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TEMPERATURE FIELD EVALUATION OF A TWO-DIMENSIONAL PARTIAL HEAT FLOW PROBLEM THROUGH GREEN'S FUNCTIONS

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Abstract. *Analytical solution of heat conduction problems is an important alternative because, among other factors, it presents a lower computational cost when compared with numerical solutions. Green's Functions method is an important method of analytical solutions because of the possibility of solving transient problems of heat conduction, with internal generation and with variable conditions of time. In other words, Green's Functions are mathematical tools used to obtain the temperature field in linear, transient or permanent heat conduction problems with internal generation. In the present work, a two-dimensional transient heat conduction problem is presented, with boundary conditions: prescribed temperature - thermal insulation on one Cartesian axis and partial prescribed heat flow - partial prescribed flow on the other axis. Analytical solution of this problem is obtained through Green´s Functions. Intrinsic verification of the solution is performed primarily by changes in the geometry and boundary conditions imposed. Subsequently, another verification is performed, comparing the original 2D problem with a 1D problem with boundary conditions prescribed temperature - thermal insulation in y axes. Results presented after the checks validate the analytical solution obtained for the original proposed problem.*

Keywords: *Analytical solution, prescribed flow, prescribed temperature, temperature field, intrinsic verification.*

1. INTRODUCTION

Analytical solutions are an important tool to obtain solutions of several thermal engineering problems and present a lower computational cost when compared to numerical solutions. In heat conduction problems, such solutions can be obtained, for example, by the method of separating variables, Laplace transform, Duhamel's Theorem or Green's Functions. Green's Functions (GF) are powerful tools to obtain linear, transient or permanent heat conduction solutions, thus they can be applied to solutions of some convection problems and to many other phenomena that are described by the same type of equations (Beck et al, 1992). In addition, GF can also be used to analyse problems with transient and non-uniform heat generation with mobile heat sources. The use of GF to analyse 2D or 3D problems makes the model closer to the real object of study. The obtainment of those solutions is, from the product of the one-dimensional GF of their respective Cartesian axes, making this tool more effective in solving heat conduction problems when compared to other tools.

Oliveira et al. (2017) used Green Functions to solve a 1D multilayer problem with flow imposed on one surface and isolated on the other. This problem was used to analyse the temperature field in a cutting tool coated in a machining process, since in these processes, tool can reach temperatures higher than 900 °C (Trent and Wright, 2000), which may have a significant effect on the metallurgical microstructure (Liu et al., 2004). Menegaz and Guimarães (2019) used Green's Functions to solve a 3D problem of bioheat transfer, whose governing equation is the Bioheat Transfer Equation, presented by Pennes (1948), aiming analysed the surface temperature field changes by the influence of the presence of an inclusion on a phantom tissue.

The authors, through this work, aim to solve a two-dimensional heat conduction problem using Green's Functions. Specifically, through the intrinsic verification method, the authors aim to perform the validation of the analytical solution of the proposed original problem temperature field.

2. MATHEMATICAL MODELING

A proposed problem is showed on Fig. 1. This is a X22Y12 problem, according Beck et al. (1992), where number 1 refers to the Dirichlet's boundary condition (prescribed temperature) and 2 to the Neumann's boundary condition (prescribed heat flow). The plate has L x W dimensions, where W = L = 0.05 m, initially at an initial temperature $T(x,y,0) = T_0 = 28$ °C, prescribed temperature in one of the surfaces $T_1 = 50$ °C, thermal conductivity $k = 0.159\ Wm^{-1}K^{-1}$ and thermal diffusivity $\alpha = 0.157\ mm^2s^{-1}$. Heat flow considered were $q_1 = 5q_0 = 500\ kWm^{-2}$.

