Stability Analysis of Incompressible Coaxial Jets Using Boundary Layer Theory

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Abstract: *A differential dispersion relations in a linear stability analysis concept is employed in order to identify the convective/absolute nature of the instability of a local base flow. In coaxial jets, curvature effects are neglected due to their momentum thickness that is much smaller than the jet diameter. The main goal is to identify the transition from convective to absolute instability as a function of the jet parameters. In order to do so, the zero group velocity conditions are applied directly to the differential dispersion relation and it becomes possible to search for arbitrary points of transition to absolute instability, as well as verify their causality. The case considered is where the inner nozzle wall separating both streams has non-negligible thickness. A base flow is created using boundary layer theory. Results obtained so far indicate an agreement with former cases analysed with matrix forming approach built with second-order finite differences.*
Keywords: *Linear Stability Analysis, Coaxial Jets, Linear Stability Theory, Convective Instability, Absolute Instability*

1. INTRODUCTION

Hydrodynamic instabilities are present in many problems related to fluid mechanics. In an aerospace engineering, the relevant applications are associated with suctioned propulsion engines such as turbojets, ramjets and scramjets, working in subsonic or supersonic state. In order to obtain aerospace propulsion, internal energy can be convert in kinetic energy inside a combustion chamber, generating the necessary impulse. The chemical reactions responsible for this energy are strongly dependent of the mixing process. Therefore, it becomes very important to understand the procedure of mixing between the fuel and the oxidant inside the combustion chamber.

The present paper focuses on injection systems used in liquid rocket engines, also known as LREs. One of the most common and largely utilized injector is the coaxial one. In this injector, the the outer jet contains fuel and the inner jet contains oxidant. This type of injectors can be found in the RS-24 engine, also know as the main engine present in the spatial station, and in the Vulcan engine present in the Ariane 5 rocket. Coaxial free jets are highly unstable shear flows, making their mixing capability highly susceptible to flow excitation. They can be produced from several sources, such as structural vibration and wall roughness. One of the main goals of the injection system designers is to avoid resonance between the jet generated noise and the combustion chamber acoustics, which can drastically modify the mixing characteristics of the jet. As a direct consequence, combustion can happen far away from its designed conditions, causing severe damage to the engine and even destruction of the entire vehicle.

Such problem has received a lot of attention in the literature. Published works focussed in the experimental part such as (Leyva *et al.*, 2007); (Rodriguez and Leyva, 2008); (Leyva *et al.*, 2012); and theoretical (Wallace and Redekopp, 1992); (Quintanilha Junior *et al.*, 2015) has been made for coaxial jets. However, in the present paper, a new novel recently developed by Alves and Hirata (2016) was employed, where the transition of stability (Drazin and Reid, 2004) could be found with less effort as the traditional methods performed by Chomaz *et al.* (1999), Delache *et al.* (2007) and Hirata and Ouarzazi (2010).

In order to investigate this transition in coaxial jets, a local linear stability analysis is employed in the governing equation model. It is organized as follows. After the introduction, these governing equations are introduced. Then, the procedure employed for the generation of the base flow is outlined. This is followed by a derivation of the linear normal mode perturbation equations. The numerical method employed for bought is introduced. Finally, a validation of the method is presented as well as the onset of the transition to absolute instabilities for a wide range of governing parameters.

2. MATHEMATICAL MODEL

2.1 Dynamical equations

The dynamical equations for this problem are the Navier Stokes equations in normalized form: the length scale is a momentum thickness δ, the velocity scale is the inner jet velocity U_1 and the Reynolds number is $\frac{U_1\delta}{\nu}$. For incompressible constant property flow, the continuity and the momentum equations can be written as:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = 0 \qquad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\frac{\partial p}{\partial \mathbf{x}} + \frac{1}{Re}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \qquad (2)$$

where $\mathbf{x} = (x, y)$ is the independent spatial variable vector.

