SOLUTION BY INTEGRAL TRANSFORMATION OF CONJUGATE HEAT TRANSFER WITH ORTHOTROPIC DUCT

Lucas C. da Silva, lucascoelho.silva@hotmail.com[1]
Daniel J. N. M. Chalhub, daniel.chalhub@uerj.br[1]
[1]Group of Environmental Studies in Reservoirs – GESAR, Department of Mechanical Engineering – PPG-EM, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, Brazil

Abstract: *There is a growing interest for applications of heat and mass transfer in orthotropic ducts. In the realm of simulation, studies for heat transfer in composite ducts, this paper proposes a hybrid solution strategies for solving the conjugated heat transfer in an axisymmetric duct made of an orthotropic material, which the conductivity vary in three main directions. Since the wall material is anisotropic, the axial diffusion is also considered in this formulation. The final result will be the analysis of temperature distribution for this case. The formulation to be employed is the Generalized Integral Transform Technique (GITT) which in the realm of hybrid analytical-numerical methods, has been playing a big role. It deals with expansions of the sought solution in terms of infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. Due to the series representation nature of the Integral Transform Technique, the estimated error can be easily obtained, which results in better global error control of the solution. The Integral Transform Technique is employed as the main solution methodology. The presented results can serve as guidance for choosing an optimum solution methodology for this type of problem.*
Keywords: *Axysimmetric ducts, Orthotropic material, Conjugate Heat Transfer, Generalized Integral Transform Technique.*

1. Introduction

The solution of the differential equation which determines the temperature distribution in a body can be obtained by analytical and numerical methods. There are a variety methods for solving partial differential equations in engineering problems. Numerical methods are very useful and commonly used in heat and mass transfer engineering problems and also in partial differential equations which can not be handled by exact analytical solutions because of their non-linearities, non-constant properties, complex boundary conditions or complex geometries (Mikhailov and Osizik, 1984). The development of scientific computation over the decades has improved the use and application of numerical methods in various areas of engineering and science. However, this paper will only focus in the analyticall method. Each method has its comparative advantages depending on the physical nature of the problem.

Many works have studied convective heat transfer problems, among them, Guedes *et al.* (1990) have achieved an analytical solution for laminar forced convection inside ducts including wall conduction effects in the axial direction, based on a radially lumped wall temperature model, and accounting for external convection. The effects of external convection and axial conduction along the wall are then investigated through consideration of typical values for Biot number and a wall-to-fluid conjugation parameter. Convergence characteristics of the approach were also examined and studied. In his work, da Silva and Cotta (1996) studied the boundary layer equations at steady regime for incompressible laminar forced convection in entrance region of a parallel plate channel. They have been solved by integral transform method, adopting the formulation of the stream function. The convergence analysis was performed by providing benchmark results for the temperature field and Nusselt number for different values of Prandtl number. Gyves and Irvine Jr (1999) analyzed the developed laminar flow and forced convection in a conjugate problem of rectangular channels, the wall average Nusselt number was presented as a conduction function. Morini (2000) used the integral transform technique in his work, and obtained an accurate solution for the temperature field, and Nusselt numbers in thermally developed region of rectangular ducts, in which a laminar velocity profile is fully developed. The analytical results were a powerful tool, which allowed the investigation of forced internal convection in incompressible fluids. In his work, Naveira *et al.* (2009) presented a study of analytical and numerical solution for forced convection transient laminar flow, where the flow varies with time, and this was a conjugate problem involving conduction and convection. The solution is proposed from the available speed distributions based on Generalized Integral Transform Technique (GITT) combined with the method of lines. Li *et al.* (2009) worked on an article about orthotropic rectangular plates. Exact solutions of these plates subjected to arbitrary loads are derived by the method of finite integral transform. In the proposed method, it was not necessary to predetermine the deformation function, because only the basic governing equations of classical theory for orthotropic plates are used. The numerical results have been presented to demonstrate the validity and accuracy of the approach, compared with those previously reported in the bibliography of the article. Knupp *et al.* (2012) presented a work of an analytical approach for conjugated conduction-convection heat transfer problems, by proposing a single domain formulation for modeling both the fluid stream and the channel wall regions. The GITT was employed in the hybrid numerical-analytical solution, and the excellent agreement between approximate and exact solutions have shown the success of the approach to deal with conjugated problems. Knupp *et al.* (2013) have studied an extension of a proposed single domain formulation of conjugated conduction–convection heat transfer problems, regarding the axial diffusion effects at both the walls and fluid regions. The single domain formulation with GITT was again employed and the converged results had confirmed the adequacy of this single domain approach in handling conjugated heat transfer problems. Braga Jr *et al.* (2014) proposed a methodology for obtaining fully analytical solutions for the extended Graetz-Brinkman problem including the effects of axial conduction in infinite and semi-infinite. The methodology was based on the GITT, and the authors used a simple eigenfunction basis in terms of Helmholtz problems.

