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An assessment of reduced order modeling techniques applied to unsteady flows

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Abstract. *High fidelity modeling of complex non-linear dynamical systems is necessary for the comprehension of several processes found in engineering systems. For example, in fluid mechanics, numerical simulations aid in the design and optimization of more efficient aircraft, automobiles, engines and wind turbines. Accurate numerical simulations of unsteady flows are required to perform such analyses and they are associated with high computational costs since high resolution spatial and temporal schemes are typically employed to resolve the broad range of spatial and temporal scales. On one hand, small time steps are required to capture the fine temporal scales of the problem. On the other hand, the simulations need to be carried out for long periods to obtain meaningful statistics. Reduced order modeling is a methodology that can considerably reduce the computational costs associated with simulations of unsteady flows. In order to be useful, the reduced order model (ROM) should be able to reproduce the relevant physical mechanisms observed in the full order model (FOM). In this work, ROMs of an incompressible flow past a cylinder are studied via Galerkin-type projection techniques of the Navier-Stokes equations. Proper orthogonal decomposition (POD) is used to generate a reduced basis space from high fidelity simulations. Although the current methodology is data driven, it is linked to the physics of the problem through direct projection of the partial differential equations governing the fluid flows. Galerkin and Petrov-Galerkin methods are employed in the ROMs and results are analyzed in terms of the relative error compared to the FOM.*

Keywords: *Reduced order modeling, proper orthogonal decomposition, Galerkin projection, Petrov-Galerkin projection, Navier-Stokes equations*

1. INTRODUCTION

The study of computational methods for fluid flows is important for the understanding of a broad range of engineering problems. For example, it is necessary to understand the physics of turbulence to improve the design of aircraft, ships, high-speed trains, automobiles, etc. In the above context, investigations of turbulent flows including the associated physical mechanisms responsible for the production, transport, and dissipation of turbulence occur since Reynolds' experiments in the 19^{th} century (Rowley *et al.*, 2012). The use of computational methods for solving engineering problems has increased due to the higher computational power achieved in the last fifty years. However, despite the increase in computer performance, in fluid mechanics, accurate numerical simulations of unsteady flows are still very costly. In such problems, high resolution spatial and temporal schemes are typically employed to resolve the broad range of spatial and temporal scales. One one hand, small time steps are required to capture the relevant temporal scales of the problem. On the other hand, the simulations need to be carried out for long periods to obtain meaningful statistics. Therefore, the computational cost for simulations of realistic engineering flows is still, in great part, unfeasible (Coutinho and Silva, 2015).

One of the most used CFD methodologies used today for solving industrial flow problems is that based on solution of the Reynolds-Averaged Navier-Stokes (RANS) equations. This type of formulation solves the temporally filtered Navier-Stokes equations and requires the use of turbulence models that work only for specific flows. Instead of resolving the turbulent scales of the flow, this methodology models the entire wavenumber range of scales, from production to dissipation. Direct numerical simulation (DNS) and large eddy simulation (LES) are methodologies that have been used mostly for studies of unsteady canonical flows at lower Reynolds numbers and for simplified configurations. The first formulation solves all the spatial and temporal scales associated with the flow while the second solves the larger, more energetic scales, and models the smaller, more isotropic and universal scales. Both methodologies present high computational costs since they require the application of parallel computers and numerical methods with high spatial and temporal resolution. Despite these drawbacks, these techniques are relevant since they can provide turbulent flow databases for the study of turbulence physics.

It is in this context that methods for construction of reduced order models (ROMs) stand out when compared to traditional methods used for computational fluid dynamics, such as LES and DNS. The application of these methods allows the construction of simpler models by reducing the dimensionality of the problem and, hence, its computational cost (Bergmann and Laurent, 2003b). In recent years, ROMs have attracted the attention from mathematicians, physicists and engineers interested in the solution of complex non-linear dynamical systems. The use of ROMs can lead to a reduction of the computational effort required to approximate a problem without a loss of precision.