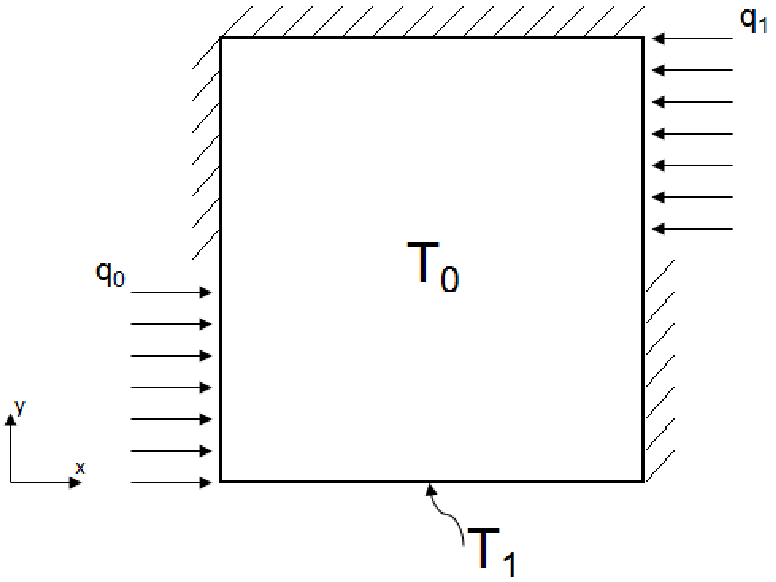

Figure 1 - Original proposed 2D problem X22Y12.

The problem represented by Fig. (1) can be described by Eq. (1). Boundary conditions were showed on Fig. (1).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

A Green's Functions based solution was used to achieve the temperature development through time for different x and y values. The Green's Function for this problem is given by Eq. (2).

$$G_{X22Y12} = \left[\frac{1}{L} + \frac{2}{L}\sum_{m=1}^{\infty} e^{\frac{-\alpha_m^2 \alpha(t-\tau)}{L^2}} \cos\left(\frac{\alpha_m x}{L}\right)\cos\left(\frac{\alpha_m x}{L}\right)\right]\frac{2}{W}\sum_{n=1}^{\infty} e^{\frac{-\beta_n^2 \alpha(t-\tau)}{W^2}} \sin\left(\frac{\beta_n y}{W}\right)\sin\left(\frac{\beta_n y'}{W}\right) \tag{2}$$

where $\alpha_m = m\pi$, and $\beta_n = (2n-1)\frac{\pi}{2}$.

For a Green's Functions based solution, the general form of the solution is the combination of the initial and boundaries conditions, changing the interval of integration according to the needs. Robin's boundary condition (heat convection boundary) doesn't appear, thus its term in the general form won't make any influence. Despite this, the third kind's condition still can be modelled by a Green's Functions based solution.

3. RESULTS AND DISCUSSIONS

3.1 Temperature field solution

Performing all the integrations in the general form and reorganizing the terms, temperature field in any position at any time is given by Eq. (3) for X22Y12 problem. Figures 3 and 4 shows a temperature field in physical domain at t = 30 s and t = 100 s, respectively.

