A DETAILED THERMAL ANALYSIS FOR INTERNAL FLOW IN MICROCHANNELS UNDER SPECIFICS BOUNDARY CONDITIONS USING INTEGRAL TRANSFORMS TECHNIQUE FOR CONVECTION DIFFUSION EQUATION

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Abstract. *Motivated by the increasing development along the last decades and ongoing studies in microfluidics and his wide range of applications on engineering such as heat transfer optimization, the present work aims a study showing the behavior of heat transfer by convection through an internal flow in parallel plates micro channels. The two-dimensional steady state convection diffusion equation in primitive variables is studied under appropriate boundary conditions, such as prescribed temperature with and without the micro-scale effects to ensure corrects physical and practical conclusions, where velocity components are given as input data. The Classical Integral Transform Technique (CITT), is then applied in the hybrid numerical-analytical solution of the proposed formulation, as a technique for solving partial differential equations, by integral transforming the governing equation in one direction, preferably the variable belonging to the eigenvalue problem, the remaining ordinary differential equation depending on the other variable, is numerically calculate by computational routines. The theory is formalized and validated simulating laminar flow for different Knudsen numbers. Results for the temperature field are compared with benchmark and other numerical methods available in the literature with excellent agreement.*

Keywords*: Microfluidics; Heat Transfer; Microchannels; Classical Integral Transform Technique*.

1. INTRODUCTION

The heat exchanger project for integrated circuits and other devices of reduced size isn't always easy. As the technology advances, those circuits become smaller and smaller. Some microprocessors concentrate billions of transistors on a small area generating a high heat concentration on a smaller area compared to the sink's area, leading to the limitation of the fin system efficiency and causing high peaks of temperature on the chip that may cause damage. The limitation on the quantity of mass in the conduction heat transport of the utilized material prevents the heat from spreading over the entire sink base causing heat concentration and temperature raise on some place close to the component making the total use of the sink impossible (Merkel and Bellosa, 2006). Some possible solutions are: use of heat pipes, spreadly used by industry, Larrodé *et al*. (2000), and the use of superconductive materials.

Therefore, the study of alternative solutions for the cooling of these equipments becomes increasingly important in view of the need for methods that can dissipate greater amounts of heat into small systems. An alternative solution which is the focus of research today is the thermal dissipation or cooling only to maintain a certain working temperature, for instance, of those chips by a flow through microchannels

The microfluidics (application of fluids mechanics at microscopic scale, Nguyen and Wereley, (2002)) applied to heat transfer is a relatively new area in engineering, since it was initially introduced by Tuckerman and Pease (1984) and with the advancement of the MEMS technology (microelectro-mechanical systems) is paying off and showing significant advances in the study and application of micro heat exchangers, which makes it relevant to study of the transport phenomena behavior that are included in these devices and thus find viable and more efficient ways to optimize them.

The equations studied here will be analyzed by integral transforms technique where Mikhailov and Özisik (1984) popularized, uniting the analytical method and the numerical method for the equations solution partial differential, which at first did not apply the classical theory of separation of variables. This technique known as Classical Integral Transform Technique (CITT), Cotta (1993) is used to solve linear problems from. The idea behind the CITT is, after finding the eigenvalue problem of Sturm-Liouville problem, transform the original partial differential equation in an infinite system of ordinary differential equations uncoupled and then numerically solve it. For non-linear problems and more complex problems an extension of CITT, known as Generalized Integral Transform Technique (GITT) is widely used until today and could be revised in Perez Guerrero and Cotta (1996), Leal *et al*. (2000) and Cotta *et al*. (2013).

2. MATHEMATICAL FORMULATION

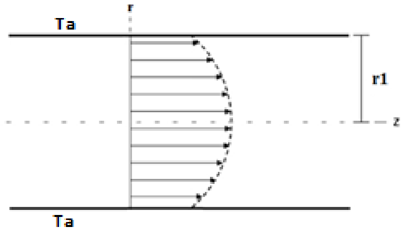

Figure 1: Internal flow through paralel plates microchannel.

Considering a laminar and fully developed flow of an incompressible fluid with constant properties at steady state between parallel plates microchannel. We have the following equations and boundary conditions:

$$\begin{cases} \mu\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial p}{\partial x} \\ \left.\frac{\partial u(y)}{\partial y}\right|_{y=0} = 0 \\ u(y_1) + \lambda\beta_v \left.\frac{\partial u(y)}{\partial y}\right|_{y=y_1} = 0 \end{cases} \tag{1}$$

where $\beta_v = \frac{2-\sigma_m}{\sigma_m}$; σ_m is the tangential momentum accommodation coefficient; λ is the molecule mean free path.