Chalhub *et al.* (2013) presented a novel methodology for solving unsteady convective heat transfer problems via the Generalized Integral Transform Technique. Instead of transforming advection terms, an upwind approximation was used previously to the integral transformation. The application had been demonstrated for a general multi-dimensional problem, and numerical results for a one-dimensional test case were calculated. Chalhub *et al.* (2016) presented an analytical solution for an extended version of the Graetz problem for slip-flow in parallel-plates channels. The problem formulation included axial heat diffusion in a semi-infinite channel with a given inlet condition and isothermal walls. The GITT was the solution methodology, a convergence analysis of the results was done, and it showed that better convergence rates are obtained for larger values of the Peclet and Knudsen numbers, even in the near-entrance region.

This work proposes an semi-analytical approach to solve the conjugate heat-transfer in a duct, where the wall is made of a composite anisotropic material. The single domain GITT formulation (Knupp *et al.*, 2012, 2013) is utilized in the present work and since the wall material is anisotropic, the axial diffusion is also considered in this formulation.

2. PROBLEM FORMULATION

In order to solve the conjugate heat-transfer problem, one needs to solve the energy equation. The flow is assumed to be hydrodynamically developed but thermally developing, with negligible heating by viscous dissipation and temperature independent of physical properties. The duct is considered to be axyssimetric, and the fluid flows with a known fully developed laminar velocity profile. The duct solid wall ($a \leq r \leq b$) has anisotropic conductivity. The inlet temperature is prescribed and there is a fluid flowing outside the duct, resulting in a Robin (third kind) boundary condition at $r = b$. The general formulation of conjugated problem in classical two-dimensional cylindrical coordinates is given by:

$$u(r)\, w_f \frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(k_r(r)\, r \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z(r)\frac{\partial T}{\partial z}\right) \qquad \text{for} \quad 0 \leq z \leq \infty \quad \text{and} \quad 0 \leq r \leq b \tag{1}$$

$$T = T_{\text{in}} \qquad \text{for} \quad z = 0 \quad \text{and} \quad 0 \leq r \leq b \tag{2}$$

$$\frac{\partial T}{\partial z} = 0 \qquad \text{for} \quad z \to \infty \quad \text{and} \quad 0 \leq r \leq b \tag{3}$$

$$\frac{\partial T}{\partial r} = 0 \qquad \text{for} \quad r = 0 \quad \text{and} \quad z \geq 0 \tag{4}$$

$$-k_r(r)\frac{\partial T}{\partial r} = h\,(T - T_f) \qquad \text{for} \quad r = b \quad \text{and} \quad z \geq 0 \tag{5}$$

$$\tag{6}$$

where $u(r)$ is the parabolic fully developed velocity profile, that flows between ($0 \leq r \leq a$), w_f is the inner fluid heat capacity, $k_r(r)$ and $k_z(r)$ are the thermal conductivities in direction r and z respectively. T_f is the external environment temperature, T_{in} is the temperature at entrance of channel and h is the heat transfer coefficient.

The following dimensionless groups are defined:

$$K_r(\eta) = \frac{k_r(r)}{k_f}; \quad K_z(\eta) = \frac{k_z(r)}{k_f}; \quad \tilde{k}_r = \frac{k_{sr}}{k_f}; \quad \tilde{k}_z = \frac{k_{sz}}{k_f}; \tag{7}$$

$$\xi = \frac{z\,k_f}{b^2\,\bar{u}\,w_f}; \quad \eta = \frac{r}{b}; \quad \beta = \frac{a}{b}; \quad \theta = \frac{T - T_f}{T_{\text{in}} - T_f}; \quad u^*(r) = \frac{u(r)}{\bar{u}}; \tag{8}$$

$$\text{Pe} = \frac{(2a)\bar{u}\,k_f}{w_f}; \quad \text{Bi} = \frac{hb}{k_{sr}} \tag{9}$$

$$u^*(\eta) = \begin{cases} 2(1 - \frac{\eta^2}{\beta^2}) & \text{if } \eta \leq \beta \\ 0 & \text{if } \eta > \beta \end{cases} \tag{10}$$

$$K_r(\eta) = \begin{cases} 1 & \text{if} \quad \eta \leq \beta \\ \tilde{k}_r & \text{if} \quad \eta > \beta \end{cases} \tag{11}$$

$$K_z(\eta) = \begin{cases} 1 & \text{if} \quad \eta \leq \beta \\ \tilde{k}_z & \text{if} \quad \eta > \beta \end{cases} \tag{12}$$