In the present work, our reduced order model strategy starts by obtaining the most energetic spatial modes of the unsteady flows through application of proper orthogonal decomposition (POD) (Bergmann and Laurent, 2003a). The POD spatial modes are then used in Galerkin-type projections to obtain ordinary differential equations in place of the strong formulation. A drastic reduction in the number of degrees of freedom for a high accuracy representation of the problem is common and there are many cases where the cost reduction is higher than two orders of magnitude (A Quarteroni and Negri, 2016). However, it is not always convenient to discard some modes with less energy. For example, there are cases where such modes represent relevant dynamical processes being responsible for the stability of the problem (Iliescu and San, 2014). Another possible complicating factor is the non-linearity of some problems. These issues should be addressed by the ROM formulation. Here, the ROM of an incompressible flow past a cylinder is studied via POD-Galerkin and Petrov-Galerkin projections of the Navier-Stokes equations. Although the current methodologies are data driven, they are linked to the physics of the problem via projection of the partial differential equations governing the fluid flow. The present results are analyzed in terms of the error relative to the FOM and details of the methodologies are provided.

2. METHODOLOGY

2.1 Proper orthogonal decomposition

In the proper orthogonal decomposition, a turbulent velocity field can be decomposed as follows

$$\mathbf{u}(\mathbf{x},t) = \langle \mathbf{u}(\mathbf{x}) \rangle + \sum_{i=1}^{N} \mathbf{\Phi_i}(\mathbf{x})\mathbf{a_i}(t)\ , \tag{1}$$

where $\mathbf{\Phi_i}$ represents the orthonormal spatial modes, a_i represents the temporal modes, which will be calculated through the projection schemes, and N is the number of data snapshots extracted from the numerical simulation or another representative problem. Here, i is the mode index. The reconstruction of the fluctuation velocity field can then be approximated by

$$\mathbf{u}'(\mathbf{x},t) \approx \sum_{i=1}^{M} \mathbf{\Phi_i}(\mathbf{x})\mathbf{a_i}(t)\ , \tag{2}$$

where M is the number of modes used in the reconstruction of the reduced order model. In practical applications, the number M is smaller than N which characterizes the ROM.

For the POD, it is usually necessary to calculate the covariance matrix

$$\mathbf{C_{i,j}} = \frac{1}{N} \int_{\Omega} \mathbf{u_i}(\mathbf{x},t_i)\mathbf{u_j}(\mathbf{x},t_j)\, dx\ , \tag{3}$$

which is symmetric positive semidefinite and, therefore, allows the use of singular value decomposition. From this matrix factorization we have the calculation of the orthonormal spatial modes that will be used in the Galerkin projection to reconstruct the system of ordinary differential equations. These equations will determine the evolution of temporal modes. The modes are calculated so that the reconstruction is optimal in the sense of the truncated mean quadratic error (4)

$$\varepsilon_m = \overline{\left\| \mathbf{u}(\mathbf{x},t) - \sum_{i=1}^{M} \mathbf{\Phi_i}(\mathbf{x})\mathbf{a_i}(t) \right\|^2}\ . \tag{4}$$

Next, the amount of Relative Information Content (RIC) of the singularity decomposition of a matrix A is defined by equation (5). This is an important parameter in determining the number of modes to be used in the construction of the reduced order model. However, it is important to point out that there are cases in which the lower energy modes are necessary to guarantee stability of the model.

$$RIC(m) = \frac{\sum_{i=1}^{m} \sigma_i}{\sum_{i=1}^{M} \sigma_i}\ . \tag{5}$$

2.1.1 Method of snapshots

The spatial complexity grows rapidly, especially when dealing with multidimensional problems, which imply large computational costs. The snapshot method is an alternative to obtain a modal basis and was introduced in Sirovich (1987).

This method consists in solving a much smaller eigenvalue problem that is proportional to the time complexity (number of snapshots). The method can be written as

$$\mathbf{X^T X}\psi_\mathbf{j} = \lambda_j \psi_\mathbf{j}, \quad \psi_\mathbf{j} \in \mathbb{R}^m, \quad m \ll n, \tag{6}$$

where $\mathbf{X^T X}$ is of dimension $m \times m$. The spatial modes Φ are recovered using the temporal modes ψ which are obtained through the reduced eigenvalue problem

$$\Phi_j = \mathbf{X}\psi_\mathbf{j}\frac{1}{\sqrt{\lambda_j}} \in \mathbb{R}^n, \quad k = 1, 2, \ldots, m, \tag{7}$$

which can also be written as follows:

$$\mathbf{\Phi} = \mathbf{X\Psi\Lambda}^{-1/2}, \tag{8}$$

where $\mathbf{\Phi} = [\Phi_1 \; \Phi_2 \; \ldots \; \Phi_m] \in \mathbb{R}^{n\times m}$ and $\Psi = [\psi_1 \; \psi_2 \; \ldots \; \psi_m] \in \mathbb{R}^{m\times m}$. This method is widely used because of its reduced computational effort and less memory usage.