The defined dimensionless groups are $U(Y) = -\mu\left(\frac{u(y)}{{y_1}^2\frac{\partial p}{\partial z}}\right)$; $Y = \frac{y}{y_1}$, leading to:

$$\begin{cases} \frac{\partial^2 U}{\partial Y^2} = -1 \\ \left.\frac{\partial U(Y)}{\partial Y}\right|_{Y=0} = 0 \\ U(1) + 2Kn\beta_v \left.\frac{\partial U(Y)}{\partial Y}\right|_{Y=1} = 0 \end{cases} \tag{3}$$

Thus:

$$U(Y) = \frac{1 + 4Kn\beta_v - Y^2}{2} \tag{5}$$

Where $Kn = \lambda/L_c$ is the Nusselt number, L_c is the characteristic length. The ratio $u(y)/u_m$ is the same as $U(Y)/U_m$. This ratio $U(Y)/U_m$ is then called $W(Y)$ and is used on heat transfer analysis. In an analog way we use the ratio $W(R)$:

$$W(Y) = C_1(1 + 4Kn\beta_v - Y^2) \tag{7}$$

where $C_1 = \frac{2}{3} + 4Kn\beta_v$.

After obtain the velocity fields, we go to the thermal analysis. We now consider constant physical properties, no energy generation, no viscous dissipation, uniform superficial temperature, uniform entry region temperature and pressure gradient on z axis is constant. Then, the PDE and the boundary condition are given as:

$$\begin{cases} u(r)\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{n}{r}\frac{\partial T}{\partial r}\right) \\ r^n \left.\frac{\partial T(r,z)}{\partial r}\right|_{r=0} = 0 \\ T(r_1, z) = Ta - 2\,Kn\,r_1\,\beta_v\beta \left.\frac{\partial T(r,z)}{\partial r}\right|_{r=r_1} \end{cases} \tag{9 a,b,c}$$

where $\beta_t = \beta_v \beta$; $\beta_t = \frac{2-\sigma_m}{\sigma_m}\frac{2\gamma}{\gamma+1}\frac{1}{Pr}$; σ_m is the thermal accommodation coefficient ; γ is the ratio of specific heat at constant pressure c_p and at constante volume c_v; $n = 0$ for parallel plates and $n = 1$ for cylindrical, where the paper focus is parallel plate configuration, with this formulation, the cylindrical approach is easily done.

The *Kn* (Knudsen number), thermal accommodation and tangential momentum accommodation coefficients are parameters widely used and necessary to simulate any flow in micro scale, they also tell us about how microscale flow distinguish from macroscale flow, their meaning are extensive for the purposes that this papers intends, where a basic knowledge about microflow is recommended, nevertheless all explanation by a physical approach could easily studied in Nguyen and Wereley, (2002).

Also, all others dimensionless groups defined here are widely used in basic theory of heat transfer (Incropera *et al.*, 2008) and found in literature as we could see in Mikhailov *et al.* (2005) as common knowledge of this field.

The defined dimensionless groups are $\theta(R,Z) = \frac{T(r,z) - Ta}{To - Ta}$; $R = r/r_1$ and $Z = z\alpha/u_m {r_1}^2 C$, leading to the following PDE, boundary and initial conditions:

$$\begin{cases} (1 + 4\,Kn\,\beta_v - R^2)\,\dfrac{\partial\theta}{\partial Z} = \left(\dfrac{\partial^2\theta}{\partial R^2} + \dfrac{n}{R}\dfrac{\partial\theta}{\partial R}\right) \quad \forall \;\; 0 \le R \le 1 \;\; e \;\; Z \ge 0 \\ R^n \dfrac{\partial\theta(R,Z)}{\partial R}\Big|_{R=0} = 0 \quad \forall \;\; Z \ge 0 \\ \theta(1,Z) + 2\,Kn\;\; \beta_v\beta \dfrac{\partial\theta(R,Z)}{\partial R}\Big|_{R=1} = 0 \quad \forall \;\; Z \ge 0 \\ \theta(R,0) = 1 \quad \forall \;\; 0 \le R \le 1 \end{cases} \qquad (10\text{ a,b,c,d})$$