$$
\begin{aligned}
T(x,y,t) = {} & 2(T_0 - T_1)\sum_{n=1}^{\infty} e^{\frac{-\beta_n^2 \alpha t}{W^2}} \sin\left(\frac{\beta_n y}{W}\right)\frac{1}{\beta_n} + \frac{4q_1W^2}{kL}\sum_{n=1}^{\infty}\sin\left(\frac{\beta_n y}{W}\right)\frac{1}{{\beta_n}^3}\left[1 - e^{\frac{-\beta_n^2\alpha t}{W^2}}\right]\sin^2\left(\frac{W_1\beta_n}{2W}\right) \\
& + \frac{8q_1W^2L}{k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} cos\left(\frac{\alpha_m x}{L}\right) sin\left(\frac{\beta_n y}{W}\right)\left[\left(\frac{\sin^2\left(\frac{W_1\beta_n}{2W}\right)}{\beta_n\left(L^2{\beta_n}^2 + W^2{\alpha_m}^2\right)}\right)\left(1 - e^{\frac{-\beta_n^2\alpha t}{W^2}} e^{\frac{-\alpha_m^2\alpha t}{L^2}}\right)\right] \\
& + \frac{2q_0W^2}{kL}\sum_{n=1}^{\infty}\sin\left(\frac{\beta_n y}{W}\right)\frac{1}{{\beta_n}^3}\left[1 - e^{\frac{-\beta_n^2\alpha t}{W^2}}\right]\cos\left(\frac{W_1\beta_n}{W}\right) + T_1 \\
& + \frac{4q_0LW^2}{k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} cos\left(\frac{\alpha_m x}{L}\right)\frac{\cos(\alpha_m)}{\beta_n\left(L^2{\beta_n}^2 + W^2{\alpha_m}^2\right)} sin\left(\frac{\beta_n y}{W}\right)\cos\left(\frac{W_1\beta_n}{W}\right)\left[1 - e^{\frac{-\beta_n^2\alpha t}{W^2}} e^{\frac{-\alpha_m^2\alpha t}{L^2}}\right]
\end{aligned}
\tag{3}
$$

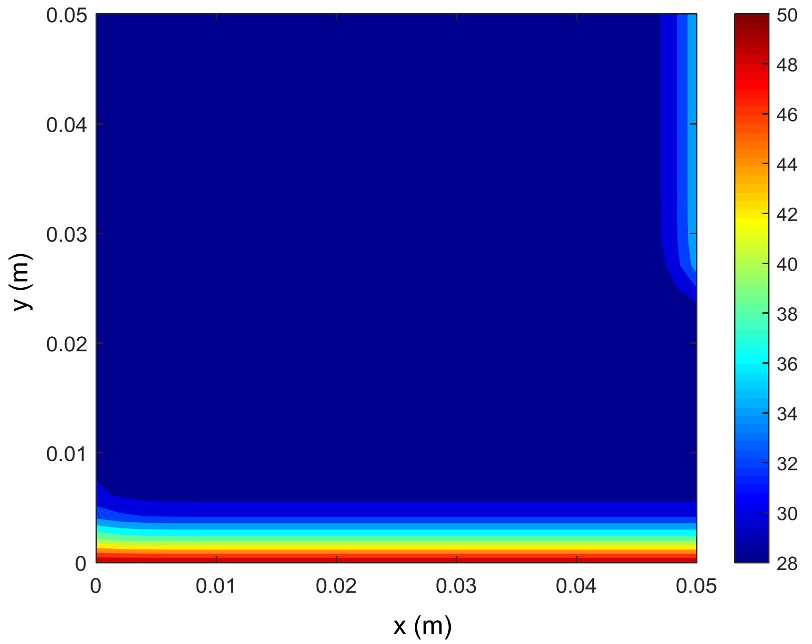

Figure 2 – Temperature field (ºC) of X22Y12 problem at time t = 30 s.