2.2 Projection Methods

Consider a system of partial differential equations $\mathbf{F}(\mathbf{u})$ defined in an connected open region whose boundary Γ is well defined

$$\begin{cases} \mathbf{F}(\mathbf{u}) = \frac{d\mathbf{u}}{dt} - \mathbf{G}(\mathbf{u}) = \mathbf{0} \quad in \quad \Omega \\ \mathbf{u}(t=0) = \mathbf{u_0} \\ \mathbf{u} = \mathbf{g} \quad on \quad \Gamma \end{cases} \tag{9}$$

Here, the term $\mathbf{u}$ is a function of space and time, and $\mathbf{G}(\mathbf{u})$ is a nonlinear operator. The orthonormal basis $\mathbf{\Phi}$ is obtained by the POD method. The state variable $\mathbf{u}$ is approximated as the linear combination of this basis vector.

A solution is sought by enforcing orthogonality of the residual $\mathbf{F}(\mathbf{u}) \approx \mathbf{R}(\hat{\mathbf{u}}) \neq \mathbf{0}$ as

$$\langle \mathbf{\Psi_i}, \mathbf{F}(\mathbf{u}) \rangle = \mathbf{0} \tag{10}$$

where $\mathbf{\Psi_i}$ is the projection test basis. Here, $\langle . \rangle$ denotes the L^2 norm. A projection method is generally called Galerkin (Petrov-Galerkin) when test $\mathbf{\Psi}$ and solution $\mathbf{\Phi}$ bases are equal (different).

$$\langle \mathbf{\Psi_i}, \mathbf{F}(\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{\Phi_i a_i}) \rangle = \mathbf{0}. \tag{11}$$

2.2.1 Galerkin projection

A projection method is, by definition, called Galerkin when the test basis is equal to the solution basis $\mathbf{\Psi_i} = \mathbf{\Phi_i}$. In this section, the POD-Galerkin methodology is applied to the two-dimensional version of the incompressible Navier-Stokes equations (12). We start from the momentum and continuity equations for an incompressible flow

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\nabla^2 \mathbf{u}\,, \tag{12a}$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,. \tag{12b}$$

The POD modes are then projected on the set of equations (12) using the standard Galerkin method. Equation (13) is obtained by an inner product of the Navier-Stokes equations with the POD spatial modes, which are orthonormal, and the pressure term is canceled by the continuity equation

$$\left\langle \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla}\mathbf{U}, \mathbf{\Phi_i} \right\rangle + \frac{1}{Re}\langle \boldsymbol{\nabla}\mathbf{U}, \boldsymbol{\nabla}\mathbf{\Phi_i} \rangle = 0\,. \tag{13}$$

The substitution of equation (1) into (13) results in the following set of nonlinear equations for the temporal coefficients

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathbf{Aa} + \mathbf{a^T N a}\,, \tag{14}$$

where the coefficients of equation (14) are given by equations (15) and (16) as

$$A_{ij} = -\frac{1}{Re}\langle \boldsymbol{\nabla}\boldsymbol{\Phi}_\mathbf{j}, \boldsymbol{\nabla}\boldsymbol{\Phi}_\mathbf{i}\rangle = -\iint_\Omega \left(\frac{\partial \boldsymbol{\Phi}_\mathbf{j}}{\partial x}\frac{\partial \boldsymbol{\Phi}_\mathbf{i}}{\partial x} + \frac{\partial \boldsymbol{\Phi}_\mathbf{j}}{\partial y}\frac{\partial \boldsymbol{\Phi}_\mathbf{i}}{\partial y} \right) dx\, dy \tag{15}$$

and

$$N_{ijk} = -\langle \boldsymbol{\Phi}_\mathbf{j} \cdot \boldsymbol{\nabla}\boldsymbol{\Phi}_\mathbf{k}, \boldsymbol{\Phi}_\mathbf{i}\rangle = -\iint_\Omega \left(\boldsymbol{\Phi}_\mathbf{j}\frac{\partial \boldsymbol{\Phi}_\mathbf{k}}{\partial x} + \boldsymbol{\Phi}_\mathbf{j}\frac{\partial \boldsymbol{\Phi}_\mathbf{k}}{\partial y} \right) \boldsymbol{\Phi}_\mathbf{i}\, dx\, dy\,. \tag{16}$$