3. SOLUTION METHODOLOGY

The solution proposed for the problem is done by the Classic Integral Transform Technique and is given by:

$$\theta(R,Z) = \sum_{i=1}^{\infty} \tilde{\Psi}_i(R)\bar{\theta}_i(Z) \qquad (11)$$

We use the Sturm-Liouville problem to solve the eigenvalues and eigenfunctions and then we define $\tilde{\Psi}_i(R)$ as a normalized eingenfunction. The next step is to solve the transformed potential $\bar{\theta}_i(Z)$ by replacing Eq. (11) in Eq. (10, a):

$$(1 + 4\,Kn\,\beta_v - R^2)\,\sum_{i=1}^{\infty} \tilde{\Psi}_i(R)\bar{\theta}_i{}'(Z) = \left(\sum_{i=1}^{\infty} \tilde{\Psi}_i{}''(R)\bar{\theta}_i(Z) + \frac{n}{R}\sum_{i=1}^{\infty} \tilde{\Psi}_i{}'(R)\bar{\theta}_i(Z)\right) \qquad (12)$$

Then we use the integral transform multiplying both sides of the equation by $\tilde{\Psi}_j(R)$ and integrating at $R^n dR$ from 0 to 1 in order to eliminate the R variable:

$$\sum_{i=1}^{\infty}\int_0^1 (1 + 4\,Kn\,\beta_v - R^2)\tilde{\Psi}_i(R)\tilde{\Psi}_j(R)R^n dR\bar{\theta}_i'(Z) = \sum_{i=1}^{\infty}\int_0^1 -\mu_i^2\tilde{\Psi}_i(R)\tilde{\Psi}_j(R)R^n dR\bar{\theta}_i(Z) \qquad (13)$$

where, from the eigenvalue problem, we introduce the orthogonality property from the Sturm-Liouville problem:

$$\int_0^1 \tilde{\Psi}_i(R)\tilde{\Psi}_j(R)R^n dR = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases} \qquad (14)$$

As the null results are not wanted, we define $i = j$. Thus Eq. (13) becomes:

$$\sum_{i=1}^{\infty} A_{ij}\bar{\theta}_i{}'(Z) = -\mu_j^2\,\bar{\theta}_j(Z) \;\; \forall \;\; Z \ge 0 \qquad (15)$$

where $A_{ij} = \int_0^1 (1 + 4\,Kn\,\beta_v - R^2)\tilde{\Psi}_i(R)\tilde{\Psi}_j(R)R^n dR$.

To solve the problem, it makes necessary to define the initial condition for $\bar{\theta}_i(Z)$. By replacing Eq. (13) in Eq. (10, d), we're given:

$$\bar{\theta}_i(R,0) = \sum_{i=1}^{\infty} \tilde{\Psi}_i(R)\,\bar{\theta}_i(0) = 1 \tag{16}$$

Again by multiplying both sides of the equation by $\tilde{\Psi}_j(R)$, integrating at $R^n dR$ from 0 to 1, then using the concept of orthogonality property to obtain the non-null values, we define the initial condition:

$$\bar{\theta}_i(0) = \int_0^1 \tilde{\Psi}_j(R) R^n dR \tag{17}$$

Other parameters can be defined to increase the study, as the medium average temperature and the Nusselt number.

4. RESULTS AND DISCUSSION

It was built a computer algorithm to solve the PDE's given by Eqs. (15) and (17). Through the solution of the equations we get the temperature field by Eq. (11) and its boundary condition (10, b) and (10, c) and initial condition (10, d) on Mathematica platform using the NDSolve subroutine. The average temperature and Nusselt number were also analytically handled and solved through symbolic manipulation packages (Wolfram, 1999) also on Mathematic platform.

Table 1: Convergence behavior of Θ.

