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STATION-KEEPING CONTROL DESIGN FOR EARTH-MOON L1 LIBRATION POINT

Thiago César Lousada Marsola
Sandro da Silva Fernandes
Instituto Tecnológico de Aeronáutica - ITA, Praça Marechal Eduardo Gomes 50 - Vila das Acácias, 12228-900, São José dos Campos - SP, Brasil
thiagoclm@gmail.com
sandro@ita.br

José Manoel Balthazar
Universidade Federal Tecnológica do Paraná - UFTPR, Avenida Monteiro Lobato - Jardim Carvalho, 84016-210, Ponta Grossa - PR, Brasil
jmbaltha@gmail.com

Abstract. *In this work, the dynamics of the circular restricted three-body problem for the Earth-Moon system is considered for a station-keeping problem. Taking into account the* L_1 *equilibrium point in this dynamics, a spacecraft moving near this point must use some correction maneuver to take and remain into this reference point. Therefore, it is proposed a station-keeping strategy applicable for this type of trajectory using the linear quadratic regulator (LQR) controller.*

Keywords: *three body problem, Earth-Moon system, libration points, LQR control.*

1. INTRODUCTION

In relation to the general three-body problem, taking into account some considerations as the non-influence of the smaller-body mass, and the motion of the greater bodies around each other being circular, the problem turns into the circular restricted three-body problem (CRTBP) (Parker and Anderson, 2014). Joseph Louis Lagrange was the first scientist to demonstrate that there are five equilibrium points associated with this system, three of them collinear in relation to each other (L_1, L_2, and L_3), and two of them triangular (L_4 and L_5). In these points the gravitational and centrifugal forces balanced out, and a spacecraft placed in one of these equilibrium points with the specified velocity will be also in equilibrium. Moreover, the collinear points are unstable and the triangular ones are stable for a given value of mass ratio μ (Szebehely, 1967).

In order to take the spacecraft to this specific point, a control effort is demanded to perform the necessary correction maneuver. The theory of optimal control deals with the idea of operating a system with a minimum cost. When a nonlinear system is linearized, it is possible to control the dynamics with a linear feedback controller, given a quadratic cost function. There are various works describing the linear quadratic regulator ((Anderson and Moore, 1971), (Kwakernaak and Sivan, 1972), (Bryson Jr and Ho, 1975)). Therefore, the main goal of this work is to introduce a station-keeping strategy around L1 libration point based on the linear quadratic regulator controller.

2. THE CIRCULAR RESTRICTED THREE-BODY DYNAMICS

Consider the xyz synodic reference frame, depicted in Fig. 1, centered at the barycenter of the Earth-Moon system, with plane xy coinciding to the inertial xy plane, with x always pointing toward Moon, y axis perpendicular to the x axis, and z axis perpendicular do the xy plane, defining a right-hand system.

2.1 Equations of motion

The equations of motion for the space vehicle in the synodic frame are the result of some simplifications from the equation in the inertial system: the total mass of the two larger bodies is normalized to one, the distance between those two bodies is normalized to one, and the angular velocity of the two larger bodies around their center of mass is normalized to one. Then, the resulted equations of motion are expressed as (Koon *et al.*, 2006):

$$\ddot{x} = 2\dot{y} + x - (1-\mu)\frac{x+\mu}{r_E^3} - \mu\frac{x-1+\mu}{r_M^3}, \tag{1}$$

$$\ddot{y} = -2\dot{x} + y - (1-\mu)\frac{y}{r_E^3} - \mu\frac{y}{r_M^3}, \tag{2}$$

$$\ddot{z} = -(1-\mu)\frac{z}{r_E^3} - \mu\frac{z}{r_M^3}, \tag{3}$$

with

$$r_E = \sqrt{(x+\mu)^2 + y^2 + z^2}, \tag{4}$$

$$r_M = \sqrt{(x-(1-\mu))^2 + y^2 + z^2}, \tag{5}$$

where the mass parameter μ is the ratio between the Moon mass to the sum of the greater bodies masses.

