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## **TWO-DIMENSIONAL FIRST INTEGRAL OF VISCO-RESISTIVE MAGNETOHYDRODYNAMICS**

**Jose I.H. Lopez**

Department of Mechanics, Mackenzie Presbyterian University, Consolacao, 930, Cep 01302-907, SP, BRAZIL  
jihlopez@mackenzie.com.br

**Rafael A. Trujillo**

Department of Automatic Control, University of Santiago de Cuba, Av. de Las Americas, CUBA  
rtrujillo@edu.uo.cu

**Abstract.** *The purpose of this article is to study some question related to the visco-resistive complex two-dimensional first integral equations of magnetohydrodynamics. In this work the authors found a way to write a first integral, for visco-resistive magnetohydrodynamics, which generalizes Bernoulli's equation and which depends on real-valued potentials. Upon inserting the stream-function representation of the flow, the first integral amounts to the two second order complex equations for four real-valued fields, i.e., the velocity and magnetic potential and stream function for the flow and flux function for the magnetic field. Most of the questions that we investigated are related to the transformations of differential magnetohydrodynamics operators from the real plane ( $\mathcal{R}$ ) to the complex plane ( $\mathcal{C}$ ). A new type of complete set of field equations appears: the first integral complex magnetohydrodynamics equations. We also calculated a special case of complex solution for these magnetohydrodynamics equations.*

**Keywords:** *MHD, Magnetohydrodynamics, Complex Navier-Stokes, visco-resistive flows, first integral equations...*

### **1. INTRODUCTION**

Magnetohydrodynamics (MHD) is a phenomenon where the velocity field  $\mathbf{u}$  and magnetic field  $\mathbf{B}$  are coupled. MHD is used to study the dynamics of electrically conducting fluids (Davison, 2001). It establishes a coupling between the Navier-Stokes equations for fluid dynamics and Maxwell's equations for electromagnetism. Magnetic fields can induce currents in moving electrically conducting and non-magnetic fluid like liquid metals, hot ionized gases (plasmas), which in turn create forces on the fluid and influence the magnetic field itself. Magnetic fields are used to pump, stir, levitate and heat liquid metals. The earth's magnetic field, which protect the surface from radiation, is generated by the motion of the earth's liquid core. The solar magnetic field generates solar flares and sunspots, and the galactic magnetic field influences the formation of stars.

The electromagnetic part of MHD, namely Maxwell's equations, the transport equation for the magnetic field  $\mathbf{B}$ , coupled with the Navier-Stokes equations -the continuity and motion equation- form the Magnetohydrodynamics equations, along with the assumptions of Newtonian fluids and incompressibility. The coupling between the two fields ( $\mathbf{u} \leftrightarrow \mathbf{B}$ ) happens because of the presence of a velocity field  $\mathbf{u}$  in the induction equation and the presence of the Lorentz force in the Navier-Stokes equations.

The first integrals, non-constant function whose derivatives vanish in the solution curves of the system, play crucial roles in studies of dynamical systems and other kinds of systems (Man, 1994). First integrals help in the definition of problems. Their values are constants that have physical significance. In many systems, they are the only computable measurement of the efficiency of the numerical method. Monitoring or computing the first integrals permit us to assess the quality of the approximation, when we do not know the exact solution. The work of Scholle (Scholle *et al.*, 2011) stated that for two dimensions ( $\mathcal{R}^2$ ) the first integral of Navier-Stokes' equations could be represented in the complex plane ( $\mathcal{C}$ ). Here, in this work, we use the same procedure for MHD equations.

## 2. NON-DIMENSIONAL FORM OF THE MHD EQUATIONS

The non-dimensional form of steady and two-dimensional Magnetohydrodynamics (MHD) equations describe the macroscopic state of the viscous incompressible and resistive fluid (Sermange and Temam, 1983a,b):

$$\left\{ \begin{array}{l} (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{R_e} \nabla^2 \mathbf{u} + \nabla p + N \nabla \left( \frac{B^2}{2} \right) - N (\mathbf{B} \cdot \nabla) \mathbf{B} = \mathbf{f} \\ (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \frac{1}{R_m} \nabla^2 \mathbf{B} = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \nabla \cdot \mathbf{B} = 0 \end{array} \right. \quad (1)$$