	Θ at R=0.5 e Z=0.1			
	n	$Kn\beta_v = 0$ e $\beta = 0$	$Kn\beta_v = 0.1$ e $\beta = 0$	$Kn\beta_v = 0.1$ e $\beta = 1$
	1	0.6778400754	0.7412640270	0.8525226681
	5	0.6270805384	0.7338742781	0.8354955487
	10	0.6266751051	0.7334465268	0.8352742113
	15	0.6266354215	0.7334062178	0.8352573444
	20	0.6266255711	0.7333961264	0.8352525208
Parallel Plates	**Θ at R=0.5 e Z=0.5**			
	n	$Kn\beta_v = 0$ e $\beta = 0$	$Kn\beta_v = 0.1$ e $\beta = 0$	$Kn\beta_v = 0.1$ e $\beta = 1$
	1	0.2177989539	0.3406309028	0.4840178720
	5	0.1985055635	0.3192566716	0.4532155861
	10	0.1983981624	0.3191372988	0.4531326083
	15	0.1983876328	0.3191256630	0.4531259060
	20	0.1983850271	0.3191228009	0.4531241663

Table 1 shows the convergence behavior for the dimensionless temperature field on several coordinates for parallel plates microchannel. It shows that 20 eigenvalues are enough to reach convergence up to at least three significant digits.

Figure 2 shows the velocity field for different values of Z and $Kn\beta_v$. For all cases, at $R = 1$ or at $R = -1$, $\theta = 0$ because the fluid temperature equals the surface temperature due the fact of the absence of temperature jump ($\beta = 0$). As Z increases, $\theta \to 0$ at all points, we also note that the higher the value of $Kn\beta_v$, higher the length Z necessary to reach $\theta \to 0$.

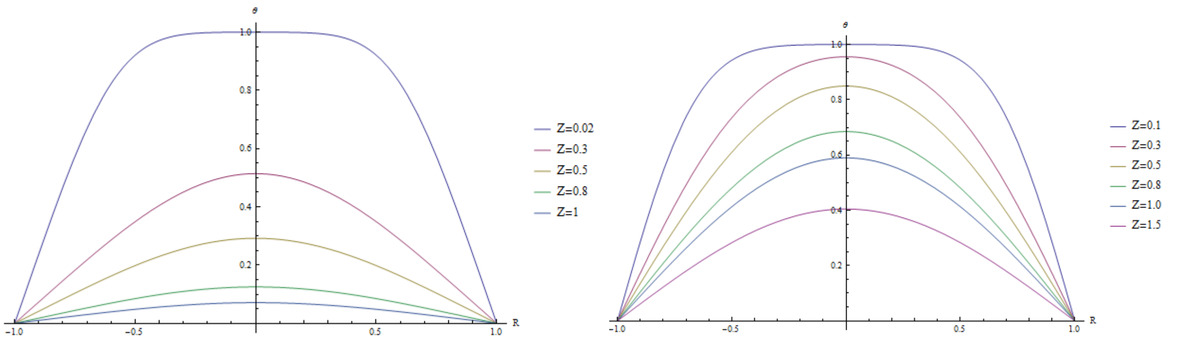

Figure 2: Temperature profile along R for different Z at $Kn\beta_v = 0$ (left), $Kn\beta_v = 0.6$ and $\beta = 0$ (both cases) for parallel plates microchannel.

Compared to the classic case, where $Kn\beta_v = 0$, we can note that at the same positions of Z, the temperature profile suffers from the microscale effects as long $Kn\beta_v$ increases. There is an enlargement of the temperature lines, so $\theta \rightarrow 0$ at higher distances because of the increase of the difference of velocity at the surface and the rarefaction of the flow.

Figure 3 shows the temperature profile for a fixed value for β and different values of and $Kn\beta_v$. For all the cases, at $R = 1$ or at $R = -1$, $\theta \neq 0$ because the fluid temperature does not equal the surface temperature due to the temperature jump which directly influences the temperature profile as θ differs from 0 at the boundaries for a longer extension. As Z increases, $\theta \rightarrow 0$ at all points.