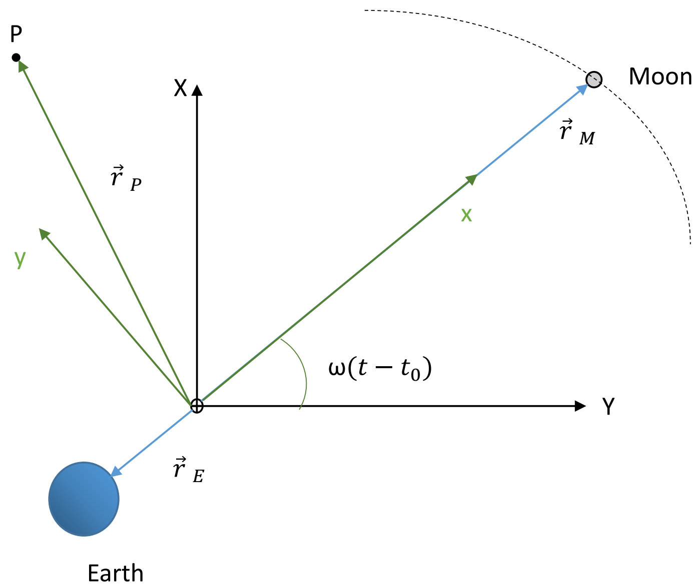

Figure 1: Synodic reference system.

2.2 Libration points

The libration points (or Lagrangian points) are equilibrium points present in the general three-body problem (and also in the circular restricted problem), where the gravitational and centrifugal forces are canceled out. In 1772, Euler discovered the existence of the collinear libration points, and later, Lagrange not only confirmed the result but also added two triangular libration points (Barrow-Green, 1996). Figure 2 shows the position of the five libration points in the Earth-Moon system, where the body with mass m_1 represents the Earth and the body with mass m_2 represents the Moon. To this particular paper, the L1 is the desired reference libration point for the station-keeping strategy.

The collinear points L_1, L_2 and L_3 are located along the x-axis, while the triangular points L_4 and L_5 form with the primaries a equilateral triangle. As equilibrium points, they remain stationary with the respect to the synodic coordinate system. The position of each point is made setting to zero all the derivatives in the equations of motion, and then solving the equations to each state as follows:

$$0 = 0 + x_{eq} - (1-\mu)\frac{x_{eq}+\mu}{r_{E,eq}^3} - \mu\frac{x_{eq}-1+\mu}{r_{M,eq}^3}, \tag{6}$$

$$0 = 0 + y_{eq} - (1-\mu)\frac{y_{eq}}{r_{E,eq}^3} - \mu\frac{y_{eq}}{r_{M,eq}^3}, \tag{7}$$

$$0 = -(1-\mu)\frac{z_{eq}}{r_{E,eq}^3} - \mu\frac{z_{eq}}{r_{M,eq}^3}, \tag{8}$$

where

$$r_{E,eq} = \sqrt{(x_{eq} + \mu)^2 + y_{eq}^2 + z_{eq}^2}, \tag{9}$$

$$r_{M,eq} = \sqrt{(x_{eq} - (1-\mu))^2 + y_{eq}^2 + z_{eq}^2}. \tag{10}$$

For the collinear points, it is known that the coordinates y_{eq} and z_{eq} are 0. Thus, Eq. 6 become as:

$$x_{eq} - \frac{(1-\mu)(x_{eq}+\mu)}{|(x_{eq}+\mu)|^3} - \frac{\mu|x-1+\mu|}{|x_{eq}-(1-\mu)|^3} = 0. \tag{11}$$