In the equations above, the indices $i, j, k = 1, \ldots, M$. The initial condition of the temporal coefficients is found by solving (17) as

$$\mathbf{a_{0i}} = \langle \mathbf{u_0}, \boldsymbol{\Phi}_\mathbf{i}\rangle = \iint_\Omega \mathbf{u_0}\boldsymbol{\Phi}_\mathbf{i}\, dx\, dy\,. \tag{17}$$

Equation (14) is then solved numerically using a fourth-order Runge-Kutta scheme to obtain the evolution of temporal modes.

2.2.2 Least-squares Petrov-Galerkin projection

The lest-squares Petrov-Galerkin (LSPG) projection method (Carlberg *et al.*, 2013, 2017; A Quarteroni and Negri, 2016) is an alternative to the traditional Galerkin projection seen previously. In this technique, the test basis $\boldsymbol{\Psi}_\mathbf{i}$ is given by

$$\boldsymbol{\Psi}_\mathbf{i} = \boldsymbol{\Phi}_\mathbf{i}^\mathbf{T}\frac{\partial \mathbf{R}(\hat{\mathbf{u}})}{\partial \mathbf{a}} \tag{18}$$

which is equivalent to solving the minimization problem

$$\underset{\hat{u}\in V}{\text{minimise}}\, \|\mathbf{R}(\hat{\mathbf{u}})\|_2. \tag{19}$$

The previous regression problem (19) may be linear or non-linear depending on the equation being solved and the time marching method. The optimal time step for the implicit LSPG is given by an intermediate value and time step refinement does not guarantee minimum error as discussed by (Carlberg *et al.*, 2013, 2017). Initial conditions for temporal modes are also given by Eq. (17).

A further approximation to the LSPG method is necessary because the problem still scales with the full order model, even though the test space is reduced. Hence, hiperreduction techniques need to be applied to reduce computational costs. Problems with strong nonlinearities or non-affine parameter dependences also require the use of such techniques. The gappy POD method (Everson and Sirovich, 1995; Willcox, 2006) and the Discrete Empirical Interpolation Method (DEIM) can be used to solve this dimensionality problem by sampling the most energetically relevant nodes. In this paper, POD modes are sampled according to the Accelerated Greedy Missing Point Estimation Procedure (Zimmermann and Willcox, 2016).

3. Results

This section compares the performance of Galerkin (G-ROM) and Petrov-Galerkin (PG-ROM) reduced order models. The main objective is to recover the full order model (CFD solution) as accurately as possible. This is viewed as a necessary first step for applying theses methods to more complex flows. The corresponding full order model is obtained by solution of equations (12) in a curvilinear O-type grid with $52,750$ nodes. A high-order spectral-like finite difference method for spatial discretization and a combination of third-order explicit and second-order implicit time marching schemes for the time integration are used.

The POD method is performed with 200 snapshots equally spaced in time. The singular values of the spatial modes are obtained and they can be seen in Fig. 1 together with their corresponding RIC in Fig. 2. The first four POD modes contain more than 98% of the energy of the system

The most energetically relevant modes are illustrated in Fig. 3. The 1st and 2nd modes are similar and paired. The 3rd and 4th are also paired but only the 3rd mode is shown. All these modes show an oscillatory pattern which is related to the von Karman vortex street formed along the cylinder wake. Such phenomenon is expected for the current flow at Reynolds number $Re_D = 100$. While the first mode pair is related to the lift fluctuations, the second one is related to drag fluctuations.