2.2 Base flow

When performing a linear stability analysis, the first step is to properly define the flow whose stability one wants to investigate. As in the coaxial jets the curvature effects are neglected due to their momentum thickness that is much smaller than the jet parameters, planar mixing layers will be considered in the present study. The flow is the one formed when two separate boundary-layer flows, merge with each other after the splitter plate ends. As here this plate has a non-negligible thickness, a wake region is formed, and it is modelled with a third boundary-layer. In the incompressible case, the most important control parameters are the velocity of each boundary-layer, the momentum thickness of the resulting mixing-layers and the thickness of the splitter plate. Boundary layer theory is used to build the base flow, but some conditions must be satisfied. In general, flow variables in a two-dimensional boundary layer depend on both spatial coordinates. Thus, different velocity profiles are found at different downstream locations. However, a similarity transformation can be used to reduce the number of independent variables to η, i.e., all downstream velocity profiles collapse into a single curve. The region of the mixing layer that behaves this way is known as similar region and its solutions are known as similar solutions (Salemi *et al.*, 2007).

2.3 Perturbations

The instability of flows to small amplitude perturbations has been analyzed using the modal approach (Juniper *et al.*, 2014). Given an operator that describes the evolution of small perturbations, this approach considers the temporal or spatial development of individual eigenmodes of that operator. The resulting linear stability theory (LST) relies on the decomposition of the flow quantity **u** and p into a steady part **U** and P, which is relative to the base flow, and an unsteady part $\tilde{\mathbf{u}}$, $\tilde{p}$:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \epsilon\tilde{\mathbf{u}}(\mathbf{x},t) \qquad p(\mathbf{x}) = P(\mathbf{x}) + \epsilon\tilde{p}(\mathbf{x}) \tag{3}$$

where ϵ is a small amplitude.

2.4 Linearized Navier Stokes equation

Substituting Eq.3 into the continuity Eq.1 and the Navier-Stokes Eq.2, subtracting the equations for the base flow, and dropping the terms of order ϵ^2 yields the linearized perturbation equations, referred to as the linearized Navier-Stokes equations - **LNSE** and continuity equation.

$$\frac{\partial\tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}}.\nabla\mathbf{U} + \mathbf{U}.\nabla\tilde{\mathbf{u}} = -\nabla\tilde{p} + \frac{1}{Re}\nabla^2\tilde{\mathbf{u}} \qquad \nabla\tilde{\mathbf{u}} = 0 \tag{4}$$

Additional simplifying assumptions are made in this study. The first one is that the Reynolds number leads to infinity, and the viscous terms of the equation 4 can be dropped.

Rearranged this equation and writing in the form of a single ordinary differential equation give us the Rayleigh equation (Quintanilha Junior *et al.*, 2015).

The second assumption is that the linear stability analysis is assumed to be local, i.e. base flow derivatives with respect to the main flow direction are zero. In other words,

$$\frac{\partial\mathbf{U}}{\partial x} = 0 \tag{5}$$

The third assumption is that the base flow is assumed to be parallel, i.e. the base flow velocity component in the non-homogeneous direction is zero.

Even though neither assumption is rigorously true for mixing layers, they are indeed approximately true. This flow satisfies the typical boundary layer assumptions and, hence, both simplifications are correct up to O($\sqrt{Re}$) (Alves, 2015).

These additional simplifications allows to assume disturbances behaviour as normal waves:

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \hat{\mathbf{u}}(y)\,\exp\,i[(\alpha_r + \alpha_i)x - \omega t)] \qquad \tilde{p}(\mathbf{x}) = \hat{p}(y)\,\exp\,i[(\alpha_r + \alpha_i)x - \omega t)] \tag{6}$$

where $\hat{\mathbf{u}}$ and $\hat{p}$ represent the normal disturbance amplitude, -α_i is the spatial amplification rate of the wave, α_r the wave-number and ω the the frequency, in a spacial stability analysis, in order to determine the onset of convective instability or search for the onset of absolute instability. Substituting Eq. (6) into the linearized disturbance Equations (4), one obtains a set of ordinary differential equations with respect to y for the normal disturbances that is know as dispersion relation. Their solution yields the normal disturbances eigenvectors and eigenvalues, which reveal the base flow (in)stability. Here, a novel procedure develop by Alves and Hirata (2016) is employed. It consists in apply the zero group velocity conditions to differential dispersion relations, enabling the use of Newton procedures to search for saddle points and verify their causality without expensive collision checks.

3. NUMERICAL METHOD

To perform a hydrodynamic stability in a mixing layer, first we need to solve the laminar base flow equations through similar equations and then solve the linearized Navier Stokes equations with the assumptions described previously. From the stability equations, an eigenvalue problem arises, where we search for the eigenvalue (in this case, α), which provides the growth ratio of the disturbances. In order to achieve this, it was developed a notebook in the commercial software *MATHEMATICA*.

