NONLINEAR INSTABILITY ANALYSIS IN STRATIFIED FLOW PATTERN IN LIQUID-LIQUID SYSTEMS

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Abstract. *In multiphase flow the determination of the spatial distribution of the phases is a necessary prerequisite. Relative knowledges about the flow pattern affects many technical aspects of engineering as the modeling process. The flow patterns transition criteria are investigated in function of the range of necessary conditions for the existence of determined flow pattern. In this article, the stratified flow pattern in liquid-liquid systems is numerically simulated in order to predict where the transition to another flow pattern occurs. Previous studies indicate that this phenomenon is related to the structure of the interfacial wave and happens in space and it is dominated by nonlinear effects. At first, the nonlinear equations of mass and momentum are obtained by the one-dimensional two-fluid model. A simplified nonlinear analysis in hydrodynamic stability is obtained by the method of characteristics and an explicit finite difference scheme, in which the flow is subjected to a solitary wave (saturated wave) with dimensions based on previous studies and its propagation is defined in time-space plane. For the development of the interfacial wave the effects caused by interface curvatures in the direction of flow and the cross section are considered. Such effects are related to interfacial tension and are essentials in liquid-liquid flow systems. Due to the inclusion of interfacial terms two new constants must be defined as functions and used to force the breakup of the wave at the point experimentally observed. The results are intended to improve the transition models between flow patterns.*

Keywords*: Liquid-liquid flow, stratified flow pattern, waves, hydrodynamic stability, transition boundaries.*

1. INTRODUCTION

Multiphase flows are observed in several natural and industrial processes. They occur through wide range of spatial distributions, which determine the flow pattern, and affect the phenomena of interfacial transfer between phases (mass, momentum and energy) by changing their hydrodynamic characteristics. Often, a flow pattern is preferable according to the applicability of the flow, where the ranges of necessary conditions for its existence are investigated through transition criteria obtained by a hydrodynamic stability analysis.

The difficulty of handling with general governing equations of multiphase flow, which are too complex to be solved analytically and sometimes also provide a high computational cost when applied to numerical methods, enabled the development of modeling theories related to the flow pattern to establish physical relations and provide an understanding of the interactions between the phases. The formulation of one-dimensional two-fluid model, developed by Ishii (1975), is the most complex way of treating a two-phase flow due to the phases being considered completely independently, but facilitated the stability analysis in liquid- liquid flow.

The stability analysis occurs by adding perturbations in the flow, which may decrease with time or space, if the flow is stable, or increase, if the flow is unstable. Stratified flow is appropriated for the study of the phenomenon due to several transition boundaries with other flow patterns. Since the instability might be generated by disturbances of different magnitudes, this paper gives preference to a simplified nonlinear approach of hydrodynamic stability. Herein, partial differential equations (PDE) that describe the flow are transformed into ordinary differential equations (ODE) in characteristic directions, which can be solved by finite difference. This method has the advantage of providing a clear understanding of the physical implications of numerical procedures, however the partial differential equations which are contained in the problem should be of the hyperbolic type (i.e. those with real and finite wave velocity). The methodology used in this approach is based on the method of characteristics (MOC), as proposed by Crowley *et al.* (1992), Barnea and Taitel (1994), Trallero (1995) and Salhi *et al.* (2010).

This article starts from a simulation of the interfacial wave imposed on a stratified two-phase flow of oil-water that is being propagated over time-space. An unstable behavior is expected by the addition and adjustments of the terms related to the interfacial tensions in the model equations by considering an interface curve in the cross section. Experimental data, obtained by Castro (2013), are used to compare the occurrence point of transition from flow regime with the results presented by the simulation in order to validate them. At this point, it is expected that the simulated waveform has a dimensionless parameter, h/D, close to one and that the spatial position is similar of that observed experimentally. The aim of the proposed study is to develop the ability to predict where will occur the change of flow pattern evaluating only the specific operating conditions, i.e., superficial velocities of the phases, fluid properties and geometry of the flow.

