

THE STATE OF THE ART OF THE THEODORSEN'S AEROELASTIC THEORY BY A FRACTIONAL CALCULUS APPROACH

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Abstract. *With the aim to analyze the typical section aerodynamic load, important aeroelastic concepts are reviewed in this paper, such as the Theodorsen and Wagner's theory of unsteady aerodynamic load. Besides, the typical section's kinematics is assessed and the study of aerodynamic loads is initiated, resulting in a integral named Duhamel's Integral, that represents all the contributions in lift of past instants. Thus, in future works we expect to develop this integral, that is part of the typical section's dynamic analysis, using the fractional calculus, which is a tool applied in engineering systems since the 70's and used by engineers ever since. Therefore, the complete State of the Art relating the fractional calculus with aeroelastic or any mechanical system about structural mechanics is presented, gathering the most relevant works related to this subject. Hence, one pretends to justify this job by means of evidences showing the small amount of work that has been done until now.*

Keywords: Aeroelasticity, fractional calculus, Duhamel and flutter.

1. INTRODUCTION

The aerodynamic load analysis is an essential part of any aircraft design. Thus, conditions must be established to ensure safety along with efficiency. Among all analysis possible to be done, maybe the most critical is the flutter study, which is a highly destructive phenomenon. The flutter phenomenon is studied in a field called aeroelasticity and according to Wright and Cooper (2015), this is a science that describes the interaction between aerodynamic, inertia and elastic forces, in a flexible structure. It has major influence over aircraft's design and performance. This science is well represented by the Collar's Triangle (Collar, 1978), presented in Fig. 1, that highlight relations between the cited forces.

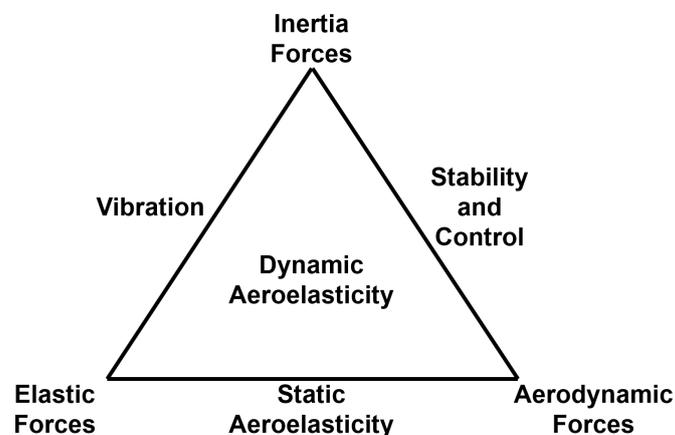


Figure 1: Collar's Triangle

Still according to Wright and Cooper (2015), aeroelastic phenomena can be classified as static or dynamic, so that the first one consider only the non-oscillatory component of the aerodynamic load, resulting in a static aeroelastic problem

which leads the structures to the divergence phenomenon. According to Bisplinghoff and Ashley (2013), divergence happens when the airfoil suffers a deflection due to static aerodynamic load, and can even cause the structural failure. On the other hand, dynamic aeroelastic phenomena consider the oscillatory effects of the aerodynamic load. In this case, the flutter phenomenon takes place, which is characterized by an unstable dynamic behavior capable to destruct the entire structure. Such an impressive instability is caused due to a self-excited motion, where two or more modes of vibration interact changing energy among them until a point where the damping becomes zero. This is the specific point where flutter occurs, and is called critical speed. To avoid it, is necessary to completely understand structure's dynamic, thus allowing solutions to mitigate the phenomenon.

When dealing with aircraft, a way to understand these conditions is characterizing the aerodynamic load of the lifting surface, revealing its influence in the dynamic behavior. There are many ways to study aerodynamic loads, a very simple and widely used approach is the typical section method, it may be a simple approach to represent a lifting surface, but it has an important academic role. According to Theodorsen and Garrick (1940), that idealized the typical section, it has capability to represent a wing by means of its discrete properties of stiffness, calculated at 3/4 of the semi span wing, and mass. Besides, according to Bisplinghoff and Ashley (2013), this approach is acceptable for a high aspect ratio, thin profile, small swept and when the cross section wing does not change importantly along the span.