The models obtained by both G-ROM and PG-ROM present a good structural agreement with the FOM as can be seen in figs. 9 to 14. The solution observed represents the flow configuration at the last time step. In these figures, the

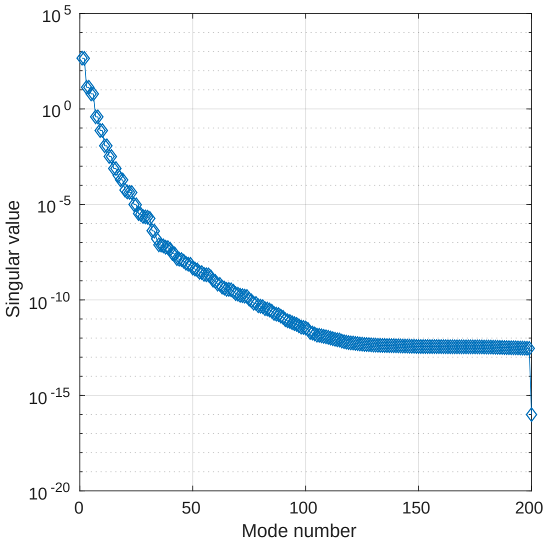

Figure 1. Singular values.

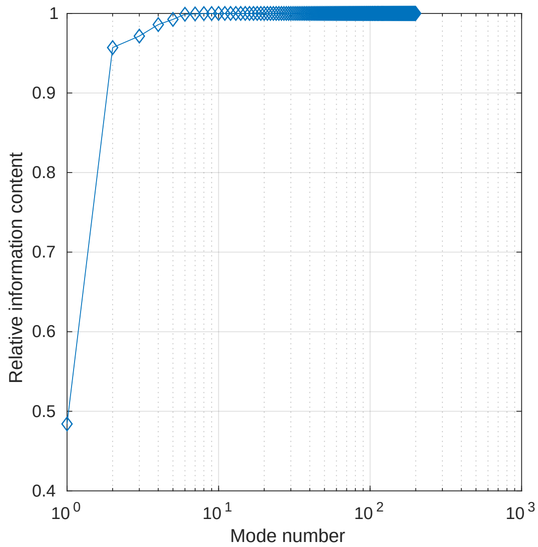

Figure 2. Relative content information.

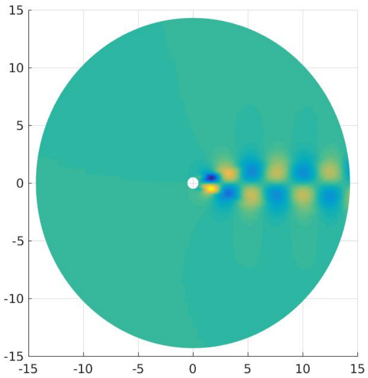

Figure 3. 1^{st} mode of **u** component.

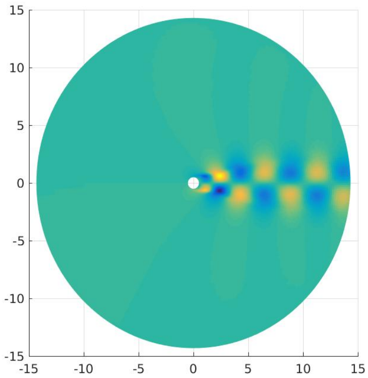

Figure 4. 2^{nd} mode of **u** component.

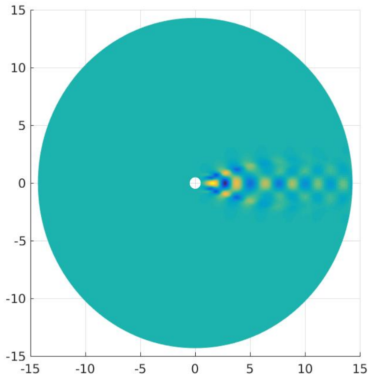

Figure 5. 3^{rd} mode of **u** component.

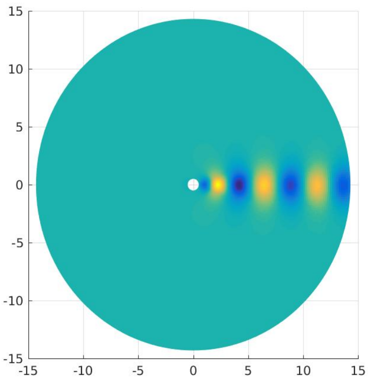

Figure 6. 1^{st} mode of **v** component.