During the development of the equations the same simplifications and correlations proposed in previous studies to a stratified gas-liquid flow pattern are used. It is noted that such considerations should add imperfections to the model equations, and possibly modify the wave propagation behavior. However, due to the lack of references about the

application of the method of characteristics to the stratified liquid-liquid flow (without this considerations), even so, herein these relations are used.

2. MODELING

The general governing equations of oil-water stratified flow are obtained through the one-dimensional two-fluid model taking into account the geometry proposed in Fig. 1 and the following assumptions: a) isothermal flow, b) no phase change, c) no mass transfer, and d) incompressible fluids.
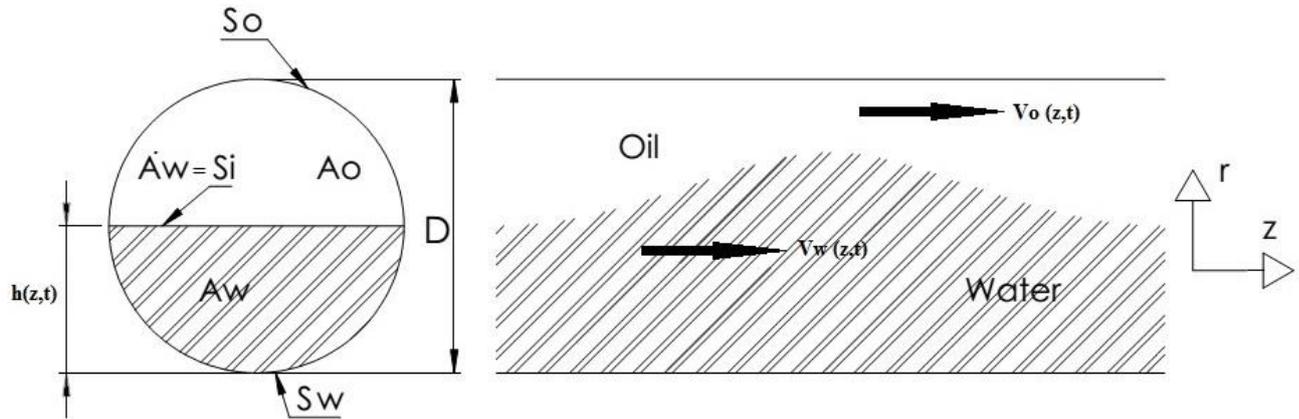

Figure 1. Geometry of the problem.

The mass conservation equations for the oil and water phases are, respectively:

$$\frac{\partial h}{\partial t} - \frac{A_0}{\dot{A_W}}\frac{\partial V_o}{\partial z} + V_o\frac{\partial h}{\partial z} = 0 \tag{1}$$

$$\frac{\partial h}{\partial t} - \frac{A_w}{\dot{A_W}}\frac{\partial V_w}{\partial z} + V_w\frac{\partial h}{\partial z} = 0 \tag{2}$$

The momentum conservation equations of the phases are coupled through the pressure gradient and using the Young-Laplace equation, resulting in:

$$\rho_w\frac{\partial V_w}{\partial t} - \rho_o\frac{\partial V_o}{\partial t} + \rho_w V_w\frac{\partial V_w}{\partial z} - \rho_o V_o\frac{\partial V_o}{\partial z} + L\frac{\partial h}{\partial z} - \sigma\left(\frac{\partial^3 h(z)}{\partial z^3} + \frac{1}{r^2 h(z)}\frac{\partial h(z)}{\partial z}\right) = fe \tag{3}$$

and

$$fe = fe(h(z), V_w, V_o) = \pm\tau_i S_i\left(\frac{1}{A_w} + \frac{1}{A_o}\right) - \frac{\tau_{ww}S_w}{A_w} + \frac{\tau_{ow}S_o}{A_o} - (\rho_w - \rho_o)g.sen\theta \tag{4}$$

$$L = (\rho_w - \rho_o)g.cos\theta \tag{5}$$

where the upper sign in Eq. (4) corresponds to phase oil faster than water. The variables A, S and V are the flow cross section, the wetted perimeter and the *in situ* velocity of the two fluids. The interfacial shear stress is represented by τ_i and the wall shear stress in the water and oil phases are represented by τ_{ww} and τ_{wo}, respectively.