Theodorsen's development of the typical section aerodynamic loading has a convolution integral that must be solved. This integral is called Duhamel's Integral, and is in Eq. (1),

$$\int_{\tau=t_0}^t \frac{dQ(\tau)}{d\tau} \Phi(t - \tau) d\tau \quad (1)$$

where, Q , is the downwash, $\Phi(s)$, is the Wagner's function, t , is the dimensionless time and τ , is a time instant.

To solve Duhamel's Integral, expressed in Eq. (1), there are several methods, which may be laborious. However, a tool that has not yet been explored, but has the potential to solve this integral and, according to Richard (2014), has great applicability in non local functions, is the fractional calculus.

Searching for simple and efficient solutions, the engineering tend to linearize the nature's behavior. But, the majority of existent systems are naturally non linear, or presents complicated behaviors and not so evident to be mathematically represented. This would be the case of non-stationary aerodynamic load, that is defined by a non local function.

Non local functions are defined by the interaction between events in different time instants or space positions. In the case of time, non local functions are characterized by the memory effect, that is described by Richard (2014) as a phenomenon wherein the actual behavior of an object is influenced not only by present events, but also by the past ones, that is, past states influence the present. However, this influence must fade according to an asymptotic behavior. Such compartment is dependent of Gamma function and Grünwald's coefficients.

One example of non locality in functions can be seen analyzing a particle's movement in a diluted gas. In the first case is present a local field, which doesn't have barriers restricting the particles movement, and in the second, is present a non local field, which has barriers, as shown in Fig. 2.

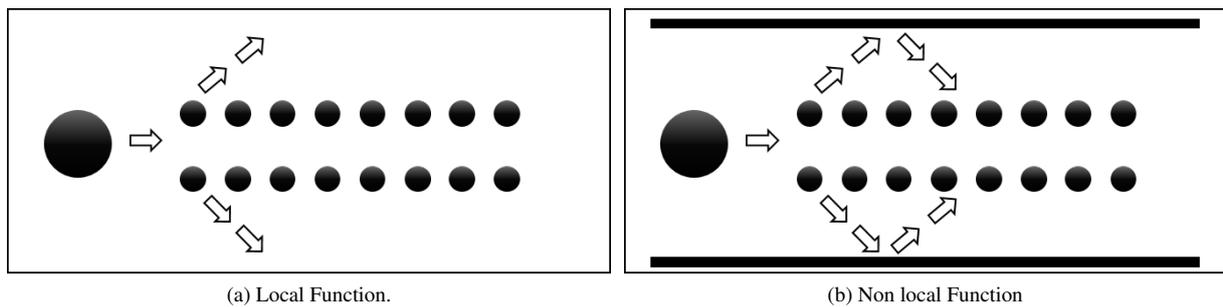


Figure 2: Example of memory effect

It is assumed that the biggest particle moves in a straight horizontal trajectory and will crash with the two first smaller particles in the exact same instant. In Figure 2a, it is understood that when the collision occurs, the smaller particles will assume any trajectory, but without collisions with the other adjacent particles. But in Fig. 2b, the barriers will cause others collisions and interfere with the entire kinematics of the system. The concept of memory, or non locality of functions is not recent and has been observed since the scholastic era, gaining much attention from Descartes. But, after Newton, according to Richard (2014), fell into oblivion. The same can be understood as a smooth transition from local theory, that is, Newtonian, to quantum theory.

According to Mainardi (2009) and Richard (2014), the fractional calculus can be described, from a historical point of view, as an expansion of the concept of derivative, so that its order ceases to be integer, to be any real or imaginary number. This concept was discussed by Leibniz (1695) and for a long time no direct application to the fractional calculus

was found, mainly due to the lack of physical understanding of its meaning. After Leibniz, an example of the application of the fractional calculus was the resolution of the problem of the tautochrone of Abel (1823). From this date, the next work involving fractional calculus was from Mandelbrot (1982) on fractal geometry, a subject that interested many physicists.

With the publication of Raspini (2000), and then by Závada (2002), the interest by wave equations which are fractional in nature was amplified. In 2002, Laskin (2002), based on the definition of Riesz (1949) of fractional derivative, presented an equation of Schrödinger with fractional derivatives. Fractional calculus is an old concept, since early mathematicians have studied it, but gained strength recently in physics and engineering, arousing the interest of researchers and showing potential to be a valuable tool.