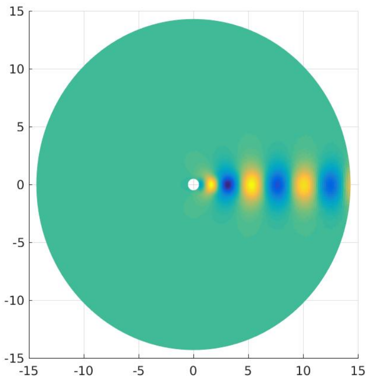

Figure 7. 2^{nd} mode of **v** component.

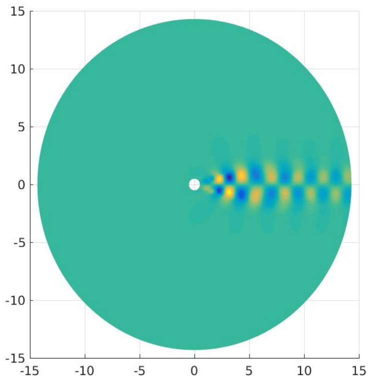

Figure 8. 3^{rd} mode of **v** component.

ROMs are reconstructed after computing the evolution of the first two temporal modes. Although the mesh has more than 52 thousand points, the PG-ROM uses only 100 points obtained via the hyper-reduction technique, what considerably reduces the cost of the minimization procedure. However, this should impact the accuracy of the model.

Error evolution is shown by Figs. 15 and 16. As can be seen, the root mean square error (RMS) increases, but stays in an acceptable range for the time period of the simulation. Additionally, it is important to note that ROMs for **v** are topologically very similar, in spite of the greater RMS error when compared to **u**. For the POD-PG method,

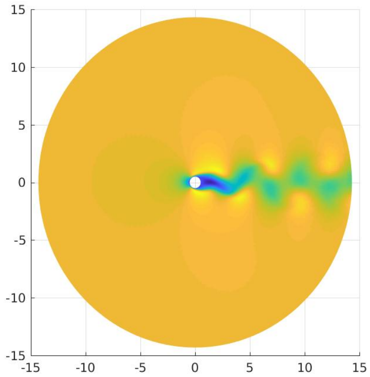

Figure 9. FOM of the **u** component.

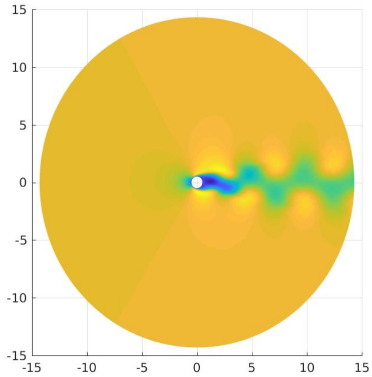

Figure 10. G-ROM of **u** component.

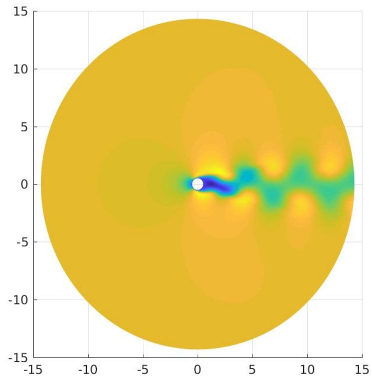

Figure 11. PG-ROM of **u** component.

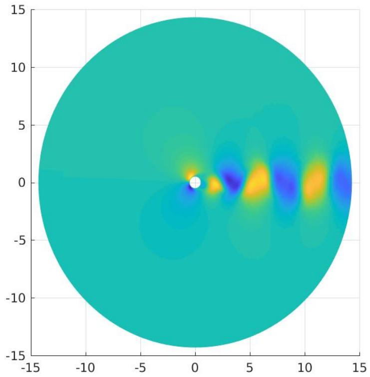

Figure 12. FOM of **v** component.

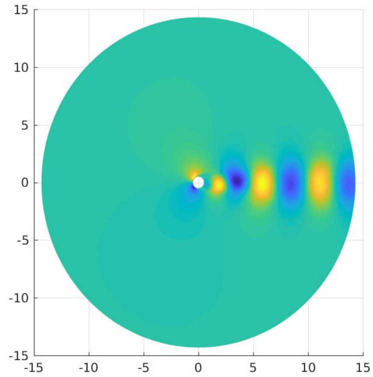

Figure 13. G-ROM of **v** component.