3. STABILITY ANALISYS – SIMPLIFIED NONLINEAR ANALISYS (METHOD OF CHARACTERISTICS)

At first, the term of interfacial tension, σ, is considered negligible to eliminate the derivative of the third-order and the term related to the curvature of the interface. Such approach ensures a hyperbolic equation, but it is only valid if the interfacial waves are long. During the analysis, there are two different situations involving slip ratio between phases, which are normally used in gas-liquid flow and provide some simplifications in the equations.

3.1 Slip ratio, S, is greater than 1

$$S = \frac{V_o}{V_w} > 1 \tag{6}$$

In this section, the oil is assumed as the fastest phase, and therefore is flowing into a *quasi*-steady state. This imposition enables neglect the time derivative of oil and achieve the following equations:

$$\int_{J_o}^{V_o} \frac{\partial V_o}{V_o} = -\int_{A}^{A_o} \frac{\partial A_o}{A_o} \tag{7}$$

$$V_o = \frac{J_o A}{A_o} \tag{8}$$

Substituting Eq. (8) in Eq. (3), and dividing all terms by the water density, we have:

$$\frac{\partial V_w}{\partial t} + V_w \frac{\partial V_w}{\partial z} + G_1 \frac{\partial h(z)}{\partial z} + E_1 = 0 \tag{9}$$

where

$$G_1 = \frac{(\rho_w - \rho_o)gcos\theta}{\rho_w} - \frac{\rho_o J_o{}^2 A^2 \dot{A}_w}{\rho_w A_o^3} \tag{10}$$

$$E_1 = -\frac{fe}{\rho_w} \tag{11}$$

Thus, for this situation, the following system of partial differential equations is obtained:

$$\begin{cases} \frac{\partial h}{\partial t} - \frac{A_w}{\dot{A}_W} \frac{\partial V_w}{\partial z} + V_w \frac{\partial h}{\partial z} = 0 \\ \frac{\partial V_w}{\partial t} + V_w \frac{\partial V_w}{\partial z} + G_1 \frac{\partial h(z)}{\partial z} + E_1 = 0 \end{cases} \tag{12}$$

Applying the method of characteristics in Eq. (12), and after some manipulations, we arrive at:

$$\begin{cases} \frac{dh}{dt} + \left(\frac{\lambda_1 + \lambda_2}{2}\right)\frac{dh}{dz} + B_1\left(\frac{\lambda_2 - \lambda_1}{2}\right)\frac{dV_w}{dz} = 0 \\ \frac{dV_w}{dt} + \left(\frac{\lambda_2 - \lambda_1}{2B_1}\right)\frac{dh}{dz} + \left(\frac{\lambda_1 + \lambda_2}{2}\right)\frac{dV_w}{dz} + E_1 = 0 \end{cases} \text{, along } \lambda_1 = V_w - \sqrt{H_1 G_1} \text{ and } \lambda_2 = V_w + \sqrt{H_1 G_1} \tag{13}$$

where

$$H_1 = \frac{A_w}{\dot{A}_w} \tag{14}$$

$$B_1 = \sqrt{\frac{H_1}{G_1}} \tag{15}$$

3.2 Slip ratio, S, is lower than 1

$$S = \frac{V_o}{V_w} < 1 \tag{16}$$

In this section, the water is assumed as the fastest phase, and therefore is flowing in a *quasi*-steady state. This imposition enables neglect the time derivative of water and perform the same considerations shown for the previous case.