where  $\rho = 1$  is the density of the fluid,  $p$  is the pressure,  $\mathbf{u}$  is the velocity of the fluid particle at point  $x$ ,  $\mathbf{B}$  is the magnetic field at point  $x$ ,  $\mathbf{f}$  is the non-dimensional volume density forces.  $R_e$  is a Reynolds number and  $\mathbf{B}, \mathbf{u}, p$  are non-dimensional quantities.  $R_m$  is magnetic Reynolds number, and the interaction parameter  $N = H_a^2 / R_e$ , where  $H_a$  is the Hartman number. The calculation can be done much more simple directly from "Eq. (1)" as follows. Setting  $\mathbf{B} = c\mathbf{u}$  for any constant  $c$  implies:

$$\mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} \quad (2)$$

so the steady induction equation reduces to:

$$\nabla^2 \mathbf{B} = 0 \quad (3)$$

One then needs to choose  $c = N^{-1/2}$  so that:

$$\mathbf{u} \cdot \nabla \mathbf{u} = N (\mathbf{B} \cdot \nabla \mathbf{B}) \quad (4)$$

and the momentum equation becomes

$$-\frac{1}{R_e} \nabla^2 \mathbf{u} + \nabla \left( p + N \frac{B^2}{2} \right) = \mathbf{f} \quad (5)$$

One knows from the induction equation that  $\nabla^2 \mathbf{B} = 0$ , which implies  $\nabla^2 \mathbf{u} = 0$ . One then has to choose  $c$  to eliminate "Eq. (2)" from the momentum equation. However, there is no restriction on  $R_e$  and  $R_m$ . Solutions of the inductions equations, and "Eq. (5)" with no body force ( $\mathbf{f} = 0$ ), are simply harmonic vector fields satisfying  $\nabla^2 \mathbf{u} = 0$ , and a pressure  $p = -NB^2/2$ . A family of such solutions is given by the potential flows  $\mathbf{u} = \nabla \phi$  with  $\nabla^2 \phi = 0$ . This is solved in 2D by taking  $\phi$  to be the real or imaginary part of a holomorphic function of  $z = \frac{1}{2}(y + ix)$ .

## 3. COMPLEX FORMULATION

MHD equations can be transformed ( $\mathcal{R}^2 \rightarrow \mathcal{C} \times \mathcal{C}$ ) by the following complex variables (Scholle *et al.*, 2011):  $z = \frac{1}{2}(y + ix)$  and  $\bar{z} = \frac{1}{2}(y - ix)$ . The complex velocity field and magnetic field are:  $u = u_x + iu_y$ ,  $B = B_x + iB_y$ . Then if multiplying the second equation for the velocity field and for the magnetic field by imaginary identity  $i$ , and sum the corresponding equations, the complex operator  $\nabla$  will satisfy the following properties:

$$\left\{ \begin{array}{l} y = z + \bar{z} \\ x = \frac{1}{i}(z - \bar{z}) \\ \frac{\partial}{\partial y} = \frac{\partial}{\partial(z + \bar{z})} \mapsto \frac{1}{2} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) \\ \frac{\partial}{\partial x} = \frac{i \partial}{\partial(z - \bar{z})} \mapsto \frac{i}{2} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \\ \nabla = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \mapsto i \frac{\partial}{\partial z} \\ \bar{\nabla} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \mapsto i \frac{\partial}{\partial \bar{z}} \end{array} \right. \quad (6)$$

then for all the gradients that participate in the equations (Spiegel, 1972):

$$-\nabla p = -\left(\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right) \mapsto -i\frac{\partial p}{\partial z} \quad (7)$$

for the divergences:

$$\text{div}(\mathbf{u}) = \Re(\bar{\nabla}\mathbf{u}) = \Im(i\bar{\nabla}\mathbf{u}) \mapsto \Im\left(\frac{\partial u}{\partial \bar{z}}\right) \quad (8)$$

and for the laplacians:

$$\nabla \cdot \nabla \equiv \nabla^2 = \Re(\bar{\nabla}\nabla) = \Im(i\bar{\nabla}\nabla) \mapsto \frac{\partial^2}{\partial \bar{z}\partial z} \quad (9)$$

The convective term is:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - (\nabla \times \mathbf{u}) \times \mathbf{u} \quad (10)$$

because the Lamb vector  $(\nabla \times \mathbf{u}) \times \mathbf{u} = \omega \times \mathbf{u}$  appears here. It is known that the kinematic identity for the Lamb vector is (Liu *et al.*, 2014):

$$\omega \times \mathbf{u} = \nabla \cdot (\mathbf{u}\mathbf{u} - \frac{1}{2}|\mathbf{u}|^2 I) \mapsto \frac{i}{2}\frac{\partial u^2}{\partial \bar{z}} - i\frac{\partial(\frac{1}{2}\bar{u}u)}{\partial z} + i\frac{\partial(\frac{1}{2}\bar{u}u)}{\partial z} \quad (11)$$

which enables the convective term to be translated into complex plane.  $I$  is the unit tensor. By mathematical manipulation the following transformations are obtained:

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{u} \mapsto \Im\left(\frac{\partial u}{\partial \bar{z}}\right) \\ \nabla \cdot \mathbf{B} \mapsto \Im\left(\frac{\partial B}{\partial \bar{z}}\right) \\ (\mathbf{u} \cdot \nabla)\mathbf{u} \mapsto i\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{u}u\right) - i\frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}u^2\right) \\ \frac{1}{Re}\nabla^2\mathbf{u} \mapsto \frac{1}{Re}\frac{\partial^2 u}{\partial \bar{z}\partial z} \end{array} \right. \quad (12)$$

if we take  $f = \left(\frac{\partial U}{\partial x} \frac{\partial U}{\partial y}\right)^T$ , then all the other terms have similar structure, which allows us to transform to:

$$\left\{ \begin{array}{l} \nabla(p + N\frac{B^2}{2} + U) \mapsto i\frac{\partial}{\partial z}(p + N\frac{B^2}{2} + U) \\ N(\mathbf{B} \cdot \nabla)\mathbf{B} \mapsto iN\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{B}B\right) - Ni\frac{\partial}{\partial \bar{z}}\left(\frac{B^2}{2}\right) \\ \frac{1}{R_m}\nabla^2\mathbf{B} \mapsto \frac{1}{R_m}\frac{\partial^2 B}{\partial \bar{z}\partial z} \\ (\mathbf{u} \cdot \nabla)\mathbf{B} \mapsto i\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{u}B\right) - i\frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}uB\right) \\ (\mathbf{B} \cdot \nabla)\mathbf{u} \mapsto i\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{B}u\right) - i\frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}Bu\right) \end{array} \right. \quad (13)$$

from "Eq. (13)" the complex non-integrable equations appear:

$$\left\{ \begin{array}{l} i\frac{\partial}{\partial z}\left(p + \frac{\bar{u}u}{2} - N|B|^2 + U\right) - \frac{i\partial}{2\partial \bar{z}}(u^2 - NB^2) = \frac{1}{R_e}\frac{\partial^2 u}{\partial \bar{z}\partial z} \\ i\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{u}B - \frac{1}{2}\bar{B}u\right) - i\frac{\partial}{2\partial \bar{z}}(uB - Bu) = \frac{1}{R_m}\frac{\partial^2 B}{\partial \bar{z}\partial z} \end{array} \right. \quad (14)$$

but the product in the complex plane is commutative:

$$uB = Bu \quad (15)$$

then,

$$\begin{cases} i \frac{\partial}{\partial z} (p + \frac{\bar{u}u}{2} - N|B|^2 + U) - \frac{i\partial}{2\partial\bar{z}} (u^2 - NB^2 + \frac{2}{iR_e} \frac{\partial u}{\partial z}) = 0 \\ i \frac{\partial}{\partial z} (\frac{1}{2}\bar{u}B - \frac{1}{2}\bar{B}u) - \frac{1}{R_m} \frac{\partial^2 B}{\partial\bar{z}\partial z} = 0 \end{cases} \quad (16)$$

Here we left with two equations of the form:

$$\begin{cases} i \frac{\partial F_u}{\partial z} - \frac{i}{2} \frac{\partial G_u}{\partial\bar{z}} = 0 \\ i \frac{\partial F_B}{\partial z} - \frac{1}{R_m} \frac{\partial G_B}{\partial\bar{z}} = 0 \end{cases} \quad (17)$$

where  $F_u$  (similar to Bernoulli potential) and  $F_B$  are real functions of  $z$ , and  $G_u$  and  $G_B$  are complex functions of  $z$ . (Scholle *et al.*, 2011) wrote:

$$\begin{cases} F_u(z) = \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \\ F_B(z) = \frac{\partial^2 \Theta}{\partial z \partial \bar{z}} \end{cases} \quad (18)$$

for some real functions  $\Phi(z, \bar{z})$  and  $\Theta(z, \bar{z})$ , which allows "Eq. (16)" and "Eq. (17)" to be written as an exact differential:

$$\begin{cases} i \frac{\partial}{\partial\bar{z}} (\frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{2}G_u) = 0 \\ i \frac{\partial}{\partial\bar{z}} (\frac{\partial^2 \Theta}{\partial z^2} - \frac{1}{iR_m}G_B) = 0 \end{cases} \quad (19)$$

which implies

$$\begin{cases} (\frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{2}G_u) = f_u(z) \\ (\frac{\partial^2 \Theta}{\partial z^2} - \frac{1}{iR_m}G_B) = f_B(z) \end{cases} \quad (20)$$

the unknown functions of integration  $f_u(z)$  and  $f_B(z)$  may be absorbed into a suitable redefinition of  $\Phi$  and  $\Theta$ , leaving  $\partial_{zz}\Phi - \frac{1}{2}G_u = 0$  and  $\partial_{zz}\Theta - G_B/(iR_m) = 0$ .