3.1 Boundary layer equation: Shooting method

To simulate the boundary layer equation for the base flow, the shooting method was employed. It consists in turn a boundary value problem into an initial value problem. The differential equation of order three provides by the boundary layer solution is rewritten as a system of three first order equations, as follows (Quintanilha Junior *et al.*, 2015):

$$f' = df \qquad df' = ddf \qquad 2ddf' + f df' = 0 \tag{7}$$

with the appropriated boundary conditions:

$$f(0) = 0 \qquad df(0) = guess1 \qquad ddf(0) = guess2 \tag{8}$$

where guess1 and guess2 will be estimated. This system is integrated towards the already known boundary condition. In order to do so, the command **NDSolve** present in the Software Mathematica was chosen to perform this integration, which has an automatic error control and switchs between more efficient methods according to the problem stiffness. The command **FindRoot** was used to search for the guesses that lead to a solution that satisfies the original boundary condition. After that, the **NDSolve** command is used again in order to solve the differential equation with the correctly calculated guesses.

As the base flow for the coaxial jet can be build as a merged of planar mixing layer, we can be combine each matched asymptotic expansion, maintained the same accuracy order of each boundary layer solution of a single layer.

In a non-dimensional form (making the velocity non-dimensional by the velocity of the first layer, and the other parameters by a chosen thickness δ_R), the velocity profile of the coaxial jet can be written as:

$$U = f'_{12}\left[\frac{1}{\delta_{12}}(y+\Delta y)\right] + R_{12} f'_{23}\left[\frac{1}{\delta_{23}}(y+\Delta y-\delta)\right] + R_{12}.R_{23} f'_{34}\left[\frac{1}{\delta_{34}}(y-\Delta y)\right] - R_{12}(1+R_{23}) \tag{9}$$

where the subscript "12, 23 and 34" means the relation between the first, second and third boundary layer, respectively. The parameter R is the velocity ratio between the subscripted layers and Δy the distance between the layers.

4. LINEAR STABILITY ANALYSIS

Performing a spatial stability analysis, the eigenvalue sought is α, as showed before, which is a complex number. The frequency ω is a real number. Here, the shooting method was again chosen. When using this method to solve the Rayleigh equation, a real frequency is fixed and an initial guess is assumed for the wave-number. This equation is numerically integrated from both sides of the far field towards the centre of the domain, where velocity continuity is verified using a Wronskian. If it is not satisfied, a new value is assigned to the wave-number in a process that is repeated until continuity is satisfied.

4.1 Dispersion relation

The problem then comes down to find an initial estimate for the wave-number that approximately satisfies the dispersion relation D(α, ω, C) = 0, where C is the control parameter of the problem. In general, this parameters are the velocity ratio between the inner and outer jet, the momentum thickness of the jets, the size of the wall between the jets, responsible for the wake effect and the size of the inner and outer jet itself.

Making the new method provide by Alves and Hirata (2016), the zero group velocity condition is applied directly to the differential dispersion relation. In general, the perturbation consists of a superposition of many waves, which interfere with each other. The patterns of constructive and destructive interference usually cause the perturbation to have an identifiable envelope (Juniper *et al.*, 2014). If the wavecrests of all wavenumbers move at the same phase velocity, this envelope also move at the same phase velocity and they do not change shape over time. If, however, the wavecrests of different wavenumbers move at different phase velocities, this interference pattern changes over time and the envelope moves at a different velocity to the wavecrests. This velocity of the envelope is known as the group velocity.

Taking the derivative of the dispersion relation with respect to α and a new Wronskian can be solved, using the **FindRoot** command to produce two complex eigenvalues α and ω. Setting zero to the imaginary part of ω allows one to calculate the critical C instead of imaginary ω. Alves and Hirata (2016) also did a validation of this method confirming that the saddle point is indeed a pinching point with a collision check.

5. RESULTS AND DISCUSSION

5.1 Base flow

Figure 1 illustrates the base flow calculated with the boundary layer asymptotic solution. The mixing layer parameters used in the present analysis are defined as the velocity ratio between the layers, (UR12 and UR23), the momentum thickness between the layers, (12, 23 and 34) and the size of the wall between the coaxial jets, (Δy). It is important to note that the three momentum thickness δ are assumed to be equal in the analysis.