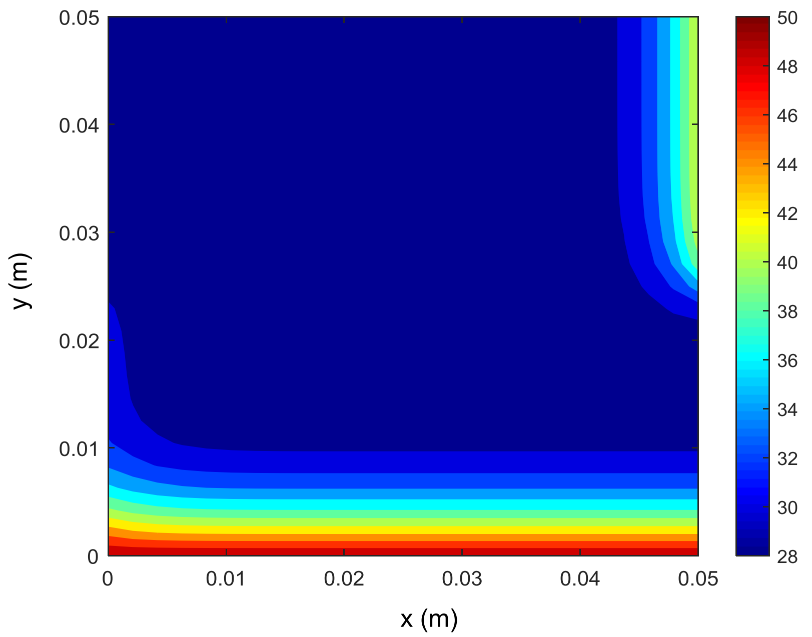

Figure 3 – Temperature field (ºC) of X22Y12 problem at time t = 100 s.

3.2 Intrinsic verification

According Cole et al (2011), intrinsic verification is the process of determining correct numerical values from an exact analytical solution, to many significant figures, in two or more independent ways. This provides assurance that the solution is correct and that the process for obtaining accurate numerical values is sound.

Intrinsic verification was performed considering the two cases shown in Fig. 4. In both, the dimension y is 10 times larger than in Fig 1, while the dimensions on the x axis are all the same. In case 1, every contour y is thermally insulated at x = 0, while a heat flux q_1 is applied to every contour y at x = L. Thus, results on Figs. 5 and 6 show the temperature difference between points A and B with those of the corresponding points in Fig. 1, respectively.

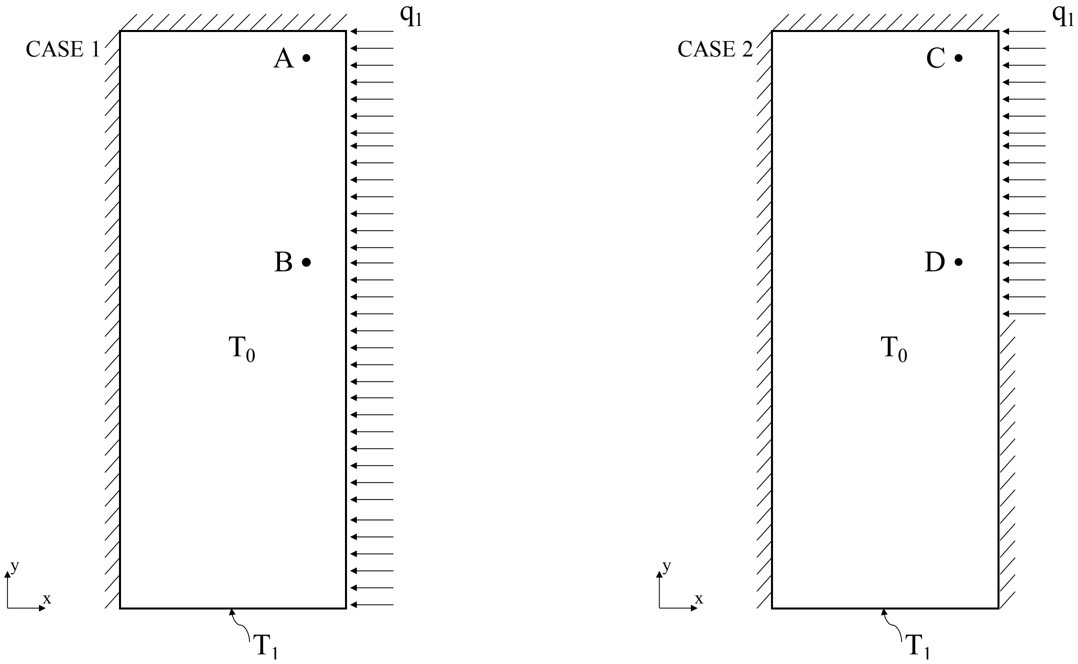

Figure 4 - Auxiliary 2D problems for intrinsic verification.