The points are ordered in relation to the distances of the primaries, and for the point considered in this work:

$$L_1 : -\mu < x_{eq} < 1-\mu, \tag{12}$$

The fifth order polynomial resulting from the equations above in terms of x_{eq} is solved numerically, with mass parameter μ for the Earth-Moon system. To calculate the L_1 position considered in Eq. 12, the equation is:

$$x_{eq}^5 + ax_{eq}^4 + bx_{eq}^3 + cx_{eq}^2 + dx_{eq} + e = 0. \tag{13}$$

with

$$\begin{aligned}
a &= 2(2\mu - 1),\\
b &= (1-\mu)^2 - 4\mu(1-\mu) + \mu^2,\\
c &= 2\mu(1-\mu)(1-2\mu) + 1,\\
d &= \mu^2(1-\mu)^2 + 2(\mu^2 - (1-\mu)^2),\\
e &= (1-\mu)^3 - \mu^3.
\end{aligned}$$

For the other collinear libration points, the calculation is very similar, changing only the position of x_{eq}. On the other hand, the calculation for the equilateral points is easier due to its position forming an equilateral triangle with the Earth-Moon system (James, 2006). Thus, the numerical real solution x_{eq} for the libration point L_1 considered in this work is:

$$L_1 \approx 0.8369373476,$$

which is the result of the L_1 libration point position in the normalized system.

Table 1 shows the position of the libration point considered in normalized and real units.

Table 1: Coordinates of the L_1 libration point in the Earth-Moon system.

Libration Point	Position in normalized units	Position in kilometers
L1	$(0.8369373476; 0; 0)$	$(321, 718.716; 0; 0)$

3. CONTROL STRATEGY

The Linear Quadratic Regulator (LQR) is designed with the objective to derive a feedback control law that drives the trajectory for a linear system (or a linearized nonlinear system) with an initial condition to a small neighbourhood around the origin, while minimizing a performance index. The optimal control problem is defined as (Anderson and Moore, 1971):

Given a linear system in state space form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{14}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{15}$$

where $\mathbf{x}$ is a $n \times 1$ state vector, $\mathbf{y}$ is a $n \times 1$ output vector, $\mathbf{u}$ is a $p \times 1$ control vector, $\mathbf{A}$ is a $n \times n$ matrix, $\mathbf{B}$ is a $n \times p$ matrix, $\mathbf{C}$ is a $n \times n$ matrix, and $\mathbf{D}$ is a $n \times p$ matrix. For the given purposes, all the matrices involved

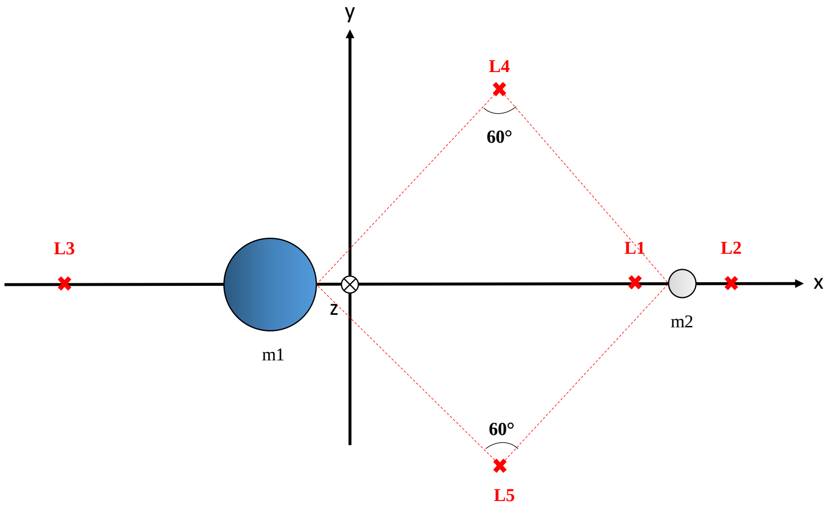

Figure 2: The five libration points in the Earth-Moon system (not in scale).