However, after applying the method of characteristics, and some manipulations, we arrive at a new and different system of equations:

$$\begin{cases} \dfrac{dh}{dt} + \left(\dfrac{\lambda_1 + \lambda_2}{2}\right)\dfrac{dh}{dz} + B_2\left(\dfrac{\lambda_1 - \lambda_2}{2}\right)\dfrac{dV_o}{dz} = 0 \\ \dfrac{dV_o}{dt} + \left(\dfrac{\lambda_1 - \lambda_2}{2B_2}\right)\dfrac{dh}{dz} + \left(\dfrac{\lambda_1 + \lambda_2}{2}\right)\dfrac{dV_o}{dz} - E_2 = 0 \end{cases}, \text{along } \lambda_1 = V_o - \sqrt{H_2 G_2} \text{ and } \lambda_2 = V_o + \sqrt{H_2 G_2} \tag{17}$$

where

$$G_2 = \frac{(\rho_w - \rho_o)g\cos\theta}{\rho_o} - \frac{\rho_w {J_w}^2 A^2 \dot{A}_w}{\rho_o A_w^3} \tag{18}$$

$$E_2 = -\frac{fe}{\rho_o} \tag{19}$$

$$H_2 = \frac{A_o}{\dot{A}_w} \tag{20}$$

$$B_2 = \sqrt{\frac{H_2}{G_2}} \tag{21}$$

4. FINITE DIFFERENCE METHODS

The systems of Eqs. (13) and (17) contains partial derivatives with respect to space and time. In a first approach the equations of the systems are approximated both simultaneously and the resulting of the discretized equations are solved through a scheme by numerical iteration. Thus, it is used a second-order centered discretization scheme in space, where it adds a dispersion error which modifies the phase of the interfacial wave, and a first-order discretization scheme in time of backward type, where it adds a dissipation error that modifies the amplitude of the interfacial wave. The negative aspects of both approaches are improved by making the stability criterion of Courant-Friedrichs-Lewy (CFL) tend to zero. This is:

$$CFL = \frac{\lambda_i \Delta t}{\Delta z} \ll 1 \tag{22}$$

Considering only the case described in section 3.2, we have:

$$\begin{cases} h_z^{t+1} = h_i^t - \dfrac{\Delta t}{4\Delta z}\big((\lambda_1 + \lambda_2)(h_{z+1}^t - h_{z-1}^t) + B_2(\lambda_1 - \lambda_2)(V_{oz+1}^t - V_{oz-1}^t)\big) \\ V_{oz}^{t+1} = V_{oz}^t - \dfrac{\Delta t}{4B_2\Delta z}\big((\lambda_1 - \lambda_2)(h_{z+1}^t - h_{z-1}^t) + B_2(\lambda_1 + \lambda_2)(V_{oz+1}^t - V_{oz-1}^t)\big) + \Delta t E_2 \end{cases} \tag{23}$$

Subsequently, a second-order scheme known as Leapfrog is tested in time discretization. In this scheme, besides the problem of associated dispersion, also we have a numerical error due to spurious root of the problem. The method is not self-beginner and because of that, for a first step in time, a backward scheme is used.

$$\begin{cases} h_z^{t+1} = h_z^{t-1} - \dfrac{\Delta t}{2\Delta z}\big((\lambda_1 + \lambda_2)(h_{z+1}^t - h_{z-1}^t) + B_2(\lambda_1 - \lambda_2)(V_{oz+1}^t - V_{oz-1}^t)\big) \\ V_{oz}^{t+1} = V_{oz}^{t-1} - \dfrac{\Delta t}{2B_2\Delta z}\big((\lambda_1 - \lambda_2)(h_{z+1}^t - h_{z-1}^t) + B_2(\lambda_1 + \lambda_2)(V_{oz+1}^t - V_{oz-1}^t)\big) + \Delta t E_2 \end{cases} \tag{24}$$