According to Swinney (1989), the use of fractional calculus allows to describe the aerodynamic load generated by a lifting surface, and thus to determine an eigenvalue problem that combines the structural equations with those of external forces. Other approaches require the creation and development of stability auxiliary parameters, such as delay states (Silva, 2016) or extended variables (Pereira, 2015). In this context, it is expected that with the use of the fractional calculus, the formulation of these aeroelastic systems becomes simpler and easier to understand.

2. METHODOLOGY

2.1 Theodorsen's Model

The typical section mathematical model can be described as a dynamic system excited by external forces of the aerodynamic kind. Theodorsen and Mutchler (1935) proposed a method to describe how those forces act in an airfoil by means of source-sink singularities, and in this way consider the non-circulatory flow. Besides, Theodorsen used the vorticity of air flow around the airfoil, that generates a wake composed of a plane vortex sheet. Basing on these concepts, he represented the circulatory flow, that is the non-stationary portion of the aerodynamic load. This model respects the Kutta condition, that is, the flow leaves the profile smoothly at the trailing edge.

2.1.1 Wagner's Model

Silva (2016) makes an excellent aerodynamics phenomena summary. According to the author, the profile oscillation generates a wake composed of alternated vortex that modifies the aerodynamic load, in a way that lift and drag do not depend only of the airfoil's instant movement, but also the wake's position and intensity, that is, the entire movement history.

Starting vortex is a non-stationary aerodynamic model in which it is supposed that the profile suffers a sudden variation in attack angle, generating a start vortex that reduces 50% the airfoil's instant load. This effect loses influence with time, once the influence of vortex into the profile has an asymptotic behavior. This vortex effect is defined by Wagner's function, $\Phi(s)$, and is presented in Results section, see Fig. 4. It provides the lift variation history in time, that is, considers the delay states.

2.2 Typical Section

The way the air moves along the wing profile creates a field pressure, $p(x, y, t)$, which is, basically, the aerodynamic load. Integrating it chord-wise will result the lifting force, that acts on the aerodynamic center. Thus, it generates a moment around the profile gravity center, forming the two external forces responsible for the unstable dynamic behavior of the typical section. It is widely known that a third force acts on the profile, the drag force. However, it is not considered in this typical section model.

Figure 3 represents a typical section with two degrees of freedom, that are the Plunge and Pitch movement, respectively $h(t)$ and $\theta(t)$. In the scheme, the point, C , is the center of mass, P , is the elastic axis, T , is a referential point distant three quarters of chord from trailing edge. The a and e parameters determine the position of C and P . The distance, r , is the distance of a infinitesimal mass, dm , to the elastic axis. The moment is M_θ , the axis, $\hat{i}_{1,2}$, are inertial axis and, $\hat{b}_{1,2}$, are non-inertial. The parameters, k_h , and k_θ represents Plunge and Pitch stiffness, respectively.