are time-invariant, which means that the LQR problem is solved only one time. The possible control laws are defined to be the form of:

$$\mathbf{u} = \phi(\mathbf{x}), \tag{16}$$

where ϕ is a continuous function and the closed loop system has a unique solution, satisfying the following equation:

$$lim_{t\to\infty}\mathbf{x}(t) = 0. \tag{17}$$

With this, along with the desired trajectory $(\mathbf{x}_d, \mathbf{u}_d)$, the objective is to design a controller described in Eq. 16 to be the form of $\mathbf{u} = (\mathbf{x}, \mathbf{x}_d, \mathbf{u}, \mathbf{u}_d)$ in order to obtain $lim_{t\to\infty}(\mathbf{x} - \mathbf{x}_d) = 0$, minimizing the following quadratic functional:

$$J(\mathbf{x}_0, \phi) = \int_{t_0}^{\infty} [\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]dt, \tag{18}$$

where $\mathbf{Q}(t)$ is a $n \times n$ positive semidefinite matrix, and $\mathbf{R}(t)$ is a $p \times p$ positive definite matrix. Therefore, the quadratic functional is dependent on the initial state condition and the decision of the control law.

This strategy applied to the spacecraft dynamics taking into account a reference point or orbit as a goal is called *station-keeping trajectory control*. Thus, in order to design the controller, it is necessary to construct the system's equations in relation to the error.

Let $\mathbf{e} = \mathbf{x} - \mathbf{x}_d$ be the state vector error and $\mathbf{e}_u = \mathbf{u} - \mathbf{u}_d$ be the control vector error. Assuming that $f(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$ is the nonlinear system with a linear input (most common), we have:

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\mathbf{x}}_d = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} - f(\mathbf{x}_d) + g(\mathbf{x}_d)\mathbf{u}_d, \\ &= f(\mathbf{e} + \mathbf{x}_d) - f(\mathbf{x}_d) + g(\mathbf{e} + \mathbf{x}_d)(\mathbf{e}_u + \mathbf{u}_d) + g(\mathbf{x}_d)\mathbf{u}_d, \\ &= F(\mathbf{e}, \mathbf{e}_u, \mathbf{x}_d(t), \mathbf{u}_d(t)). \end{aligned} \tag{19}$$

It is possible to make the assumption that, in steady state, the error is approximately equal to zero, i.e., the system's output is sufficiently close to the given set-point. Thus, the system is linearized around $\mathbf{e} = \mathbf{0}$ as follows:

$$\mathbf{0} \approx \mathbf{A}(t)\mathbf{e} + \mathbf{B}(t)\mathbf{e}_u, \tag{20}$$

where

$$\mathbf{A}(t) = \frac{\partial F}{\partial \mathbf{e}}|_{\mathbf{x}_d(t),\mathbf{u}_d(t)}, \quad \mathbf{B}(t) = \frac{\partial F}{\partial \mathbf{e}_u}|_{\mathbf{x}_d(t),\mathbf{u}_d(t)}. \tag{21}$$

With this approach, the linear system described by Eqs. 14 and 15 becomes:

$$\begin{aligned} \mathbf{0} &= \mathbf{A}(t)\mathbf{e} + \mathbf{B}(t)\mathbf{e}_u, \\ \mathbf{r}(t) &= \mathbf{C}(t)\mathbf{e} + \mathbf{D}(t)\mathbf{e}_u, \end{aligned} \tag{22}$$

where $\mathbf{r}$ is a $n \times 1$ vector, described as the reference trajectory. It is convenient rewrite Eq. 22 in a matrix form to simplify some calculation procedure. Therefore,

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{r}(t) \end{bmatrix}_{(n+n)\times 1} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}_{(n+n)\times(n+p)} \begin{bmatrix} \mathbf{e} \\ \mathbf{e}_u \end{bmatrix}_{(n+p)\times 1} \tag{23}$$

Rearranging the equation above in terms of the variables $\mathbf{e}$ and $\mathbf{e}_u$, and assuming the invertible matrix (or the pseudo-inverse matrix), we obtain:

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{e}_u \end{bmatrix}_{(n+p)\times 1} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}^{-1}_{(n+p)\times(n+n)} \begin{bmatrix} \mathbf{0} \\ \mathbf{r}(t) \end{bmatrix}_{(n+n)\times 1}, \tag{24}$$

and, with reference vector $\mathbf{r}$ in evidence:

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{e}_u \end{bmatrix}_{(n+p)\times 1} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}^{-1}_{(n+p)\times(n+n)} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}_{(n+n)\times n} \mathbf{r}(t)_{n\times 1}. \tag{25}$$

where $\mathbf{0}$ is a null vector with dimension $n \times n$, and $\mathbf{I}$ is a $n \times n$ identity matrix .