Where k_f is the fluid thermal conductivity, k_{sr} is the solid thermal conductivity in r-direction, k_{sz} is the solid thermal conductivity in z-direction, $\bar{u}$ is the fluid average velocity ($0 \leq r \leq a$), a is the inner radius, and b is the outer radius, Bi is the Biot number, Pe is the Péclet number, $K_r(\eta)$, $K_z(\eta)$, θ, ξ, η and $u^*(\eta)$ are nondimensional versions of $k_r(r)$, $k_z(r)$, T, z, r and $u(r)$ respectively and β is the aspect ratio.

After nondimensionalization, equation (1) can be rewritten as follows, with proper boundary conditions for the problem:

$$u^*(\eta)\,\frac{\partial\theta}{\partial\xi} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(K_r(\eta)\,\eta\frac{\partial\theta}{\partial\eta}\right) + \frac{4\,K_z(\eta)\,\beta^2}{\mathrm{Pe}^2}\frac{\partial^2\theta}{\partial\xi^2} \qquad \text{for} \qquad 0 \le \xi \le \infty \qquad \text{and} \qquad 0 \le \eta \le 1 \tag{13}$$

$$\theta(0,\eta) = 1; \qquad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=1} = 0; \qquad \left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0} = 0; \qquad \left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} = -\mathrm{Bi}\,\theta(\xi,1). \tag{14}$$

For this problem, the local Nusselt number can be calculated using the following formulation:

$$\mathrm{Nu}(\xi) = \frac{2\beta(\partial\theta/\partial\eta)_{\eta=\beta}}{\theta(\xi,\beta) - (2/\beta^2)\int_0^\beta u^*\theta\eta\,\mathrm{d}\eta} \tag{15}$$

2.1 Generalized Integral Transform Technique (GITT)

One alternative for handling the proposed problem is now more closely considered. The formulation is based on the Generalized Integral Transform Technique (GITT) and the approach begins with the choice of an eigenfunction expansion that is originated from an auxiliary eigenproblem and for the current problem, the Bessel eigenproblem is chosen due to the nature of the coordinate system. The transformation pair is then defined, and afterward the equation is transformed multiplying by the eigenfunction and integrating in the domain. The resulting transformed equation is then solved using an matrix algebra approach. Finally, making use of the inversion formula, the final step is to obtain the analytical solution of each potential (Chalhub, 2011).

In order to solve the temperature distribution problem with GITT, the appropriate Bessel eigenvalue problem is considered:

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}R_n}{\mathrm{d}\eta}\right) + \lambda_n^2 R_n = 0 \tag{16}$$

$$R_n'(0) = 0; \qquad R_n'(1) = -\mathrm{Bi}\,R_n(1)\,. \tag{17}$$

where R_n are the eigenfunctions, λ_n are the eigenvalues, and the solution to the Bessel equation is given by:

$$R_n(\eta) = J_o(\lambda_n\eta) \tag{18}$$

The eigenvalues can now be calculated by a numerical method from the following equation below:

$$\lambda_n = \frac{\mathrm{Bi}\,J_o(\lambda_n)}{J_1(\lambda_n)} \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{19}$$

The transformation pair is defined by:

$$\text{Transformation} \quad \Rightarrow \quad \bar{\theta}_n(\xi) = \int_0^1 \theta(\xi,\eta)R_n(\eta)\eta\,\mathrm{d}\xi \tag{20}$$

$$\text{Inversion} \quad \Rightarrow \quad \theta(\xi,\eta) = \sum_{n=1}^{\infty}\frac{\bar{\theta}_n(\xi)R_n(\eta)}{N_n} \tag{21}$$

where the norms N_n are defined by:

$$N_n = \int_0^1 R_n^2\,\eta\,\mathrm{d}\eta \tag{22}$$

The transformation of the given problem is accomplished by multiplying equation (13) by ηR_n, integrating within $0 \le \eta \le 1$, and applying the inversion formula to the non-transformable terms. This process yields the following coupled system of ODEs:

$$\sum_{m=1}^{\infty} B_{n,m}\bar{\theta}_m''(\xi) = \sum_{m=1}^{\infty} A_{n,m}\bar{\theta}_m'(\xi) + \sum_{m=1}^{\infty} C_{n,m}\bar{\theta}_m(\xi) \tag{23}$$