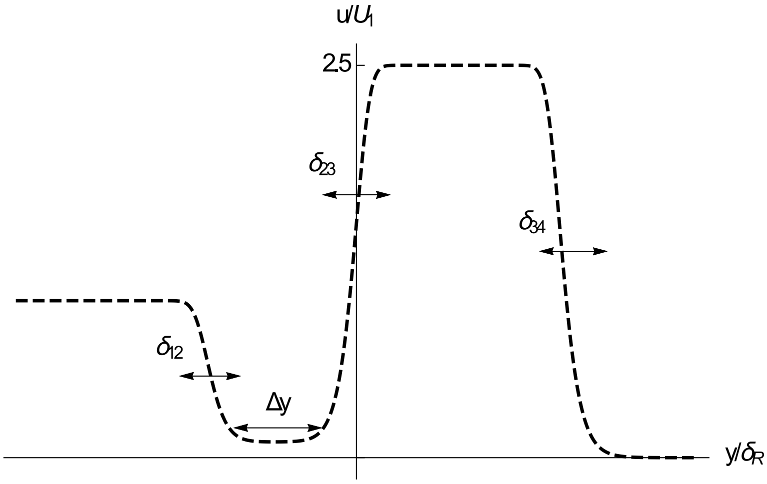

Figure 1: **Base flow for the coaxial jet analysed**

5.2 Validation

In order to do a qualitative validation of the results, a case was perform where we assume $\Delta y = -25$, $\delta = 1/40$, varying the outer jet velocity UR23. Figure 2 shows the imaginary part of the frequency ω versus the variation of the outer jet velocity. As the frequency goes to a positive number between the approximated velocity 10 to 33, we have a transition of instability. That means, the convectively behaviour expected to jets becomes absolutely and the disturbances begin to grow not only in space but also in time.

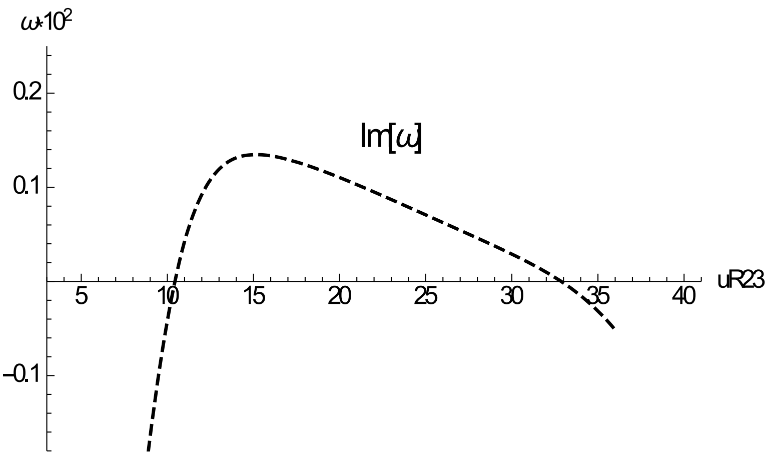

Figure 2: **Imaginary frequency versus outer jet velocity for the new method**

A MatrixForming approach performed by Quintanilha Junior *et al.* (2015) was used to check the pinching point of one of this velocities (in this paper, uR23 = 25). A grid with 481 points with second order finite differences was chosen. Figure 3 shows the full spectrum calculated and the red point indicate the value provided by the new method. In this case, in a convectively point of view, tree unstable modes are expected to be found. However, as we can see, there is a region where an unstable mode collapse with a stable mode, that comes from above. When they collapse, they split themselves into two news curves, indicating the instability transition to absolutely. The point provided by the new novel indicate accurately the region where this curves collapse and start to behaviour as two new curves.