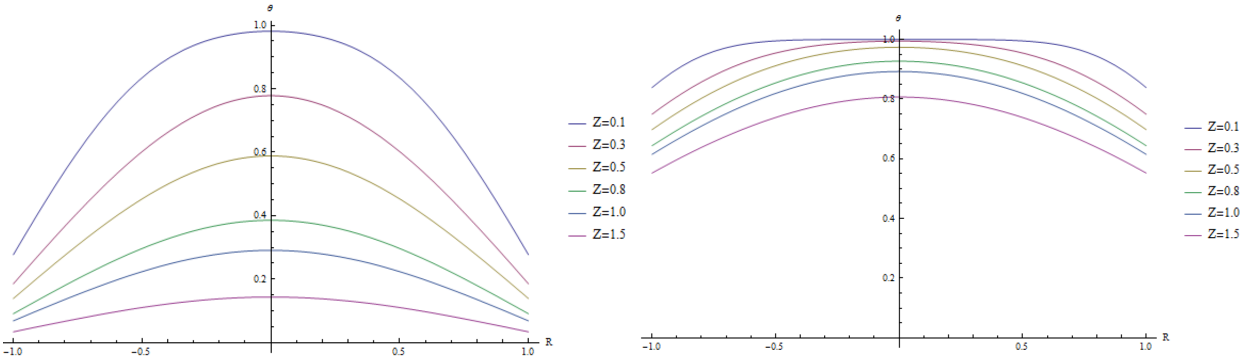

Figure 3: Temperature profile along R for different Z at $Kn\beta_v = 0.1$ (left), $Kn\beta_v = 0.6$ and β =1(both cases) for parallel plates microchannel.

Figure 4 shows the Nusselt number at different values of Z at different β and $Kn\beta_v$. Once again we could note that the flow becomes thermally fully developed at a Z lower than the one where $\theta \rightarrow 0$ at all cases. For the cases where $\beta = 0$ the Nusselt Number stays constant and the higher $Kn\beta_v$, higher the Nusselt number which means that the slip-flow effects accentuate the convection heat transfer in order of the conduction heat transfer due to the rarefaction effects or the decreases of the aspect ratio of the channel.

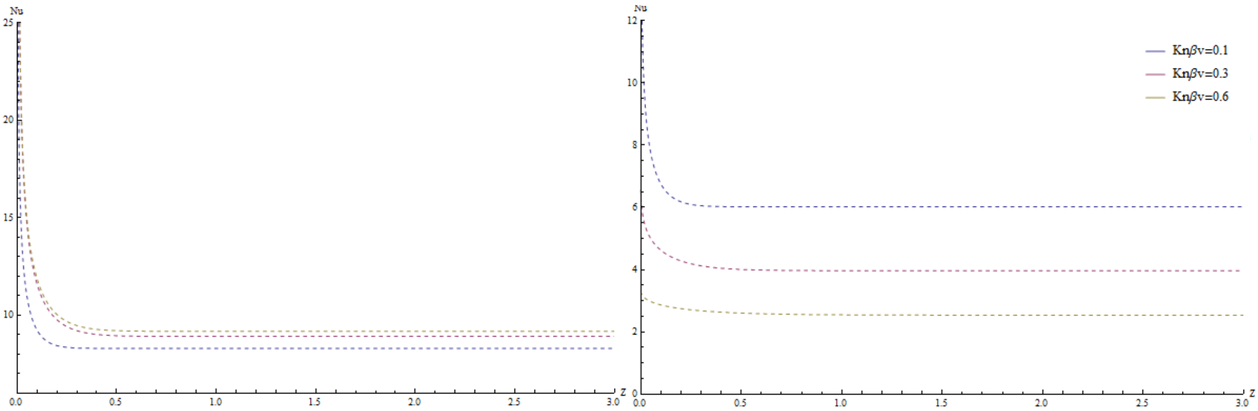

Figure 4: Nusselt number along Z for different $Kn\beta_v$ at $\beta = 0$ (left) and $\beta = 1$ for parallel plates microchannel.

For $Kn\beta_v = 0.1$, 0.3 and 0.6, at the developed region, the values of Nusselt are 8.2739, 8.8971 and 9.2553. For the cases where $\beta = 1$ we could note the great influence of the temperature jump over the heat transfer at the problem. It makes that the lowest value of $Kn\beta_v$ shows the highest value of Nusselt number (the opposite of the last case) as the flow becomes thermally fully developed. For $Kn\beta_v = 0.1$, 0.3 and 0.6, at the developed region, the values of Nusselt are 6.018, 3.9635 e 2.533 for parallel plates. The values obtained are close to the ones obtained numerically by Mikhailov *et al.* (2005), (where we need to set n=0 to simulate flow between parallel plates) who found for $\beta = 0$, 8.2512, 8.8629 e 9.2273 and for $\beta = 1$, 6.2376, 3.9892 e 2.6421, respectively.

Figure 5 shows some 3D graphics showing the behavior of θ in function of Z and R. All the graphics were purposely plotted until Z = 3 for better notion of the influence of $Kn\beta_v$ and β over the flow. As expected by Figure 2, the temperature behavior complete the analysis of how the temperature changes along the plates.