$$\bar{\theta}_n(0) = \int_0^1 R_n(\eta)\,\eta\,\mathrm{d}\eta = b_n \tag{24}$$

$$\bar{\theta}_n(\xi \to \infty) = 0 \tag{25}$$

where the coefficients $A_{n,m}$, $B_{n,m}$ e $C_{n,m}$ are given by:

$$A_{n,m} = -\frac{1}{N_m}\int_0^1 u^* R_m R_n \eta \,\mathrm{d}\eta \tag{26}$$

$$B_{n,m} = \frac{4\,\beta^2}{\mathrm{Pe}^2 N_m}\int_0^1 K_z R_m R_n \eta \,\mathrm{d}\eta \tag{27}$$

$$C_{n,m} = \frac{1}{N_m}\left[\mathrm{Bi}\,\tilde{k}_{sr}\, R_n(1)\, R_m(1) + \int_0^1 K_r R'_m R'_n \eta \,\mathrm{d}\eta\right] \tag{28}$$

All coefficients above are computed analytically except $A_{n,m}$ which needs to be calculated numerically due to the non-existent analytical solution.

This system of equations can be rewritten in the matrix form:

$$\mathbf{B}\boldsymbol{\theta}'' = \mathbf{A}\boldsymbol{\theta}' + \mathbf{C}\boldsymbol{\theta} \tag{29}$$

$$\boldsymbol{\theta}(0) = \boldsymbol{b}; \qquad \boldsymbol{\theta}(\xi \to \infty) = \mathbf{0} \tag{30}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrix representations of the coefficients $A_{n,m}$, $B_{n,m}$ and $C_{n,m}$ respectively, $\boldsymbol{\theta}$ and $\boldsymbol{b}$ are vector representations of $\bar{\theta}_m$ and b_m respectively and $\mathbf{0}$ is the null vector.

In order to solve this system of equations, an analytical solution is proposed. A change of variable is now introduced:

$$\boldsymbol{y}\,(\xi) = \left(\bar{\theta}_1\,(\xi)\,,\bar{\theta}_2\,(\xi)\,,\bar{\theta}_3\,(\xi)\,,...,\bar{\theta}_{n_{\max}}\,(\xi)\,,\bar{\theta}'_1\,(\xi)\,,\bar{\theta}'_2\,(\xi)\,,\bar{\theta}'_3\,(\xi)\,,...,\bar{\theta}'_{n_{\max}}\,(\xi)\right) \tag{31}$$

Equation (29) can be rewritten in the following modified form:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\xi} = \mathbf{M}\boldsymbol{y} \tag{32}$$

Where the matrix $\mathbf{M}$ is defined as:

$$\mathbf{M} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{E} & \mathbf{D} \end{pmatrix} \tag{33}$$

where $\mathbf{O}$ is the null matrix, $\mathbf{I}$ is the identity matrix and $\mathbf{E}$ and $\mathbf{D}$ are given by:

$$\mathbf{D} = \mathbf{B}^{-1}\mathbf{A} \tag{34}$$

$$\mathbf{E} = \mathbf{B}^{-1}\mathbf{C} \tag{35}$$

One can obtain the solution of the modified system by integrating analytically. The eigenvalues and eigenvectors of $\mathbf{M}$ are calculated so that the solution of the components of $\boldsymbol{y}$ can be written in the following form:

$$y_n\,(\xi) = \sum_{m=1}^{2n_{\max}} G_{n,m} c_m \exp\left(\omega_m\,\xi\right), \quad \text{for} \quad n = 1, 2, 3, \ldots, 2n_{\max} \tag{36}$$

in which $G_{n,m}$ are the coefficients of a matrix $\mathbf{G}$ containing the eigenvectors of $\mathbf{M}$ as columns, ω_m are the eigenvalues of $\mathbf{M}$, and c_m are arbitrary constants. The matrix $\mathbf{M}$ yields $n_{\max}$ positive eigenvalues and $n_{\max}$ negative eigenvalues and since the solution when ξ gets larger must converge to a finite value, one can say that:

$$c_n = 0 \qquad \text{if} \qquad \omega_n > 0 \qquad \text{for} \qquad n = 1, 2, 3, ..., 2n_{\max} \tag{37}$$

The algebraic system allows to calculate the remaining c_n values, according to inlet conditions:

$$\sum_{m=1}^{2n_{\max}} G_{n,m} c_m = b_n \qquad \text{for} \qquad n = 1, 2, 3, ..., n_{\max} \tag{38}$$