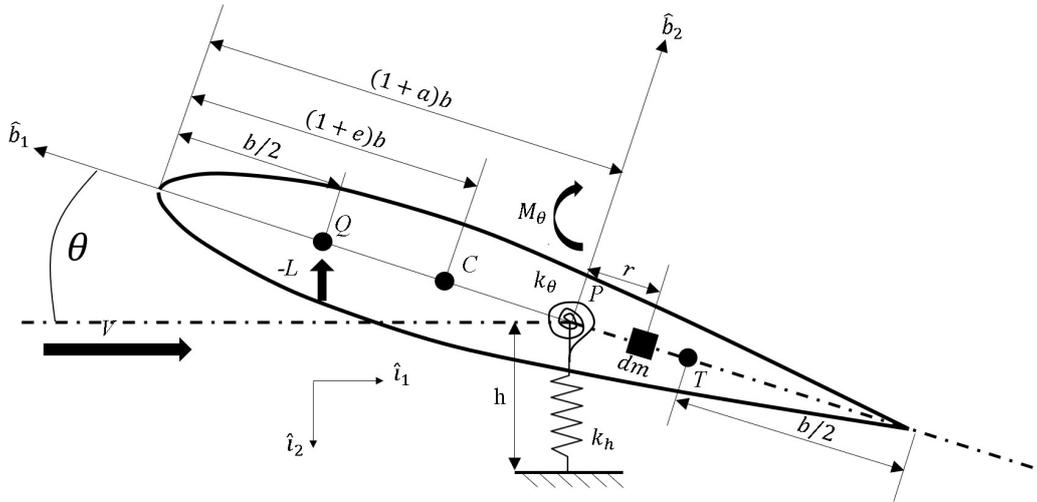


Figure 3: 2 DOF Typical Section

3. Fractional Calculus

Several approaches have been developed along the years such as Riemann (Eq. (4)), Liouville (Eq. (5)), Fourier (Eq. (6)), Caputo (Eq. (7)) and Grunwald-Letnikov (GL) (Eq. (3)) fractional derivatives. The approximation of GL is being widely used in Finite Element formulation due to its easy adaptability (L. Bagley and Torvik, 1979), (Schmidt and Gaul, 2001). The Riemann Liouville fractional operator is shown in Eq. (2), followed by GL approximation (Eq. (3)).

$$D^\alpha[f(t)] = \frac{d}{dt} \int_0^t \frac{f(\tau)}{\Gamma(1-\alpha)(t-\tau)^\alpha} d\tau \quad (2)$$

$$D^\alpha[f(t)] \approx \Delta t^{-\alpha} \sum_{j=1}^{Nt} A_{j+1} f(t-j\Delta t) \quad (3)$$

where, $A_{j+1} = \frac{j-\alpha-1}{j} A_j$, are the Grünwald's coefficients, Γ , is the Gamma function and, α , is the derivative order.

The Riemann fractional operator:

$$f(x) = x^{\alpha-1} \sum_{k=0}^{\infty} a_k x^{k\alpha} \quad (4a)$$

$$D^\alpha[f(x)] = x^{\alpha-1} \sum_{k=0}^{\infty} a_{k+1} \frac{\Gamma((k+2)\alpha)}{\Gamma((k+1)\alpha)} x^{k\alpha} \quad (4b)$$

The Liouville fractional operator:

$$f(x) = \sum_{k=0}^{\infty} a_k e^{kx} \quad (5a)$$

$$D^\alpha[f(x)] = \sum_{k=0}^{\infty} a_k k^\alpha e^{kx} \quad (5b)$$

The Fourier fractional operator:

$$f(x) = a_0 \sum_{k=1}^{\infty} a_k \text{sen}(kx) + \sum_{k=1}^{\infty} b_k \text{cos}(kx) \quad (6a)$$

$$D^\alpha[f(x)] = \sum_{k=1}^{\infty} a_k k^\alpha \text{sen}\left(kx + \frac{\pi}{2}\alpha\right) + \sum_{k=1}^{\infty} b_k k^\alpha \text{cos}\left(kx + \frac{\pi}{2}\alpha\right) \quad (6b)$$

The Caputo fractional operator:

$$f(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha} \quad (7a)$$

$$D^\alpha[f(x)] = \sum_{k=0}^{\infty} a_{k+1} \frac{\Gamma(1 + (k+1)\alpha)}{\Gamma(1 + k\alpha)} x^{k\alpha} \quad (7b)$$

3.1 State of the Art

Over time, several works, as well as Swinney (1989), contributed to the application of the fractional calculus in aeroelasticity. Therefore, the works that relate to the resolution of Duhamel's integral through the fractional calculus are collected, in order to gather information that facilitate its resolution. This State of the Art does not contain absolutely all the contributions already made, in order to filter the content and avoid repetitions.

1. "A Fractional Calculus Model of Aeroelasticity Thesis" by David V. Swinney (1989). Using the typical section model, David Swinney uses the fractional calculus to simplify the instability equations due to aerodynamic load, using them to perform flutter and divergence analyzes. The equations of motion are used with a single eigenvalue problem involving fractional derivatives. In addition, this methodology allows to find the eigenvalues directly, while through other methods it is necessary to create other parameters to create the model of the typical section. The fractional derivative used in the mathematical development is the extended fractional derivative of Riemann Liouville, represented by Eq. (2)

Throughout the paper the author makes comparisons between Theodorsen's function and approximation models created, concluding that the fractional calculus approximation is the most accurate to represent it. In addition, he concludes that the fractional calculus method requires less data than the other methods analyzed, besides being simpler to manipulate mathematically. He also suggests its use in the adaptation of existing models of three-dimensional bodies with aeroelastic behavior, such as the turbine blade and the helicopter rotor dynamics.