Alternatively, using an implicit scheme of Crank-Nicolson, the system of Eqs. (17) are solved in matrix form through the time-marching method, the mathematical formulation is developed in Lomax *et al.* (1999). Thus the equations are written in the following form:

$$\begin{cases}
\begin{aligned}
& -\frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} h_{z-1}^{t+1} + h_z^{t+1} + \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} h_{z+1}^{t+1} - \frac{B_2 \Delta t(\lambda_1 - \lambda_2)}{8\Delta z} V_{z-1}^{t+1} + \frac{B_2 \Delta t(\lambda_1 - \lambda_2)}{8\Delta z} V_{z+1}^{t+1} \\
& = \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} h_{z-1}^{t} + h_z^{t} - \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} h_{z+1}^{t} + \frac{B_2 \Delta t(\lambda_1 - \lambda_2)}{8\Delta z} V_{z-1}^{t} - \frac{B_2 \Delta t(\lambda_1 - \lambda_2)}{8\Delta z} V_{z+1}^{t} \\
& -\frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} V_{z-1}^{t+1} + V_z^{t+1} + \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} V_{z+1}^{t+1} - \frac{\Delta t(\lambda_1 - \lambda_2)}{8B_2\Delta z} h_{z-1}^{t+1} + \frac{\Delta t(\lambda_1 - \lambda_2)}{8B_2\Delta z} h_{z+1}^{t+1} \\
& = \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} V_{z-1}^{t} + V_z^{t} - \frac{\Delta t(\lambda_1 + \lambda_2)}{8\Delta z} V_{z+1}^{t} + \frac{\Delta t(\lambda_1 - \lambda_2)}{8B_2\Delta z} h_{z-1}^{t} - \frac{\Delta t(\lambda_1 - \lambda_2)}{8B_2\Delta z} h_{z+1}^{t} + E_2
\end{aligned}
\end{cases} \tag{25}$$

or

$$\begin{cases}
A h^{t+1} + B v^{t+1} = C h^t + D v^t \\
A v^{t+1} + \frac{1}{B_2^2} B h^{t+1} = C v^t + \frac{1}{B_2^2} D h^t
\end{cases} \tag{26}$$

where A, B, C, D are $N \times N$ sparse tridiagonal matrices and h, v are N vectors representing the function values in the mesh points and time.

5. PRELIMINARY ANALYSIS

The initial behavior expected during the simulation can be analyzed in terms of characteristic velocity λ_1. If $\lambda_1 > 0$, the flow is said supercritical, being the interfacial wave propagated only downstream. If $\lambda_1 < 0$, the flow is said subcritical, being the interfacial wave propagated in both directions, up and downstream. In both cases it is possible to occur two opposite phenomena: stability, where the wave propagates with a constant or decreasing amplitude; and instability where the wave amplitude grows indefinitely. In the latter case, the characteristics velocities become imaginary and the system becomes ill-posed, therefore needing the addition of relative functions about the interfacial tension in the equations.

The simulation begins through the imposition of a solitary wave to initial flow. His form was proposed by Lamb (1932) and indicated by Brauner and Maron (1992), and Trallero (1995), i.e.:

$$\eta = a\, sech^2 \left(\frac{z}{b} \right) \tag{27}$$

Figure 2 shows the output generated by the system of Eqs. (23), which has high-frequency vibrations generated in the propagated downstream wave that tend to destabilize the simulation. It is also important to notice that the downstream wave already presents an unstable behavior.
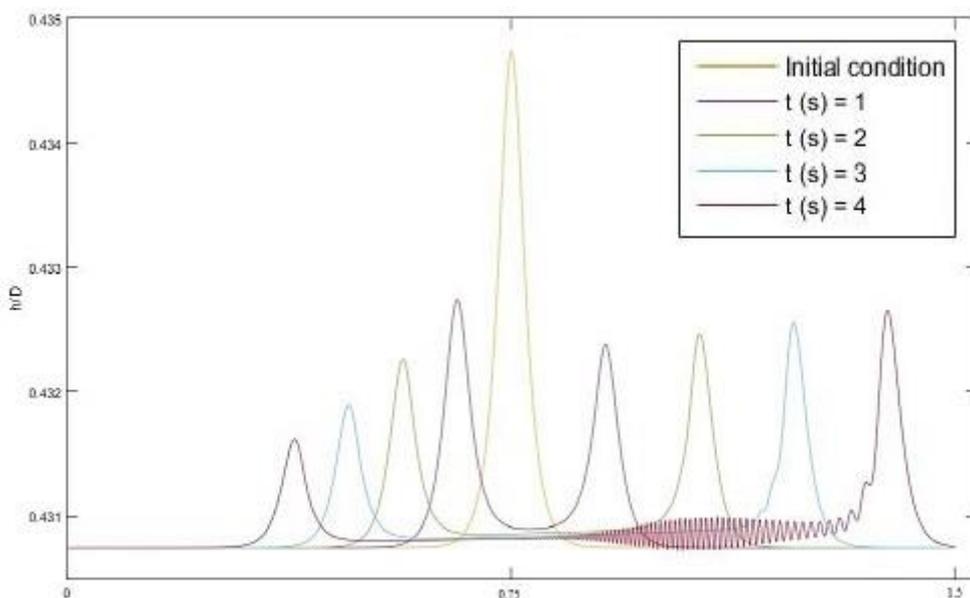

Figure 2. Wave propagation using a centered scheme in space and backward in time.

The problem of high-frequency vibrations is eliminated by increasing the order of approximation of the time derivative using the Leapfrog numerical method. Thus, Fig. 3 shows the output generated by the system of Eqs. (24), where there are two simulated cases for different superficial velocities of water.
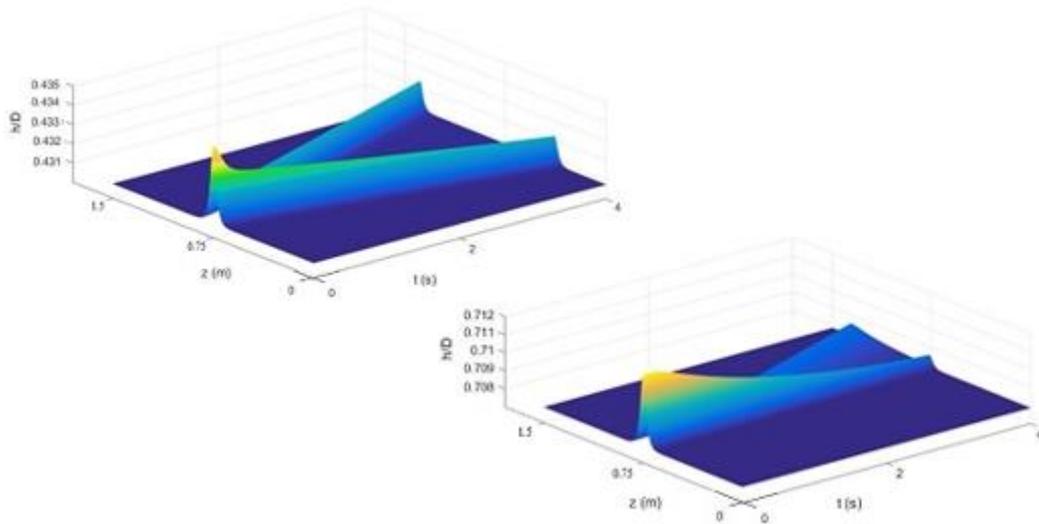
Figure 3. Wave propagation using Leapfrog method: a) for superficial velocity of water and oil, $Jw = 0.04$ and $Jo = 0.02$, respectively, with $\lambda_1 < 0$, b) $Jw = 0.11$ and $Jo = 0.02$, with $\lambda_1 > 0$.