The solution of transformed potential is obtained, and then the temperature field is calculated by applying the inversion formula and the Nusselt number can be obtained directly from the transformed temperatures using the following expression:

$$\mathrm{Nu}(\xi) = \frac{\sum_{n=1}^{\infty} \bar{\theta}_n \frac{2\beta}{N_n} R'(\beta)}{\sum_{n=1}^{\infty} \frac{\bar{\theta}_n}{N_n}\left[R_n(\beta) - \frac{2}{\beta^2}\int_0^\beta u^* R_n \eta \,\mathrm{d}\eta\right]} \tag{39}$$

3. Results and discussion

In this section, a discussion on the numerical results solved by the problem formulation presented in the previous section is made. The goal here is to evaluate solution convergence by GITT. Results presented in the following tables have been analyzed mainly for different Biot and Peclet numbers , according the axial position (ξ), truncation orders ($n_{\max}$) and two combinations of $\tilde{k}_z$ and $\tilde{k}_r$. The Nusselt number is chosen for the convergence analysis due to its importance in this kind of heat transfer problem.

Table 1 introduces the main results for GITT, it is analyzed the local Nusselt number convergence at different axial

Table 1: Nusselt convergence solved by GITT for Pe= 1, β= 0.8, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	24.9386	8.73310	4.59526	4.07013	59.3387	12.7663	8.09697	7.34972
20	16.6396	8.86891	4.60797	4.07462	52.9510	10.0700	8.07524	7.34203
30	13.0605	8.93626	4.60793	4.07225	38.9362	9.51772	8.03717	7.31200
40	11.4494	8.96457	4.60399	4.06759	29.9504	9.35841	7.99681	7.27766
50	10.6344	8.97535	4.59823	4.06179	24.2268	9.28162	7.95492	7.24100
60	10.1871	8.97664	4.59131	4.05521	20.3801	9.22285	7.91115	7.20213
70	9.92637	8.97182	4.58341	4.04790	17.6561	9.16620	7.86468	7.16052
80	9.76651	8.96220	4.57443	4.03971	15.6269	9.10620	7.81402	7.11493
90	9.66558	8.94766	4.56385	4.03018	14.0046	9.03792	7.75585	7.06237
100	9.66424	8.92294	4.54882	4.01676	11.9168	8.93935	7.67148	6.98587
	Bi=10							
10	38.3267	11.1952	4.38735	3.76267	60.3647	18.9176	8.60404	7.22643
20	33.5521	11.3415	4.37869	3.75031	103.777	13.6426	8.61157	7.25189
30	24.1883	11.4465	4.37421	3.74462	95.0136	12.0450	8.57293	7.22679
40	19.1150	11.4958	4.36862	3.73889	76.3690	11.6154	8.52947	7.19407
50	16.4139	11.5182	4.36216	3.73279	60.7474	11.4569	8.48415	7.15827
60	14.9104	11.5259	4.35497	3.72625	49.0647	11.3640	8.43690	7.12003
70	14.0381	11.5241	4.34705	3.71919	40.3727	11.2860	8.38686	7.07897
80	13.5166	11.5150	4.33821	3.71143	33.7381	11.2077	8.33244	7.03393
90	13.2079	11.4989	4.32793	3.70247	28.3613	11.1207	8.27005	6.98195
100	13.2781	11.4689	4.31342	3.68994	21.2508	10.9969	8.17962	6.90615