In sequence, we will analyze sections of this article that refer to Theodorsen and Wagner's function, in which the fractional calculus is implemented.

Theodorsen's function describes the reduction and phase change of lift with the reduced frequency increment (k), presented in Eq. (8).

$$k = \frac{\omega b}{U} \quad (8)$$

where " ω " is the circulatory frequency, " b " is the semi-chord and " U " is the air speed.

Theodorsen's function is used to denote the aerodynamic circulating forces resulted from the harmonic oscillation of a section in an incompressible flow. The equation (9a) is the Theodorsen's function on the imaginary axis. This function operates in the complex plane if " ik " is replaced by a complex variable " \bar{s} ".

$$C(ik) = \frac{K_1(ik)}{K_0(ik) + K_1(ik)} \quad (9a)$$

$$C(k) = \frac{H_1^{(2)}(k)}{iH_0^{(2)}(k) + H_1^{(2)}(k)} \quad (9b)$$

The equation presented by Swinney (Eq. (9a)) uses a modified Bessel function of the third type that has domain on the imaginary axis, whereas the formulation used here Eq. (9b) is a combination of Bessel's first and second types which is function only of the reduced frequency (k).

Theodorsen's function ranges $[1.0, 0.5]$ while its argument ranges $[0, \infty[$, so to model this function using polynomials, it is necessary that the polynomial fit the indicated interval. Adding parameters to improve the fit of integer power polynomials means adding higher powers of the argument. By introducing a fractional power as a parameter, the function must be manipulated by reducing the order of the polynomial to a fractional value, as in the Eq. (10).

$$\hat{C}(\bar{s}) = \frac{1 + F\bar{s}^\beta}{1 + 2F\bar{s}^\beta} \quad (10)$$

adopting $\beta = 5/6$ to simplify the equation's treatment and the fractional coefficient $F = 2.19$ to cover a larger range of frequencies.

Wagner's function, however, describes the time-dependent circulatory lift of an airfoil in incompressible flow. This function is the convolution of Theodorsen's function with Heaviside's function.

The fractional calculus model of Theodorsen's function leads to a simple representation of Wagner's function by means of the inverse Laplace transform. To find this approximation, it is necessary to employ a binomial expansion of the fractional model of the Theodorsen's function, as shown in Eq. (11).

$$\widehat{C}(\bar{s}) = \frac{1 + F\bar{s}^\beta}{1 + 2F\bar{s}^\beta} = \frac{1}{2} + \frac{1}{4F\bar{s}^\beta} \left[\frac{1}{1 - \left[\frac{-1}{2F\bar{s}^\beta} \right]} \right] = 1 - \frac{1}{2} \sum_{n=0}^{\infty} \left[\left[\frac{-1}{2F\bar{s}^\beta} \right]^n \right] \quad (11)$$

In this formulation, all terms have known inverse Laplace transforms, so the convolution integral can be made by a multiplication in the Laplace domain. The Wagner's function can be defined as:

$$k_1(\sigma) = L^{-1} \left(\frac{C(\bar{s})}{\bar{s}} \right) \quad (12)$$

where $\sigma = \frac{tU}{b}$ is the dimensionless time.

That is,

$$k_1(\sigma) = L^{-1} \left[\frac{1}{\bar{s}} - \frac{1}{2} \sum_{n=0}^{\infty} \left[\left[\frac{-1}{2F\bar{s}^{\beta+1}} \right]^n \right] \right] = u_{-1}(\sigma) - \frac{1}{2} \sum_{n=0}^{\infty} \left[\left[\frac{1}{2F} \right]^n \frac{\sigma n \beta}{\Gamma(1 + n\beta)} \right] \quad (13)$$