One would integrate "Eq. (16)" and absorb arbitrary functions of integrations into the potentials  $\Phi$  and  $\Theta$  to obtain "Eq. (23)". In particular, it requires some thought to establish that one can absorb the complex functions  $f_u(z)$  into the real  $\Phi$  and  $f_B(z)$  into the real  $\Theta$ , by adding a function of the form  $F_u(z) + \overline{F_u(z)}$  and  $F_B(z) + \overline{F_B(z)}$ , when treating  $z$  and  $\bar{z}$  as independent variables. The two second derivatives of potentials,  $\Phi(z, \bar{z})$  and  $\Theta(z, \bar{z})$ , permit "Eq. (16)" to resolve this problem. With a similar procedure the potential  $\Theta(z, \bar{z})$  appears and plays the same role in the integrability of magnetic parcel of MHD equations:

$$\begin{cases} (p + \frac{\bar{u}u}{2} - N|B|^2 + U) = \frac{\partial^2 \Phi}{\partial\bar{z}\partial z} \\ (\frac{1}{2}\bar{u}B - \frac{1}{2}\bar{B}u) = \frac{\partial^2 \Theta}{\partial\bar{z}\partial z} \end{cases} \quad (21)$$

the pressure is obtained from the first equation of (21):

$$p = -\frac{\bar{u}u}{2} + N|B|^2 - U + \frac{\partial^2 \Phi}{\partial\bar{z}\partial z} \quad (22)$$

That is, when substituting "Eq. (21)" with "Eq. (16)", as a result we obtain:

$$\begin{cases} i \frac{\partial^2 \Phi}{\partial z^2} - \frac{i}{2} (u^2 - NB^2) = \frac{1}{R_e} \frac{\partial u}{\partial z} \\ i \frac{\partial^2 \Theta}{\partial z^2} = \frac{1}{R_m} \frac{\partial B}{\partial z} \\ \Im(\frac{\partial u}{\partial\bar{z}}) = 0 \\ \Im(\frac{\partial B}{\partial\bar{z}}) = 0 \end{cases} \quad (23)$$

Equation (23) is the first integral of magnetohydrodynamics equations in complex form, a complete set of field equations for the real values potentials  $\Theta(z, \bar{z})$ ,  $\Phi(z, \bar{z})$  the complex velocity  $u$ , and the complex magnetic field  $B$ . These equations are connected thanks to the square of magnetic fields  $B^2$  in the velocity equation (23).

#### 4. PARALELL FLOWS WITH $R_e = R_m$

From the equations (23) when  $R_m = R_e$  another way exist to decouple. By summing and subtracting both equations in (23) and multiplying the second equation by  $N^{\frac{1}{2}}$  we obtain:

$$i\left(\frac{\partial\Phi^2}{\partial z^2} \pm \sqrt{N}\frac{\partial^2\Theta}{\partial z^2}\right) - \frac{i}{2}(u^2 - (\sqrt{N}B)^2) = \frac{1}{R_e}\left(\frac{\partial u}{\partial z} \pm \sqrt{N}\frac{\partial B}{\partial z}\right) \quad (24)$$

this is equivalent to:

$$i\frac{\partial^2}{\partial z^2}(\Phi \pm \sqrt{N}\Theta) - \frac{i}{2}(u^2 - (\sqrt{N}B)^2) = \frac{1}{R_e}\frac{\partial}{\partial z}(u \pm \sqrt{N}B) \quad (25)$$

if introducing the well-known Elsasser variables (Elsasser, 1950; Davison, 2001):

$$\begin{cases} \Phi_1 = \Phi + \sqrt{N}\Theta \\ \Phi_2 = \Phi - \sqrt{N}\Theta \\ u_1 = u + \sqrt{N}B \\ u_2 = u - \sqrt{N}B \end{cases} \quad (26)$$

then the following system of equations is obtained:

$$\begin{cases} i\frac{\partial^2\Phi_1}{\partial z^2} - \frac{i}{2}u_1u_2 = \frac{1}{R_e}\frac{\partial u_1}{\partial z} \\ i\frac{\partial^2\Phi_2}{\partial z^2} - \frac{i}{2}u_1u_2 = \frac{1}{R_e}\frac{\partial u_2}{\partial z} \end{cases} \quad (27)$$

this system "Eq. (27)" is coupled and every equation cant be solved separately. Both are equations in complex form. If the originals solutions  $u_1$ ,  $\Phi_1$  and  $u_2$ ,  $\Phi_2$  are found, then  $u$  and  $B$  are found as:

$$\begin{cases} u = \frac{1}{2}(u_1 + u_2) \\ B = \frac{1}{2\sqrt{N}}(u_1 - u_2) \end{cases} \quad (28)$$

#### 5. PARALELL FLOWS WITH $u_1 = u_2$ AND $u_1 = -u_2$

In this cases when  $u_1 = u_2$  then  $u$  is solution of a individual complex first integral of Navier-Stokes equation and  $B = 0$ , There is the fields are decouples and magnetic field is null. The opposite case when  $u = u_1 = -u_2$  the situation change something  $u = 0$  and  $B$  is another solution of this same special kind of equation. The equations for  $u_1$  and  $u_2$  "Eq. (27)" are decoupled too in both situations. The solution of the both individual equations from "Eq. (27)" for example for  $u_1$  is a family of solutions which disconnect the velocity field, and we can write them in the following form:

$$\begin{cases} u = \frac{i}{R_e(F(z) - \frac{z}{2})} & B = 0 \\ B = \frac{1}{\sqrt{N}R_m(F(z) - \frac{z}{2})} & u = 0 \end{cases} \quad (29)$$

where  $F(z)$  any holomorphic function, and the second derivative of velocity-magnetic potential is express like this (for both situations):

$$\frac{\partial^2\Phi_1}{\partial z^2} = -\frac{\frac{\partial F(z)}{\partial z}}{R_e^2(F(z) - \frac{z}{2})^2} \quad (30)$$

and the velocity potential:

$$\Phi_1 = -\frac{1}{R_e^2} \int \int \frac{dF}{(F(z) - \frac{z}{2})^2} dz + Q_1(z) + \overline{Q_2(z)} \quad (31)$$

This second anti-derivatives must be chosen carefully because the potential  $(\Phi_1(z, \bar{z}), \Phi_2(z, \bar{z}))$  must fulfill:

$$\Phi_1, \Phi_2 : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{R} \quad (32)$$

The solutions (29) satisfy the (27) equations, with the potentials (31) and (32). When the inviscid flow is irrotational the term:  $\frac{\partial^2 \Phi}{\partial \bar{z} \partial z} = -N|B|^2$  complement Bernoulli's equation. The fluid will satisfy the Bernoulli equation or not, being kept viscous in both cases. This is a new aspect that appears as result of taking things to the complex plane, where there are viscous fluids that meet the complex Bernoulli equation. The found solution keeps this validity for a region in the parameter space where the triple product remains equal to unity.

## 6. CONCLUSION

For the case of steady, two-dimensional visco-resistive flow, and using a complex variable formulation, together with the introduction of a new two scalar potentials  $(\Phi, \Theta)$ , allows for a first integral of the magnetohydrodynamics equations: given conveniently as a complex-valued equation.

The complex variable formulation for two-dimensional viscous flow and electrically conducting fluid with two new scalars  $(\Phi$  and  $\Theta)$ , i.e. for the MHD equations, allows this problem, in  $\mathcal{R}^2$ , to be successfully brought to the complex plane  $\mathcal{C}$ . This first integral complex equation is particular because it opens a new range of complex solutions for an old problem, which is not necessarily rich in quadrature solutions. The applicability of this situation is not so clear, but from the theoretical point of view some benefits seem to be evident, both in the examination of the dynamic and stability, or other problems in fluid dynamics.

A problem that remains open is related to the three-dimensional case ( $\mathcal{R}^3$ ), which is to find a way of taking the set of connected equations (MHD) to arrival at an abundance of solutions. How to attempt this is not trivial, because we do not know the manifold in which to put this set of equations. We think that they could open new ways to study visco-resistive magnetohydrodynamics equations. But apparently certain difficulties interfere in this paradigm, the majority associated with the treatment of the vectorial operators in  $\mathcal{R}^3$  and at the same time encounter a complex representation. As of yet, we do not know how address those challenges.

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