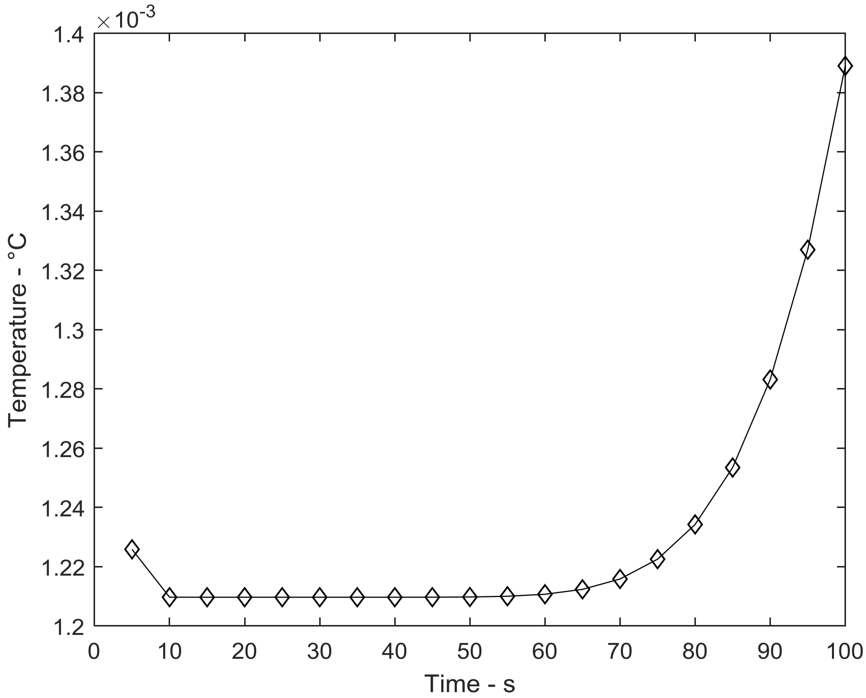

Figure 5 - Absolute error obtained in intrinsic verification with 2D auxiliary problem at point A.

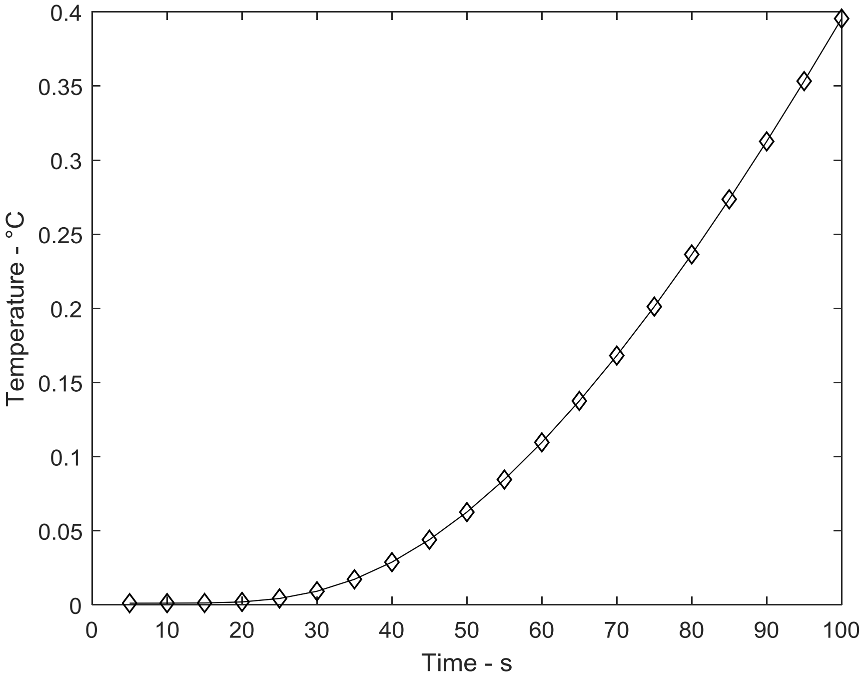

Figure 6 - Absolute error obtained in intrinsic verification with 2D auxiliary problem at point B.