Table 2: Nusselt convergence solved by GITT for Pe= 10, β= 0.8, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	8.79019	4.24575	3.44971	3.44971	13.5674	7.80217	6.28704	6.28701
20	8.86173	4.23896	3.44112	3.44112	11.6492	7.83561	6.31029	6.31026
30	8.89617	4.23253	3.43481	3.43481	11.2722	7.81772	6.29521	6.29519
40	8.90681	4.22568	3.42871	3.42871	11.1662	7.78809	6.27113	6.27110
50	8.90678	4.21846	3.42252	3.42251	11.1092	7.75311	6.24285	6.24282
60	8.90083	4.21082	3.41610	3.41610	11.0577	7.71432	6.21155	6.21152
70	8.89086	4.20266	3.40932	3.40931	11.0020	7.67176	6.17726	6.17723
80	8.87744	4.19373	3.40196	3.40195	10.9388	7.62442	6.13912	6.13909
90	8.86003	4.18350	3.39357	3.39357	10.8635	7.56924	6.09469	6.09466
100	8.83319	4.16932	3.38200	3.38199	10.7501	7.48813	6.02939	6.02937
	Bi=10							
10	10.7146	4.02746	3.26900	3.26900	17.3373	7.75672	6.01095	6.01095
20	10.7943	4.00833	3.25229	3.25229	13.5830	7.81416	6.05585	6.05584
30	10.8635	4.00033	3.24520	3.24520	12.5117	7.79792	6.04412	6.04411
40	10.8937	3.99326	3.23913	3.23913	12.2288	7.76808	6.02157	6.02156
50	10.9047	3.98617	3.23317	3.23317	12.1203	7.73270	5.99454	5.99453
60	10.9051	3.97881	3.22706	3.22706	12.0490	7.69358	5.96449	5.96448
70	10.8984	3.97101	3.22063	3.22063	11.9820	7.65077	5.93152	5.93151
80	10.8861	3.96251	3.21366	3.21366	11.9097	7.60325	5.89484	5.89484
90	10.8680	3.95280	3.20573	3.20573	11.8253	7.54796	5.85212	5.85211
100	10.8375	3.93930	3.19474	3.19474	11.6999	7.46670	5.78924	5.78923

Table 3: Nusselt convergence solved by GITT for Pe= 1, β= 0.9, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	-24.9032	11.8685	4.77098	4.00867	292.464	9.05233	8.71448	7.50106
20	25.8121	12.3238	4.80417	4.02972	72.0341	14.4669	8.63079	7.44034
30	8.58828	12.4807	4.81216	4.03386	-38.2978	13.0112	8.56437	7.38832
40	18.3809	12.5577	4.81225	4.03262	44.7887	13.2422	8.50752	7.34213
50	12.9224	12.5965	4.80881	4.02895	-11.3675	13.0378	8.45427	7.29795
60	16.3312	12.6146	4.80333	4.02383	30.9638	12.9724	8.40169	7.25377
70	14.2496	12.6197	4.79635	4.01760	-1.48503	12.8691	8.34772	7.20805
80	15.6137	12.6152	4.78794	4.01026	23.7744	12.7753	8.29012	7.15898
90	14.7446	12.6018	4.77769	4.00146	3.15032	12.6673	8.22471	7.10302
100	15.4236	12.5733	4.76291	3.98893	18.3959	12.5194	8.13165	7.02309
	Bi=10							
10	815.415	15.7255	4.50087	3.62432	86.3093	8.72404	9.03581	7.07751
20	45.6724	16.2721	4.49508	3.61661	112.879	19.8168	9.03687	7.08931
30	3.72534	16.4895	4.49511	3.61519	-152.590	16.6098	8.98032	7.05089
40	30.2469	16.6048	4.49228	3.61212	93.2234	17.3239	8.92428	7.01032
50	15.3625	16.6663	4.48753	3.60782	-50.6483	16.9232	8.86976	6.96968
60	24.7440	16.6979	4.48147	3.60262	63.4281	16.8717	8.81517	6.92830
70	18.9663	16.7102	4.47431	3.59662	-22.1934	16.7200	8.75882	6.88511
80	22.7733	16.7086	4.46598	3.58975	45.5964	16.5981	8.69848	6.83853
90	20.3730	16.6943	4.45604	3.58163	-9.26427	16.4531	8.62983	6.78522
100	22.3424	16.6586	4.44187	3.57014	31.5850	16.2578	8.53171	6.70865

Table 4: Nusselt convergence solved by GITT for Pe= 10, β= 0.9, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	11.5356	4.45525	3.39984	3.39981	10.4834	8.35705	6.38527	6.38515
20	11.9049	4.47013	3.40800	3.40797	15.1630	8.32587	6.35740	6.35728
30	12.0262	4.47165	3.40791	3.40788	13.9792	8.27996	6.32234	6.32222
40	12.0836	4.46871	3.40502	3.40499	14.1941	8.23433	6.28772	6.28760
50	12.1106	4.46369	3.40080	3.40078	14.0253	8.18845	6.25288	6.25277
60	12.1209	4.45738	3.39574	3.39571	13.9671	8.14131	6.21704	6.21693
70	12.1207	4.45002	3.38995	3.38992	13.8715	8.09172	6.17931	6.17920
80	12.1125	4.44156	3.38336	3.38333	13.7799	8.03792	6.13833	6.13822
90	12.0968	4.43155	3.37563	3.37560	13.6716	7.97607	6.09120	6.09109
100	12.0671	4.41747	3.36482	3.36479	13.5181	7.88706	6.02335	6.02324
	Bi=10							
10	14.8838	4.20730	3.19321	3.19320	9.93832	8.20466	5.96452	5.96448
20	15.3400	4.19445	3.18350	3.18350	18.6137	8.24153	5.98720	5.98715
30	15.5207	4.19169	3.18104	3.18104	16.1234	8.20474	5.96074	5.96069
40	15.6155	4.18761	3.17771	3.17770	16.6997	8.16138	5.92971	5.92967
50	15.6648	4.18231	3.17353	3.17353	16.3874	8.11634	5.89738	5.89733
60	15.6886	4.17607	3.16870	3.16869	16.3423	8.06965	5.86375	5.86370
70	15.6960	4.16897	3.16323	3.16323	16.2137	8.02041	5.82820	5.82815
80	15.6913	4.16090	3.15705	3.15705	16.1045	7.96693	5.78953	5.78948
90	15.6754	4.15140	3.14980	3.14979	15.9725	7.90542	5.74499	5.74495
100	15.6400	4.13802	3.13962	3.13961	15.7892	7.81664	5.68065	5.68061

