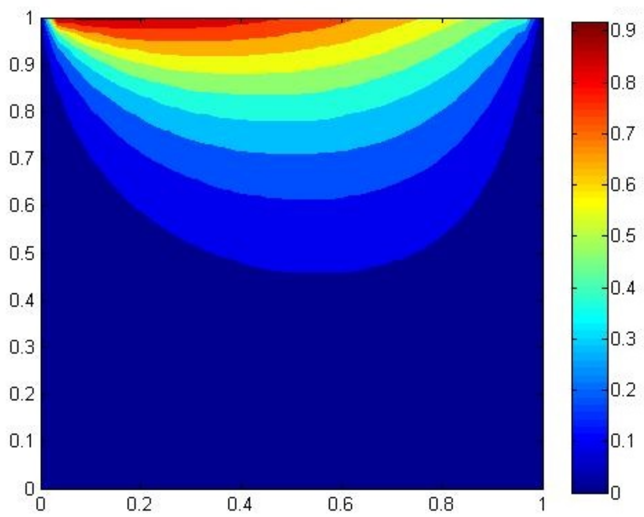


Figure 3. Analytic solution to the convection diffusion problem in steady state

The numerical solution to the permanent convection diffusion problem are shown below Figure 4

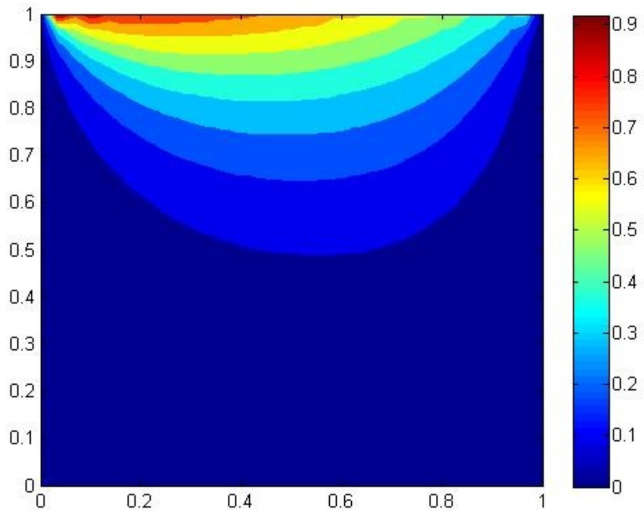


Figure 4. Numerical solution to the convection diffusion problem in steady state via LSFEM

For the validation of the numerical solution will be made the comparison between a column of the numerical solution and a column of the analytical solution in Figure 5.

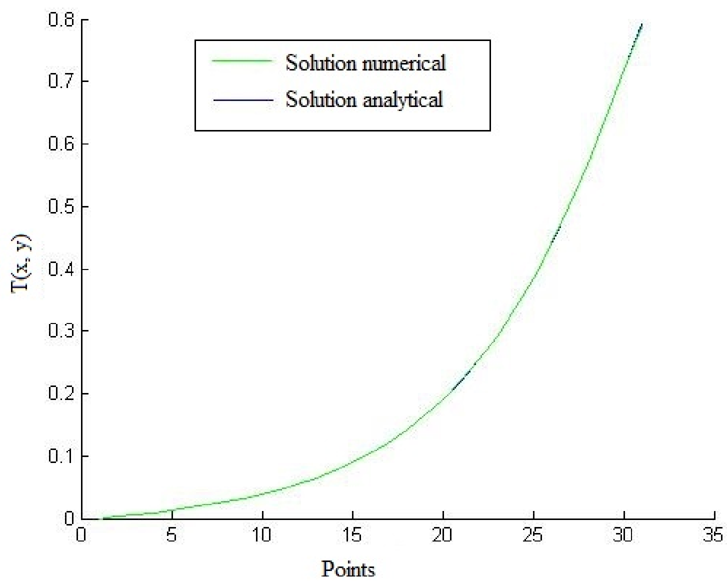


Figure 5. Comparison between Numerical and Analytical Solutions.

The Table 1 presents the values of the analytical solution (sol_{anal}), numerical solution (sol_{num}) and the error given by the difference between them comparing a column.