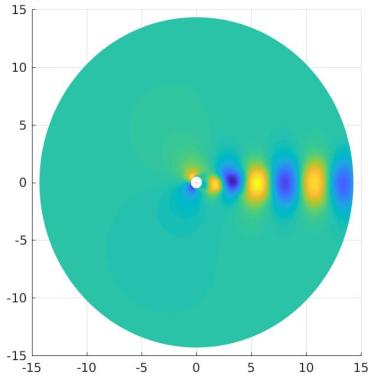

Figure 14. PG-ROM of **v** component.

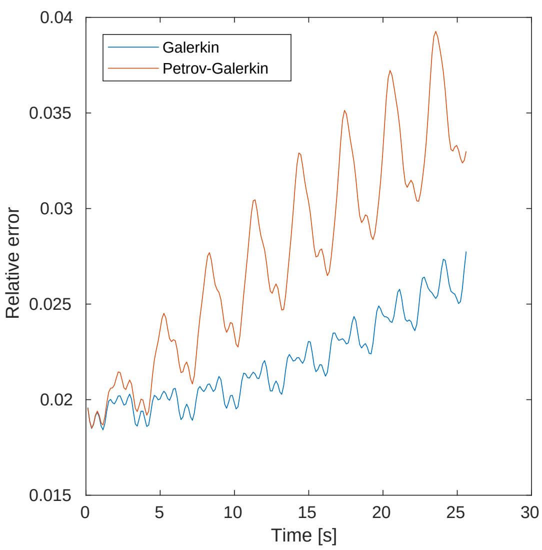

Figure 15. Relative error of **u**.

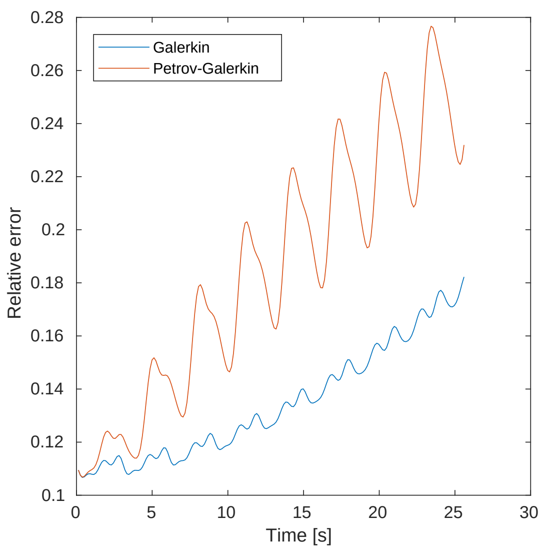

Figure 16. Relative error of **v**.

The temporal coefficients obtained by the ROM procedures are presented and compared to the FOM temporal coefficients in figure figs. 17 and 18. Again, it is clear that the present ROMs reproduce the dynamics of the limit cycle oscillations in this problem; however, they exhibit an unstable character and would need to be modeled for longer time integrations.

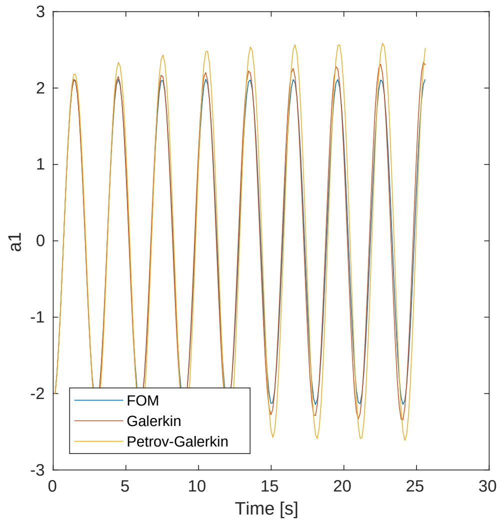

Figure 17. Evolution of first temporal mode for the FOM, Galerkin and Petrov-Galerkin projections.