2. "The Fractional Order State Equations for the Control of Viscoelastically Damped Structures" by Bagley and Calico (1991). In this paper the authors use the fractional calculus to represent the behavior of large damped space structures as a viable alternative to replace the incorrect or very complicated models that were used at that time. With the fractional calculus, it is possible to incorporate the effects of inertia, damping, elastic and control. In the same way as Swinney (1989), the fractional derivative used is the extended fractional derivative of Riemann Liouville Eq. (2). Using the fractional calculus with the equations of motion, the accelerations that describe the effects of inertia can be written in function of the same operators used to describe the viscoelastic effects. According to the authors, the fractional model is a generalization of the classical initial value model and has the potential to be an alternative to describe linear systems.
3. "Fractional Calculus and Waves in Linear Viscoelasticity" by Francesco Mainardi (2009). The main purpose of this book is to show that the fractional calculus is a suitable method to describe viscoelastic behavior. Among the presented content, the author demonstrates viscoelastic models through the fractional calculus, including some physical interpretations of its application. In addition, the fractional derivatives of Caputo are compared with the fractional derivatives of Riemann Liouville, concluding that both methods coincide for the case where the system is at rest for $t < 0$.
4. "A State-Space Approach to Dynamic Stability of Fractional-Order Systems: The Extended Routh-Hurwitz Theorem" by E. Bologna, L. Deseri and M. Zingales (Bologna *et al.*, 2017). In this article, a generalization of the Routh-Hurwitz's criterion is formulated for a 2 DOF Beck's column system, which corresponds to the case where an elastic beam is subjected to a constant loading at its free end. In order to react with the vertical force, supports in the base of the beam of intermediate behavior between a spring and a damper are used, being that the constitutive equation of this support involves derivatives of fractional order. The stress and strain equations involve Duhamel's integrals, that are solved with the Laplace transform of the functions involved. Then, experimental data are used to find an equation to represent the relaxation function of the material, which is fractional and allows to simplify Duhamel's integral for the fractional derivative of Caputo.
5. "Frequency-Dependent Material Damping Using Augmenting Thermodynamic Fields (ATF) with Fractional Time Derivatives" by David S. Hansen (1990). A damping model of a material is developed using the Augmenting Thermodynamics Fields (ATF), whose equilibrium equations have fractional order derivatives. This model was compared with the viscoelastic model of Bagley and Torvik, which also use fractional derivatives. In addition, it was possible to conclude that through the fractional derivatives, a more generalized ATF model was obtained for

the loss factor and damping rate, which are functions of the frequency, in the case of a uniaxial rod. In this way, the ATF model accurately describes the behavior of several materials, due to the correspondence between such model and other models of thermoelasticity and viscoelasticity.

6. “Dynamics of Viscoelastic Structures” by Buhariwala and Hansen (1986). This work presents a general method for modeling damped materials in dynamic systems. Initially a viscoelastic dissipation model is assumed and for that, the constitutive relations are reviewed and the fractional derivatives are used, as suggested by Caputo. Then the equations of motion of a structure made of an anisotropic viscoelastic material are formulated, starting from the virtual working principle. These equations results in the Duhamel’s integral, which is solved through the Laplace transform, obtaining an eigenvalue problem.
7. “Applications of Fractional Calculus in Solving Abel-Type Integral Equations: Surface-Volume reaction Problem” by Ryan M. Evans, Udit N. Katugampola and David A. Edwards (Evans *et al.*, 2016). In this document approximations are elaborated to generalize the tautochronic’s problem of Abel. In addition, the document implements solutions that use fractional derivatives in real applications, such as the problem of surface-volume reactions, whose governing equations tend toward a non-linear equation that contains a fractional integral.
8. “Quadratic Optimal Control Theory For Viscoelastically Damped Structures Using a Fractional Derivative Viscoelasticity Model” by Richard N. Walker (1989). The aim of this thesis is to develop a control law for passively damped structures using viscoelastic materials, which are modeled by a stress-strain law using fractional calculus. In addition, this law should also cover structures incorporated by active damping due to applied forces and moments. For that, the optimal quadratic control theory is modified to fit with systems with fractional derivatives in the state vector.

To perform this approximation, it is necessary to expand the structure’s equation of motion to a fractional state space, this system of order $1/n$, being $n \in \mathbb{I}$ based on the damping constitutive law of the material. Thus, this approximation restricts the theory to only materials that have fractional derivatives in their constitutive laws.

9. “Fractional Calculus Formulation of the Quasi-Static Viscoelastic Problem” by Joseph B. McCullough (1989). The purpose of this thesis is to incorporate the temperature and frequency dependence of the behavior of viscoelastic materials in the dynamic module with the use of fractional calculus. The fractional derivatives and the temperature change function allow a dynamic modulus model that can be used in a finite element formulation to analyze the system response of a structure containing elastic and viscoelastic members. This model is limited for low frequencies and use of the dynamic modulus in the transition region, but its response is relatively simple and has few parameters. In addition, the solutions were developed in the Laplace domain to facilitate the manipulation of fractional derivatives. For that, the reduced variables method was used together with the use of the WLF equation.