Table 1. Values of analytical solution (sol_{anal}), numerical solution (sol_{num}) and error $(sol_{anal} - sol_{num})$

points	sol_{anal}	sol_{num}	$error$	points	sol_{anal}	sol_{num}	$error$
1	0.00000	0.00000	0.00000	17	0.12140	0.12141	0.00001
2	0.00280	0.00279	0.00001	18	0.14120	0.14123	0.00003
3	0.00580	0.00578	0.00002	19	0.16390	0.16400	0.00010
4	0.00910	0.00900	0.00010	20	0.18990	0.19000	0.00010
5	0.01270	0.01260	0.00010	21	0.21960	0.21990	0.00030
6	0.01680	0.01680	0.00000	22	0.25350	0.25370	0.00020
7	0.02130	0.02140	0.00010	23	0.29200	0.29230	0.00030
8	0.02640	0.02638	0.00002	24	0.33550	0.33570	0.00020
9	0.03220	0.03221	0.00010	25	0.38440	0.38490	0.00050
10	0.0388	0.03890	0.00010	26	0.43910	0.43980	0.00070
11	0.04630	0.04650	0.00020	27	0.49990	0.50000	0.00010
12	0.05490	0.05510	0.00020	28	0.56700	0.56720	0.00020
13	0.06480	0.06500	0.00020	29	0.64050	0.64040	0.00010
14	0.07610	0.07630	0.00020	30	0.72020	0.71940	0.00040
15	0.08920	0.08940	0.00020	31	0.79130	0.78630	0.00500
16	0.10420	0.10470	0.00050				

According to the Figure 5 and the table 1 we can observe that the numerical solution presents a good approximation to the analytical solution, the largest error obtained was equal to 0.0050 approximately 0.5 %.

6. CONCLUSIONS

The value of the numerical error was very small, so it can be concluded that the implemented code presents a good numerical solution for the equation convection diffusion two diemnsional using LSFEM.

7. ACKNOWLEDGEMENTS

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8. REFERENCES

Argyris, J. and Kelsey, S., 1963. “Energy theorems and structural analysis”. *Butterworth*.

Bergstrom, R., 2000. “Least-squares finite element methods for electromagnetic applications.”

Bon, N.J., 1998. *The least - squares finite element method: theory and applications in computational fluid dynamics and electromagnetics*. Springer.

Camprubi, N.F., Colominas, I., Navarrina, F. and Casteleiro, M., 2000. “Galerkin, least square and galerkin least square numerical approaches for convective diffusive transport problems”. In *Europen congress on compuattional in aplied science and engineering*. Barcelona, Espanhal, p. 11–14.

Chen, T., Cox, C. and H., L., 2010. “Least-squares finite element methods for generalized newtonian and viscoelastic flows.” *Applied Numerical Mathematic*, p. 1024 – 1040.

Clough, R., 1960. “The finite element method in plane stress analysis”. *American Society of Civil Engineers*.

Oden, J. and Wellford Junior, L., 1972. “Analysis of flow of viscous fluids by the finite element method”. *AIAA*, Vol. 10, No. 12, p. 1590–1599.

Perrreira, V.D., 2005. *Simulação numérica de escoamentos de fluidos pelo método de elementos finitos de mínimos quadrados*. Ph.D. thesis, Universidade Estadual Paulista, Ilha Solteira, Brazill.

Reddy, J.N., 1993. *An introduction to the finite element method*. Mcgraw-Hill.

Romão, E.C., 2004. *Variantes do método dos elementos finitos para solução de Fenômenos convectivo-difusivos bidimensionais*. Ph.D. thesis, Universidade Estadual Paulista, Ilha Solteira, Brazill.

Turner, J., 1956. “Stiffness and deflection analysis of complex structures”. *Journal of The Aeronautical Sciences*, Vol. 23, No. 9, pp. 805– 823.

Zienkiewicz, C. and Cheung, Y., 1965. “Finite elements in the solution of field problems”. *The Engineer*, p. 507–510.

9. RESPONSIBILITY NOTICE