Equation 25 shows a direct relation between the error variables, the system's matrices and the desired reference trajectory. Another strategy to simplify some mathematical calculations is defining two variable matrices, $\mathbf{N}_x$ with dimension $n \times q$, and $\mathbf{N}_u$ with dimension $p \times q$, such that:

$$\begin{aligned} \mathbf{e} &= \mathbf{N}_x \mathbf{r}, \\ \mathbf{e}_u &= \mathbf{N}_u \mathbf{r}, \end{aligned} \tag{26}$$

in which,

$$\begin{bmatrix} \mathbf{N}_x \\ \mathbf{N}_u \end{bmatrix} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}. \tag{27}$$

Now, if we design a state feedback controller with law $\mathbf{Ke}$, we can regulate the controlled system using the following feedback control:

$$\mathbf{u}_d = -\mathbf{K}\mathbf{x}_d, \tag{28}$$

with gain $\mathbf{K}$ obtained from the linear optimal control. Thus, substituting the last equations in the control error definition, the controller becomes:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_d + \mathbf{e}_u \\ &= -\mathbf{K}\mathbf{x}_d + \mathbf{N}_u\mathbf{r} \\ &= \mathbf{N}_u\mathbf{r} - \mathbf{K}(\mathbf{x} - \mathbf{e}) \\ &= -\mathbf{K}\mathbf{x} + \mathbf{N}_u\mathbf{r} + \mathbf{K}\mathbf{N}_x\mathbf{r} \\ &= -\mathbf{K}\mathbf{x} + (\mathbf{N}_u + \mathbf{K}\mathbf{N}_x)\mathbf{r}. \end{aligned} \tag{29}$$

Now, with the control strategy stated, the station-keeping method is performed for the CRTBP dynamics, and the numerical simulations are presented in the next section.

4. RESULTS

For the equations of motion in the CRTBP, defined by Eqs. 1, 2, and 3, it is considered three control inputs, $\mathbf{u}(t)$, $i = 1, 2, 3$, which are basically forces applied to the spacecraft in each coordinate axis with the goal to produce the necessary correction maneuvers. The terms r_E and r_M are defined in Eqs. 4 and 5. Hence, the controlled equations of a low-thrust propulsion system are described as (Nazari *et al.* (2016)):

$$\ddot{x} = 2\dot{y} + x - (1-\mu)\frac{x+\mu}{r_E^3} - \mu\frac{x-1+\mu}{r_M^3} + u_1, \tag{30}$$

$$\ddot{y} = -2\dot{x} + y - (1-\mu)\frac{y}{r_E^3} - \mu\frac{y}{r_M^3} + u_2, \tag{31}$$

$$\ddot{z} = -(1-\mu)\frac{z}{r_E^3} - \mu\frac{z}{r_M^3} + u_3. \tag{32}$$

Now, the linear control implementation follows the strategy presented in the previous section, where matrices **A** and **B** are obtained by a linearization procedure in terms of the error between the desired position point and the present state position, described as:

$$\mathbf{A}(\mathbf{x},\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 2 & 0 \\ 0 & \beta & 0 & -2 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

$$\mathbf{B}(\mathbf{x},\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{34}$$

with

$$\alpha = \frac{\partial e(\ddot{x})}{\partial e(\dot{x})}, \tag{35}$$

$$\beta = \frac{\partial e(\ddot{y})}{\partial e(\dot{y})}, \tag{36}$$

$$\gamma = \frac{\partial e(\ddot{z})}{\partial e(\dot{z})}, \tag{37}$$

where $e()$ represents the error for each state.