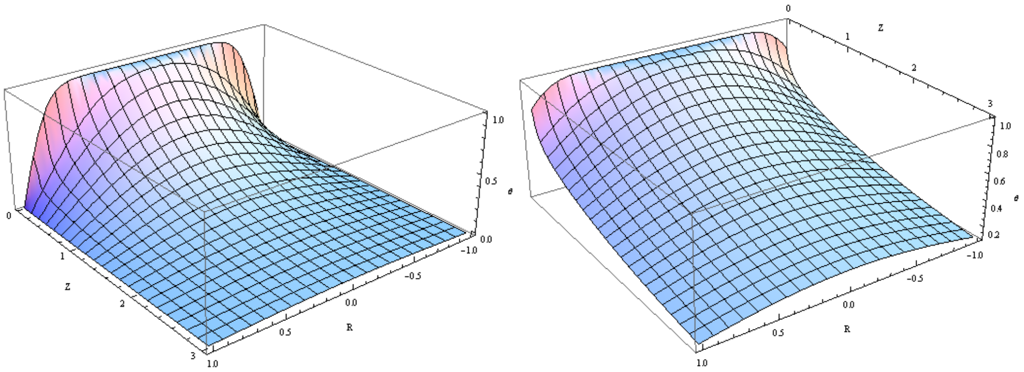

Figure 5: Temperature profile along R and Z for $Kn\beta_v = 0.3$ at $\beta = 0$ and $\beta = 1$ for parallel plates microchannel.

5. CONCLUSION

The results presented here demonstrate good agreement with benchmark. We can conclude that the Classical Integral Transform Technique (CITT) revealed itself as a great tool with good performance for solving some different kinds of linear PDE such as the convection diffusion equation applied to microchannels.

6. REFERENCES

Cotta, R.M., 1993. Integral Transforms in Computational Heat and Fluid Flow. CRC Press, Boca Raton, USA.

Cotta, R.M., Knupp, D.C., Naveira-Cotta, C.P., Sphaier, L.A., Quaresma, J.N.N., 2013. "Unified Integral Transforms Algorithm For Solving Multidimensional Nonlinear Convection-Diffusion Problems". Num. Heat Transf., Part A, Vol.63, pp. 840-866.

Incropera, F. P., De Witt, D. P., Bergman, T., 2008. Fundamentos de Transferência de Calor e Massa. 6. ed., LTC, Rio de Janeiro.

Larrodé, F.E., Housiadas, C., Drossinos, Y., 2000. "Slip-Flow Heat Transfer in Circular Tubes". Int. J. Heat Mass Transfer, vol. 43(15), pp. 2669-2680.

Leal, M.A., Machado, H.A., and Cotta, R.M., 2000. "Integral Transform Solutions of Transient Natural Convection in Enclosures with Variable Fluid Properties", Int. J. Heat & Mass Transfer, V.43, no.21, pp.3977-3990.

Merkel, A., Bellosa, F., 2006. "Balancing power consumption in multiprocessor systems". ACM SIGOPS, Operational Systems, Review 40, pp. 403-414.

Mikhailov, M.D., Cotta, R.M., Kakaç, S., 2005. "Steady State and Periodic Heat Transfer in Micro Conduits". In: Kakaç, S., Vasiliev, L.L., Bayazitoglu, Y., Yener, Y., Microscale Heat Transfer Fundamentals and Applications. Springer-Verlag, Dordrecht, The Netherlands.

Mikhailov, M.D. and Özisik, M.N., 1984. Unified Analysis and Solutions of Heat and Mass Diffusion. John Wiley & Sons, New York.

Nguyen, N.T., Wereley, S., 2002. Fundamental and Applications of Microfluidics. Artech House, Boston, MA.

Perez Guerrero, J.S. and Cotta, R.M., 1996, Benchmark Integral Transform Results for Flow Over a Backward - Facing Step, Computers & Fluids, 25, 527-540.

Tuckerman, D.B. and Pease, R.F.W., 1984. "Microchannel heat transfer: early history, commercial applications, and emerging opportunities". In Proceedings of the ASME 2011 9th International Conference on Nanochannels, Microchannels, and Minichannels. Edmonton, Alberta, Canada. Paper no. 58308.

Wolfram, S., 1999. The Mathematica Book, 4th ed., Wolfram Media, Cambridge.

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