positions, ranging from 0.01 up to 10, for different Biot numbers and thermal conductivities. Nusselt values were presented for different truncation orders, so that it is possible to analyze the convergence behavior. Converged three digits are noticed for position ξ = 0.1, in which k_z=0.5 and k_r=1.5, of table 1 among 50 and 70 terms in the series, for Bi=1. Meanwhile for position ξ = 1, we have three digits converged among 20 and 40 terms in the series. On the table 2, it's realized three converged digits for Bi=10, in which k_z=0.5 and k_r=1.5, for position ξ=0.01 among 70 and 100 terms in the series, and at position ξ=0.1, also three digits converged behavior among 10 and 30 terms, and for 60 and 70 terms as well. On table 3, for Bi=1 is possible to see that convergence occurs for positions ranging from ξ= 0.1 until ξ=10 just for 2 meaningful digits, except for position ξ=0.1, which we have four converged digits, among 60 and 80 terms. When Bi is

Table 5: Nusselt convergence solved by GITT for $\mathrm{Pe} = 10^3$, β= 1 and $\mathrm{Bi} = 10^6$.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
10	7.49530	4.00603	3.65744	3.65744	7.49543	4.00611	3.65751	3.65751
20	7.48211	4.00499	3.65682	3.65682	7.48239	4.00514	3.65696	3.65696
30	7.48065	4.00484	3.65672	3.65672	7.48108	4.00508	3.65693	3.65693
40	7.48023	4.00478	3.65667	3.65667	7.48081	4.00509	3.65695	3.65695
50	7.48004	4.00473	3.65662	3.65662	7.48077	4.00512	3.65698	3.65698
60	7.47991	4.00468	3.65659	3.65659	7.48079	4.00516	3.65702	3.65702
70	7.47980	4.00464	3.65655	3.65655	7.48083	4.00520	3.65705	3.65705
80	7.47973	4.00460	3.65651	3.65651	7.48087	4.00523	3.65709	3.65709
90	7.47969	4.00456	3.65647	3.65647	7.48091	4.00527	3.65713	3.65713
100	7.47965	4.00453	3.65644	3.65644	7.48094	4.00530	3.65716	3.65716

increased to 10, there is no convergence of four digits.

On table 4 the convergence of 2 digits is not obtained at position ξ= 0.01 among 20 and 100 terms in the summation. But this occurs for Bi=10. And also 2 digits from 10 until 100 terms at position ξ=0.1, that does not happen for when Bi=10. For both tables it was not still noticed fully-converged six digits. In regards to table 5, one can confirm what the literature (Shah and London, 1978) claim about Nusselt local number to be equal to 3.66 for larger ξ, Péclet numbers, Biot Numbers and negligible duct wall. One should note that for larger Bi the boundary condition approaches to the prescribed temperature in $\eta = 1$. The result is that neglecting the axial diffusion, due the high value of Péclet number, the local Nusselt number converges to 3.66.

This effect of range of convergence rate seems to be strongly dependent on both the Peclet number and the Biot number. For larger Pe values, notably better convergence rates are seen and it also seems to improve for larger Bi values. In tables 2 and 4 in spite of larger Péclet Number, a low convergence rate occurs in the entrance of channel in which in which k_z=1.5 and k_r=0.5. In all tables we can see a worse convergence rate for Pe=1 due to greater influence of the axial diffusion term. On the other hand, when Peclet number increases for 10, the convergence rate improves considerably. For all tables, at the entrance of channel at position ξ= 0.01, is verified the worst convergence rate due to the boundary condition discontinuity.