The major problem to proceed with the nonlinear stability analysis through adding related terms of the interfacial tension and thereby validate the proposed simulation as showed in Fig. 3, is the anomalous behavior in the *in situ* velocity of oil, presented in Fig. 4b, which should have a proportional/similar behavior to the dimensionless parameter h/D, but behaves antagonistically during the downstream propagation of interfacial wave. It is known that the use of the Leapfrog method is not the most desirable for transient problems, and due to this fact, it is necessary to check this behavior through other types of numerical schemes.
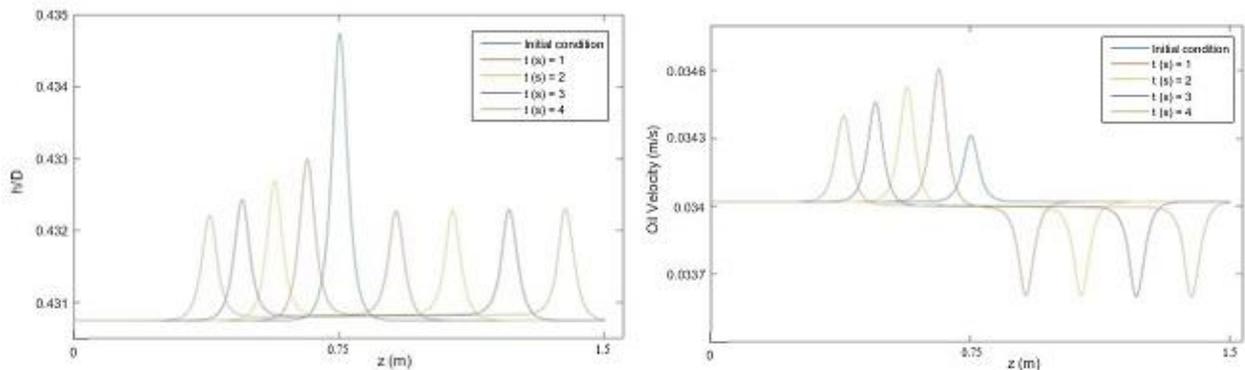
Figura 4. Wave propagation using Leapfrog method ($Jw = 0.04$ and $Jo = 0.02$, with $\lambda_1 < 0$): a) dimensionless parameter h/D, b) anomaly in the *in situ* velocity of oil.

Through the system of Eqs. (25) it is implemented an implicit scheme of Crank-Nicolson, which is numerically unconditionally stable for the time-marching method. However, with preliminary results, the simulation showed an incorrect behavior for $\lambda_1 < 0$. The interfacial wave propagates only downstream, as shown in Fig. 5a. However, it is checked that the anomalous behavior in the *in situ* velocity of oil persisted, as shown in Fig. 5b.
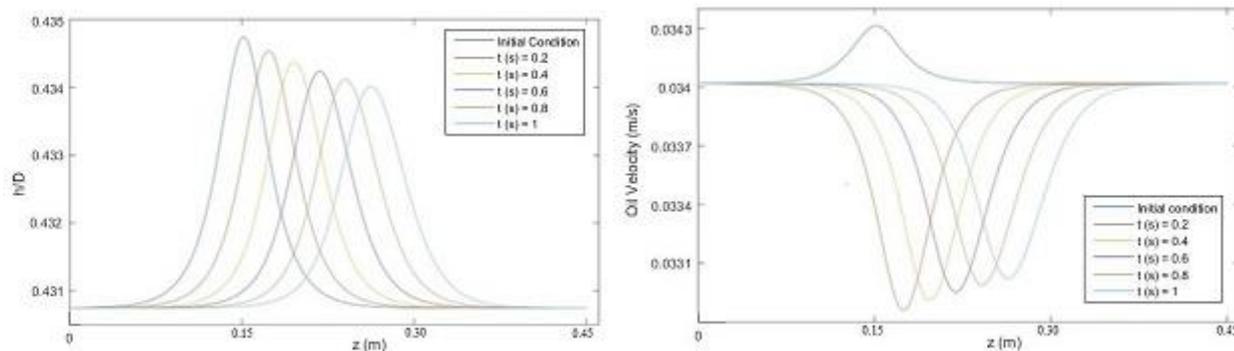
Figure 5. Wave propagation using Crank-Nicolson method ($Jw = 0.04$ and $Jo = 0.02$, with $\lambda_1 < 0$): a) dimensionless parameter h/D, b) anomaly in the *in situ* velocity of oil.