Similar procedure is performed in case 2. However, at x = L the heat flux is applied only to half of the contour y — the other half of this contour is isolated. At x = 0 the condition is the same as in case 1. Temperatures at points C and D are compared with temperatures at geometrically proportional points in the case shown in Fig. 1. The graphs of Figs 7 and 8 show the difference between temperatures of points C and D with those of the geometrically corresponding points in Fig. 1.

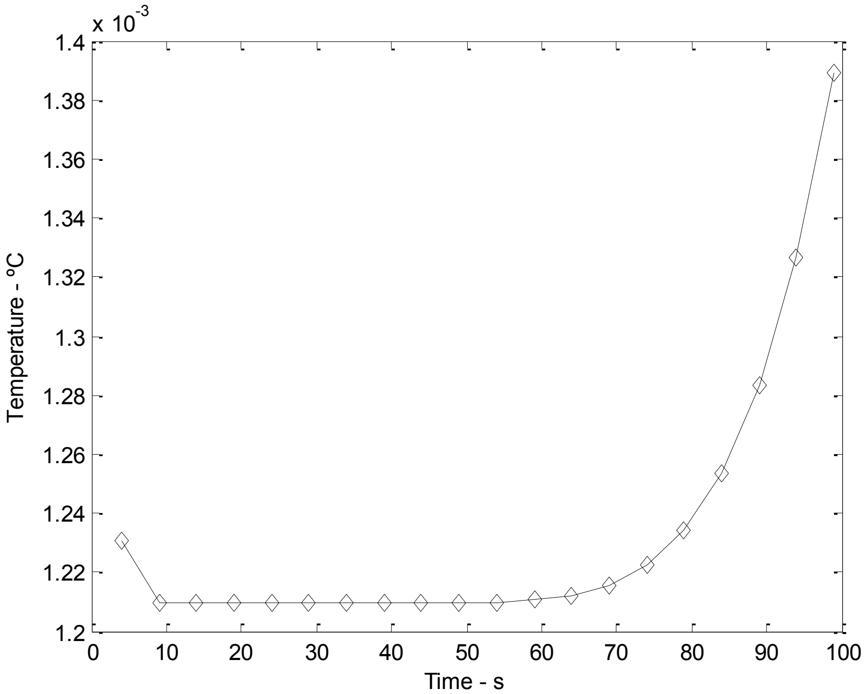

Figure 7 - Absolute error obtained in intrinsic verification with 2D auxiliary problem at point C.

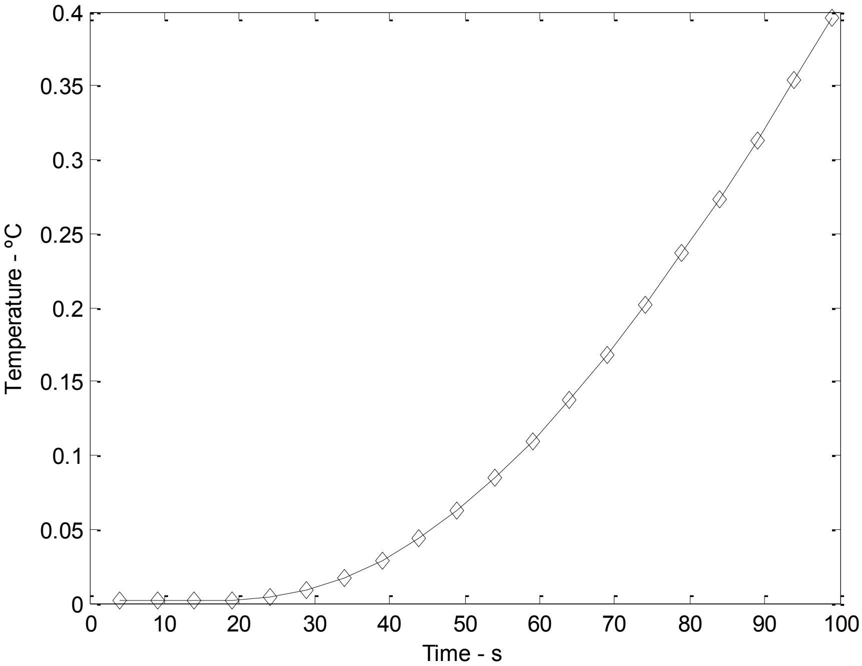

Figure 8 - Absolute error obtained in intrinsic verification with 2D auxiliary problem at point D.