The solution was made using a lattice with 9 degrees of freedom, composed of aluminum and rubber stems of neoprene subject to the variation of temperature.

10. “Modelling and Analysis of Kernel Function and Development of Equivalent Theodorsen Function for Three-Dimensional Aeroelastic Analysis” by John K. Ryder (1990). In this thesis, the integral in equation of pressure-downwash is integrated analytically, using mathematical simplifications by means of fractional calculus. In addition, the three-dimensional aerodynamic instability problem was simplified by developing an equivalent Theodorsen’s function. This development allows aerodynamic loads, produced by time-dependent motion, of a finite airfoil, to be written in a more compact mathematical form and easily applicable in the design of a control system.

In addition, considering a rectangular wing in an incompressible subsonic flow, a fractional calculus model is developed for the Kernel’s function, which relates the aerodynamic load with the downwash on the wing. The same is defined in the complex s -plane, but due to complicated mathematics, it is normally analyzed on the imaginary axis. This restricts the use of the Kernel’s function for stability analyzes such as flutter.

The equivalent Theodorsen’s function is deduced by relating the lift coefficients of the two-dimensional and three-dimensional cases, in an incompressible flow. The modeling was motivated by Swinney Swinney (1989) that successfully modeled Theodorsen’s function using fractional calculus, for the two-dimensional case. The form of the equivalent Theodorsen’s function allows to include the three-dimensional effects in the equations of motion of the two-dimensional case.

11. “Applications of Generalized Derivatives to Viscoelasticity” by Ronald L. Bagley (1979). According to the author, generalized fractional order derivatives are used to construct stress-strain relationships for viscoelastic materials, based on sine wave behavior of observed materials. In addition, the constitutive relations of generalized derivatives are able to describe the dependence with the frequency, rigidity and damping of materials.

The focus of this work is the finite elements formulation and the equations of motion’s solutions of structures that contain elastic and viscoelastic components.

12. "Formulation and Inversion of Transfer Functions of Combined Elastic/Viscoelastic Structures" by S. B. Skaar, G. A. Nariboli and L. Tang (Skaar *et al.*, 1989). In the document a method is developed for construction systems of transfer functions, for structures which are assemblies of rigid, elastic or viscoelastic members, connected by joints. For that, a triangular structure is used. First an elastic version is made and the response frequency is checked by comparison with a finite elements model. Then one of the limbs is replaced by another highly damped viscoelastic limb, modeled with a constitutive law of fractional derivatives.
From this, the two responses are compared and a method is developed in which the impulse response of the elastic/viscoelastic structure can be determined by reversing the transfer function. It is interesting to invert this function because it contains the Laplace variable s with fractional powers.
13. "Improved Solution Techniques for the Eigenstructure of Fractional Order Systems" by Michele Lynn Devereaux (1989). In this work we consider the problem of a damped viscoelastic rod, in which a viscoelastic model of fractional derivatives of four parameters is used to describe the stress-strain relationship. Throughout the paper, it is demonstrated step by step how to apply the fractional derivatives in the equations of viscoelastic materials, a demonstration that facilitates the understanding of how to manipulate such tool.
14. "Fractional State Feedback Control of Undamped and Viscoelastically-Damped Structures" by David L. Yang (1990). The purpose of this work is to develop and demonstrate a control theory, incorporating the structure response in the form of fractional derivatives. Since the theory of control was developed with a finite elements formulation of a structure, which can be damped or not. It is shown that there is a relationship between the traditional state vector including displacements and structural velocities, with the fractional state vector including fractional derivatives of structural responses. This relation allows to modify the linear quadratic regulator theory to include the application of fractional order states in response control.
15. "Finite Element Formulation of Viscoelastic Constitutive Equations Using Fractional Time Derivatives" by Schmidt and Gaul (2001). Fractional derivatives are used to deduce a generalization of the viscoelastic constitutive equation of differential operator. These equations result in an improvement of curves fit, mainly for experimental data of long intervals or that cover several frequencies. In addition, this method uses fewer parameters than non-fractional derivatives methods.