6. CONCLUSIONS

During the stability analysis of stratified water-oil flow in a nonlinear approach, using a formulation obtained by the method of characteristics, it was checked through various numerical finite difference schemes some inconsistent physical aspects. This article suggests that the difficulty in propagate the interfacial wave of flow properly is due the used of strong simplifications. In this type of approach, always some of the phases of the flow will have his in situ velocity necessarily considered as *quasi* permanent. This peculiarity, widely used in stratified gas-liquid flows, it may be distancing the observed behavior in the simulation of the real situation.

The method of characteristics applied to a simplified momentum equation (terms related to interfacial tension are neglected) is a plausible condition in gas-liquid flows. However, as mentioned in Rodriguez and Castro (2013) and Rodriguez and Bannwart (2006), the terms related to interfacial tension are very important, since they can stabilize or destabilize the flow. Thus, the solution to the inconsistencies shown in this article may be not to use the method of characteristics and avoid, in this way, the strong simplifications. Another suggested approach is to use the general governing equations of the flow in a diverging shape, since close to regime transition the flow will tend to generate discontinuities, which can provide complications if used in nonconservative equations.

7. REFERENCES

Barnea, D. and Taitel, Y., 1994. Non-linear Interfacial Instability of Separated Flows, Chemical Engineering Science, Vol. 49, no. 14, pp. 2341-2349.

Brauner, N. and Maron. D.M., 1992. Stability analysis of stratified liquid-liquid flow, International Journal of Multiphase Flow, v. 18, n. 1, pp. 103- .

Castro, M.S., 2013. Fenômeno de Transição Espacial do Escoamento Óleo Pesado-Água no Padrão Estratificado.

Crowley, C.J., Wallis, G.B., and Barry, J.J., 1992. Validation of a One-Dimensional Wave Model for the Stratified to Slug Flow Regime Transition, With Consequences for Wave Growth and Slug Frequency, International Journal of Multiphase Flow, vol. 18, no. 2, pp. 249-271.

Ishii, M., 1975. Thermo-fluid dynamic theory of two-phase flow, Paris: Eyroles, pp. 248.

Lamb, H., 1932. Hydrodynamics, Cambridge: Cambridge University Press, pp. 738.

Lomax, H., Pulliam, T.H., and Zingg, D.W., 1999. Fundamentals of Computational Fluid Dynamics.

Rodriguez, O.M.H., and Bannwart, A.C., 2006. Analytical model for interfacial waves in vertical core flow, Journal of Petroleum Science and Engineering, v. 54, pp. 173-182.

Rodriguez, O.M.H., and Castro, M.S., 2013. Stability Analysis of Wave-stratified Heavy Oil-water Flow, 8th International Conference on Multiphase Flow.

Salhi, Y., Si-Ahmed, E., Legrand, J., and Degrez, G., 2010. Stability analysis of inclined stratified two-phase gas-liquid flow, Nuclear Engineering And Design, v. 240, pp. 1083-1086.

Trallero, J.L., 1995. Oil-Water Flow Patterns in Horizontal Pipes, Ph.D thesis, The University of Tulsa, Tulsa, Oklahoma, USA.

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