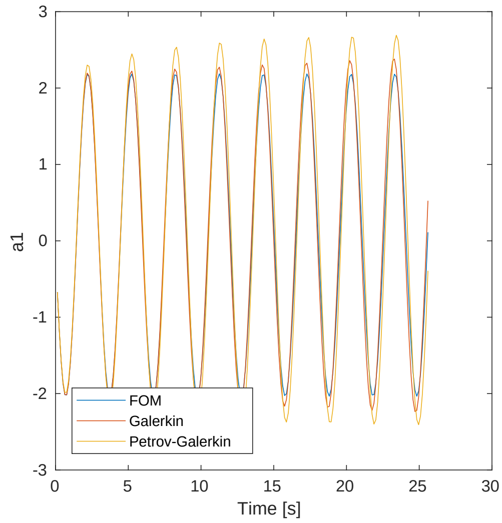

Figure 18. Evolution of second temporal mode for the FOM, Galerkin and Petrov-Galerkin projections.

4. CONCLUSIONS

Reduced order model (ROM) techniques are analyzed for the solution of the unsteady flow past a cylinder. The methods are built using proper orthogonal decomposition (POD) and different projection methods of the Navier-Stokes equations. The Galerkin and Petrov-Galerkin projection techniques are tested and they reduce the original set of partial differential equations that govern the fluid flows to a dynamical system represented by a set of ordinary differential equations. From the current analysis, one can observe that the projection techniques present instabilities and accuracy issues for long time integrations of the ROMs. The Petrov-Galerkin method could be made more accurate by sampling more mesh points in the Gappy POD, but the cost would also increase. For example, using the full mesh information on the reconstruction would lead to a more accurate ROM when compared to Galerkin projection. Future research will focus in stabilization techniques for long time integration.

5. Acknowledgements

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6. References

A Quarteroni, A.M. and Negri, F., 2016. *Reduced Basis Methods for Partial Differential Equations: An Introduction*. Springer, Switzerland, 1st edition.

Bergmann, M. and Laurent, C., 2003a. "Proper orthogonal decomposition: An overview". Lecture Series 2003-04 von Karman Institute for Fluid Dynamics.

Bergmann, M. and Laurent, C., 2003b. "Two typical applications of pod: coherent structures education and reduced order modelling". Lecture Series 2003-04 von Karman Institute for Fluid Dynamics.

Carlberg, K., Barone, M. and Antil, H., 2017. "Galerkin v. least-squares petrov-galerkin projection in nonlinear model reduction". *Journal of Computational Physics*, Vol. 330, pp. 693–734.

Carlberg, K., Farhat, C., Cortial, J. and Amsallem, D., 2013. "The gnat methodfor nonlinear model reduction: Effective implementation and application to computational fluid dynamics and turbulent flows". *Journal of Computational Physics*, Vol. 242, pp. 623–647.

Coutinho, A.L.G.A. and Silva, D.F.C., 2015. "Practical implementation aspects of galerkin reduced order models based on proper orthogonal decomposition for computational fluid dynamics". *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, Vol. 37, No. 1, pp. 1309–1327.

Everson, R. and Sirovich, L., 1995. "Karhunun-loève procedure for gappy data". *Optical Society of America*, Vol. 12, pp. 1657–1664. doi:10.1364/JOSAA.12.001657.

Iliescu, T. and San, O., 2014. "Proper orthogonal decomposition closure models for fluid flows: Burgers equation". *International Journal of Numerical Analysis and Modeling, Series B*, Vol. 5, pp. 217–237.

Rowley, C.W., Holmes, P., Lumley, J.L. and Berkooz, G., 2012. *Reduced Basis Methods for Partial Differential Equations:*

An Introduction. Cambridge, New York, 2nd edition.

Sirovich, L., 1987. "Turbulence and the dynamics of coherent structures. Part 1: Coherent structures". *Quarterly of Applied Mathematics*, Vol. 45, pp. 561–571. doi:10.1090/qam/910463.

Willcox, K., 2006. "Unsteady flow sensing and estimation via the gappy proper orthogonal decomposition". *Computers & Fluids*, Vol. 35, pp. 208–226. doi:10.1016/j.compfluid.2004.11.006.

Zimmermann, R. and Willcox, K., 2016. "An accelerated greedy missing point estimation procedure". *SIAM Journal on Scientific Computing*, Vol. 38, pp. A2827–A2850. doi:10.1137/15M1042899.

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