Results of case 1, Fig. 5 and 6 show that the difference between the temperature of point A and that of its corresponding point in the original problem is about 3 orders of magnitude lower than the temperature of point B and that of corresponding point in the original problem. Similar behaviour also occurs in case 2 verification (Fig. 7 and 8). This is due to the fact that the conditions of the points near the vertices (A and C) in both cases are more similar to the original problem than points B and D.

Another possibility of intrinsic verification is to assume that heat fluxes q_0 and q_1 tend to zero, making the original problem, Fig. 1, into a 1D heat conduction problem, specifically a Y12 problem, as shown in Fig. 9. Thus, comparing the temperatures of points E and F with the geometrically corresponding points in Fig. 1, Fig. 10 and 11 are respectively obtained by Separation of Variables.

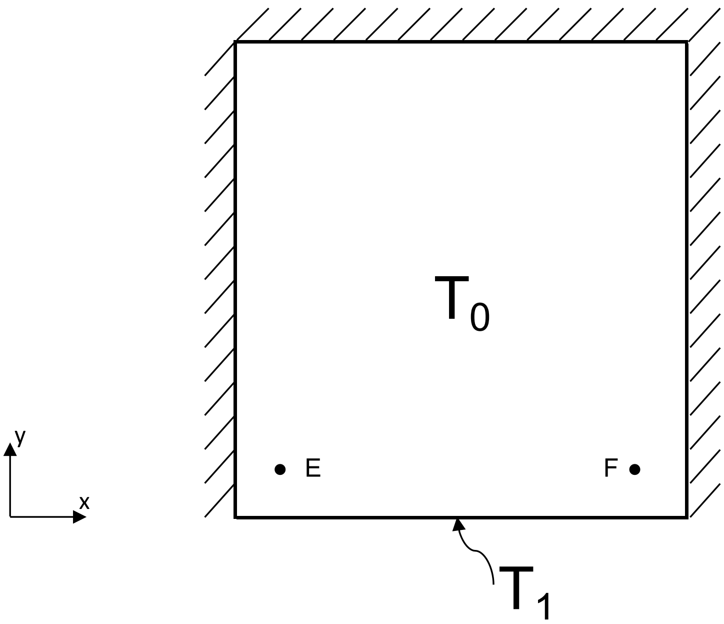

Figure 9 - Auxiliary 1D problem for intrinsic verification

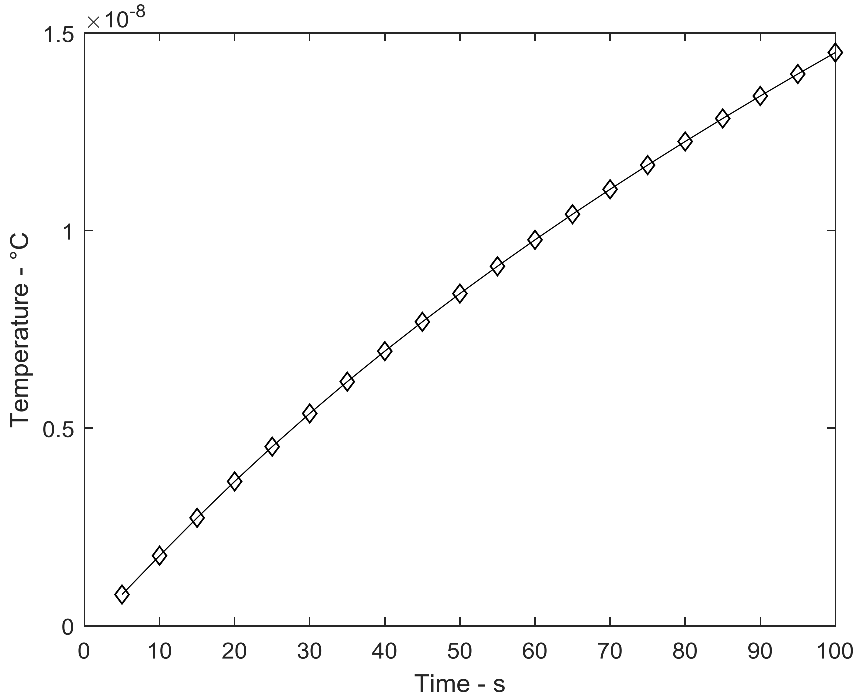

Figure 10 - Absolute error obtained in intrinsic verification with 1D auxiliary problem at point E.

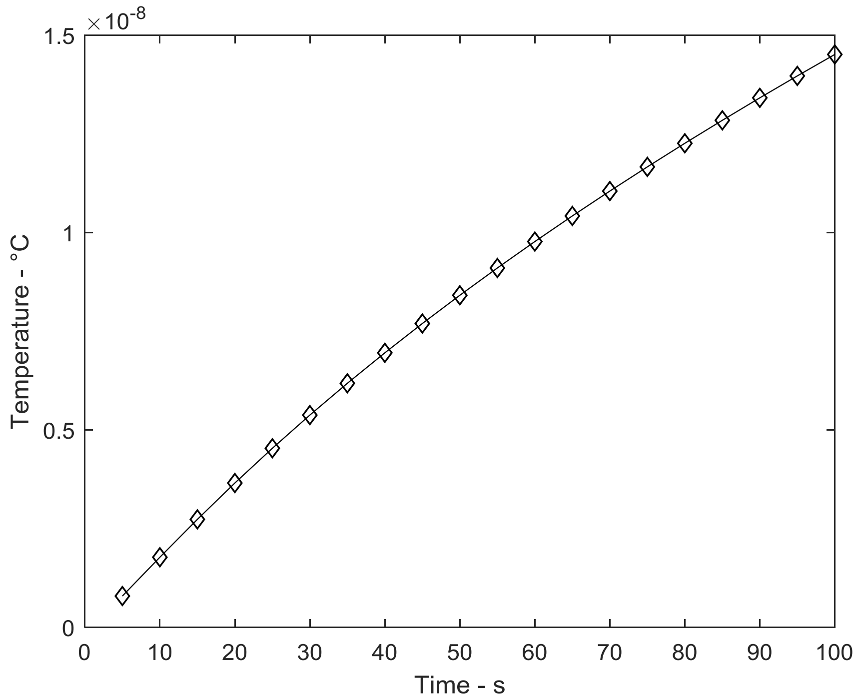

Figure 11 - Absolute error obtained in intrinsic verification with 1D auxiliary problem at point F.

4. CONCLUSIONS

In this work, the temperature field was evaluated by solving a direct 2D problem, of type X22Y12, with partial flows prescribed in the contours in y. Initially, the solution obtained was verified through geometric changes and the imposed flow conditions of the original problem. Thus, two cases were verified, which, when compared to the original problem, present a greater agreement of the temperature field in regions close to the vertices of the physical domain.

Another verification was made considering the partial heat fluxes imposed on y as null, thus rendering the one-dimensional problem of type Y12. Thus, temperature fields solutions of the original and 1D problems were compared. It was found that in regions near the lower vertices of the physical domain, the temperature difference between the two cases is of the order of 10^{-8} ºC, which validates the analytical solution of the original problem.

Future work will be undertaken to estimate the partial heat fluxes studied here using the inverse problem methodology.

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