All the simulations are performed using the LQR function from MATLAB©. The control problem is integrated, as the linear control, from 0 to 4.5 units of time, at every $\Delta t = 8 \times 10^{-5}$, with the same vector error given by $[-8.2 \times 10^{-5}, 0, 0, 0, 0, 0]$. The weight matrices **Q** and **R** were chosen to be constant, but is common to use all time-varying matrices in the time-varying system. Hence, the selected weight matrices are:

$$\mathbf{Q} = 1000 \times \mathbf{I}_{6\times6}, \quad \mathbf{R} = 10 \times \mathbf{I}_{3\times3}, \tag{38}$$

where **I** is the identity matrix.

Thus, the control effort is designed to drive the difference between an initial position and the desired position to zero $(\mathbf{x} - \mathbf{x_d})$, where $\mathbf{x}$ and $\mathbf{x_d}$ are in dimensionless units, respectively:

$$\begin{aligned} \mathbf{x} &= [0.8370, 0, 0, 0, 0, 0], \\ \mathbf{x_d} &= [0.836918, 0, 0, 0, 0, 0], \end{aligned} \tag{39}$$

where $\mathbf{x}$ is the position of the spacecraft and $\mathbf{x_d}$ is the coordinate of the L_1 libration point (reference).

After the integration, the errors of each state is shown in Fig. 3. It is possible to see that, for every state variable, the controller lead the error to zero, which means that the control law is effective in taking the present vector state $\mathbf{x}$ into the reference point $\mathbf{x}_d$. Figure 4 presents each control input value in function of time. With this, in Fig. 5 is possible to see the controlled trajectory, taking the spacecraft from initial point $(321, 742.8$ km) into the desired point $(321, 711.28$ km).

To obtain a numerical value about the control consumption of a low-thrust propulsion system, the performance index (PI) for the LQR controller is calculated by the following equation:

$$PI = \frac{1}{2} \int_0^{tf} (\mathbf{u}^T \mathbf{u}) dt, \tag{40}$$

representing the fuel consumption necessary that leads the spacecraft from the initial trajectory to the reference point (Marec, 1979). For the LQR controller with references given by Eq. 39, the maneuver take approximately 4.5 days, and the performance index is 1.51308×10^{-3} in dimensionless values, and it is possible to see in Fig. 6 the consumption in relation to time for this controller.

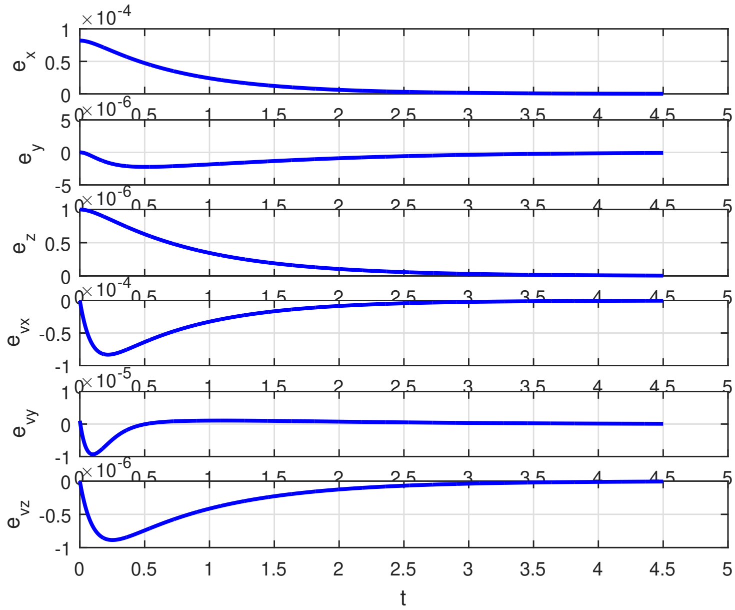

Figure 3: State errors.