4. Conclusions

This paper presented a semi-analytical solution for the conjugate heat transfer problem in a duct with anisotropic wall material considering axial diffusion. It was shown the convergence behavior of local Nusselt. In this work, the axial diffusion has been considered and the solution methodology was based on the Generalized Integral Transform Technique, and a simple eigenvalue problem with analytical solution was employed for the transformation. Although a coupled ODE system was obtained, the equations could be solved analytically by rewriting this system in a modified form and employing a matrix method. The results were verified by a comparisons with the datas from the literature and it achieved good agreement. A convergence analysis of the solution showed that very good converge rates are seen for larger Peclet, Biot and aspect ratio values. On the other hand, a worse convergence behavior was seen for the beginning of the duct length. This is due to the boundary condition discontinuity at the entrance of the channel. A worse convergence behavior was also observed for smaller values of Peclet and Biot. Regarding the convergence analysis, illustrative results were presented, showing the variation of the local Nusselt number with axial positions for different values of Peclet, Biot, β and thermal conductivities.

5. ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support provided by FAPERJ, CAPES and Universidade do Estado do Rio de Janeiro.

6. REFERENCES

Braga Jr, N.R., de Barros, L.S. and Sphaier, L.A., 2014. “Generalized integral transform solution of extended graetz problems with axial diffusion”. *ICCM*.

Chalhub, D.J.N.M., 2011. *Desenvolvimento de Soluções Para Problemas de Advecção-Difusão Combinando Tranformação Integral e Métodos Discretos*. Master’s thesis, Universidade Federal Fluminense, Niterói-RJ.

Chalhub, D.J.N.M., Sphaier, L.A. and de B. Alves, L.S., 2013. “Integral transform solution of convective heat transfer problems using upwind approximations”. *Numerical Heat Transfer*, Vol. 63, pp. 167–187.

Chalhub, D.J.N.M., Sphaier, L.A. and de B. Alves, L.S., 2016. “Integral transform solution for thermally-developing slip-flow within isothermal parallel plates”. *Computational Thermal Sciences*.

da Silva, E.F. and Cotta, R.M., 1996. “Benchmark results for internal forced convection through integral transformation”. *International Communications in Heat and Mass Transfer*, Vol. 23, pp. 1019–1029.

Guedes, R.O.C., Cotta, R.M. and Brum, N.C.L., 1990. "Heat transfer in laminar flow with wall axial conduction and external convection". *J. Thermophysics*, Vol. 5.

Gyves, T.W. and Irvine Jr, T.F., 1999. "Laminar conjugated forced convection heat transfer in curved rectangular channels". *International Journal of Heat and Mass Transfer*, pp. 3953–3964.

Knupp, D.C., Naveira-Cotta, C.P. and Cotta, R.M., 2012. "Theoretical analysis of conjugated heat transfer with a single domain formulation and integral transforms". *International Communications in Heat and Mass Transfer*, Vol. 39, pp. 355–362.

Knupp, D.C., Naveira-Cotta, C.P. and Cotta, R.M., 2013. "Conjugated convection-conduction analysis in microchannels with axial diffusion effects and a single domain formulation". *International Journal of Heat and Mass Transfer*, Vol. 135.

Li, R., Zhong, Y., Tian, B. and Liu, Y., 2009. "On the finite integral transform method for exact bending solutions of fully clamped orthotropic rectangular thin plates". *Applied Mathematics Letters*, pp. 1821–1827.

Mikhailov, M.D. and Osizik, M.N., 1984. *Unified Analysis and Solutions of Heat and Mass Diffusion*. John Wiley & Sons, New York.

Morini, G.L., 2000. "Analytical determination of the temperature distribution and nusselt numbers in rectangular ducts with constant axial heat flux". *International Journal of Heat and Mass Transfer*, pp. 741–755.

Naveira, C.P., Lachia, M., Cotta, R.M. and Padet, J., 2009. "Hybrid formulation and solution for transient conjugated conduction-external convection". *International Journal of Heat and Mass Transfer*, pp. 112–123.

Shah, R.K. and London, A.L., 1978. *Laminar flow forced convection in ducts: A source book for compact heat exchanger analytical data. In T. F. Irvine Jr. and J. P. Hartnet, editors, Advances in Heat Transfer*. Academic Press, New York, USA.

7. RESPONSIBILITY NOTICE

The following text, properly adapted to the number of authors, must be included in the last section of the paper: The author(s) is (are) the only responsible for the printed material included in this paper.