The three-dimensional constitutive fractional equations based on the Grünwaldian formulation are implemented in an elastic finite element model. Then, the parameters of the fractional model in the domain of time and frequency are compared with the non-fractional constitutive equations. Soon after, the material model is used to analyze a viscoelastic structure through a finite element model.

4. RESULTS

The matrix equation of motion for a two DOF typical section is represented in Eq. (14).

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} Q_h \\ Q_\theta \end{Bmatrix} \quad (14)$$

where, m , is the mass, S_θ , is the static moment of mass, I_θ , is the angular inertia moment around the elastic center, c , is the structural damping and, $k_{h,\theta}$, are the stiffness. The, Q , is the aerodynamic load and is separated in circulatory and non circulatory components.

$$Q_h = - \left(L^{(c)} + L^{(nc)} \right) \quad (15a)$$

$$Q_\theta = M_{ea}^{(c)} + M_{ea}^{(nc)} \quad (15b)$$

where, L , is lift and, M_{ea} , is the pitching moment.

Equations (16a) and (16b) present the circulatory lift and pitching moment in frequency domain, using the Theodorsen's function, $C(k)$. While Eqs. (17a) and (17b) are in time domain, using Wagner's function, $\Phi(s)$. Equation (17a) introduces the Duhamel's Integral, whose solution is not a simple task to be undertaken. This paper will introduce the basic concept involving the fractional calculus approach to solve convolution integrals. The major objective of the research project is to solve the Duhamel's integral by means of fractional calculus. The non circulatory components can be found in Silva (2016).

$$L^{(c)}(f) = 2\pi\rho V b Q C(k) \quad (16a)$$

$$M_{ea}^{(c)}(f) = 2\pi\rho V b^2 \left(a + \frac{1}{2} \right) Q C(k) \quad (16b)$$

$$L^{(c)}(t) = 2\pi\rho Vb \int_{\tau=t_0}^t \frac{dQ(\tau)}{d\tau} \Phi(t-\tau) d\tau \quad (17a)$$

$$M_{ea}^{(c)}(t) = -2\pi\rho Vb^2 Q \frac{1}{2} + \left[a + \frac{1}{2} \right] L^{(c)}(t) \quad (17b)$$

where, b , is the semi-chord, ρ , is air density and, V , the air speed.

Jones suggested an approximation for Wagner's function, which is presented in Eq. (18) and its curve in Fig. 4.

$$\Phi(s) = 1 - 0.165e^{-0.041s} - 0.335e^{-0.32s} \quad (18)$$

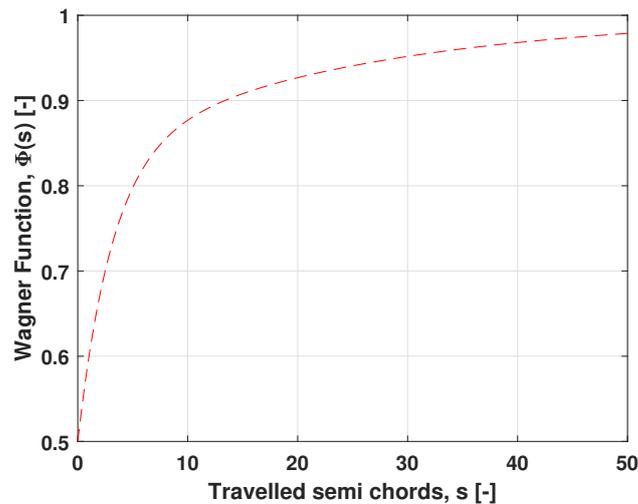


Figure 4: Wagner's Function

5. DISCUSSION

This work introduces the reader into aeroelasticity of a typical section, presenting relevant concepts and tools in flutter's prediction and, in parallel, fractional calculus is revised. The State of the Art presented brings together the main documents that relate fractional calculus and the Duhamel's Integral. This review emphasizes the small amount of work about fractional calculus in aeroelasticity, thus justifying the implementation of fractional derivative to solve Duhamel's Integral. We hope to, in future works, use fractional derivatives to determine the non stationary aerodynamic load in a Matlab environment, since, as shown, in this context several studies used fractional calculus only to represent viscoelastic systems.

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7. RESPONSIBILITY NOTICE

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