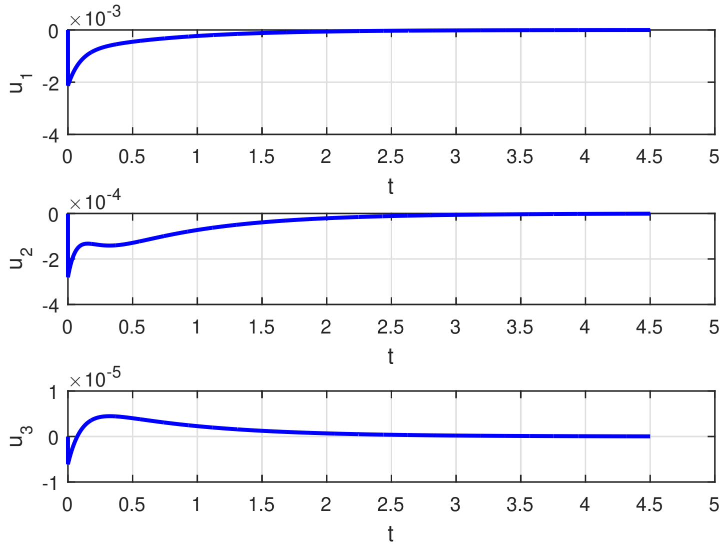

Figure 4: Control signals.

5. CONCLUSIONS

This work concludes that the LQR control is effective in taking the spacecraft in the CRTBP to the reference L1 libration point. The implementation is based on a reference tracking method, which the objective of the controller is to lead the system's output to follow a reference input signal. The use of a reference tracking is well suited for the station-keeping problem.

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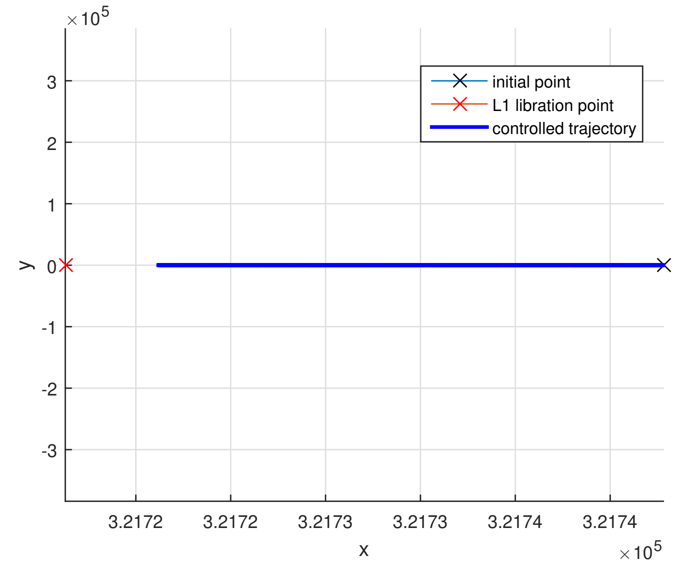

Figure 5: Controlled trajectory.

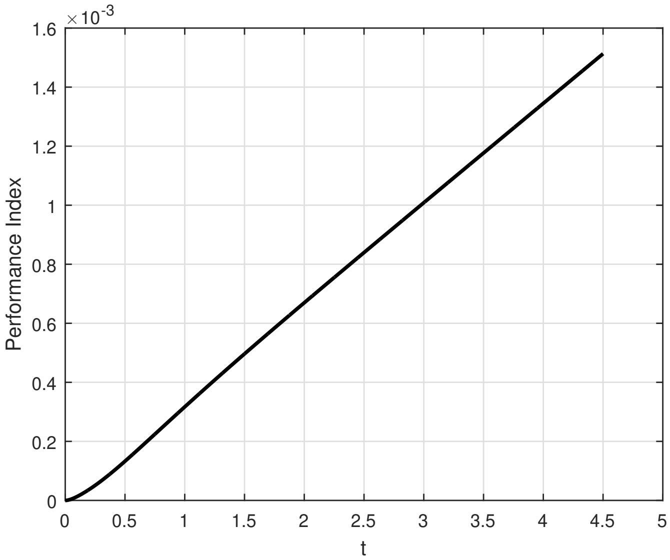

Figure 6: Performance index.

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