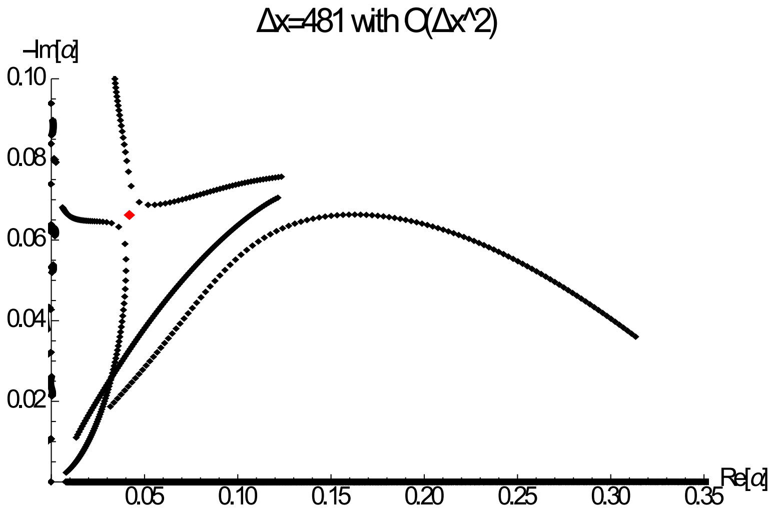

Figure 3: **Full sprectrum calculated with the MatrixForming approach**

5.3 Spatial Analysis

The next step was the analysis of the sign of the imaginary frequency varying some control parameters. In the first case, the size of the wall between the jets was increased (-22 to -30) and in the second one the momentum thickness of the mixing layers was varied between 1/20 to 1/80. Figure 4 and 5 show this two cases.

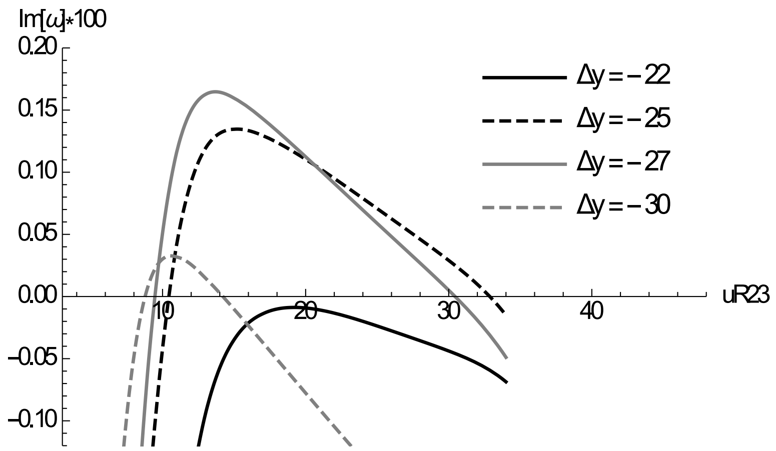

Figure 4: **Imaginary frequency versus outer jet velocity increasing the wall between the jets**

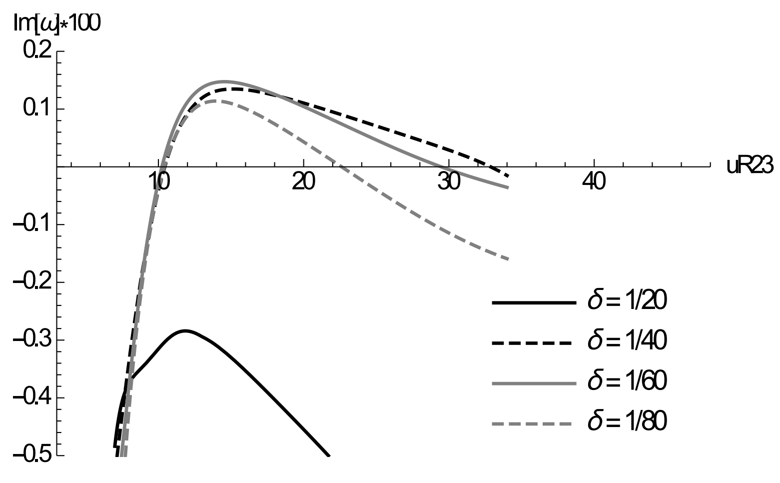

Figure 5: **Imaginary frequency versus outer jet velocity varying the momentum thickness of the jets**

Some interesting results were found. In the first one, we can see that the size of the wall between the jets has a directly interference in the instability transition. Although in the first size ($\Delta y = -22$) the transition is not found, in the $\Delta y = -25$ and $\Delta y = -27$ this transition occurs for a large range of outer jet velocity. However, if we keep increasing this size, the jet behaviour tends to come back to the standard convectively unstable case expected to jets. So, if the wall is to short, the wake effect does not make any influence in the instability transition. However if the size is to big, the jets do not interfere in each other behaviour and no instability transition is found.

In the second one, varying the the momentum thickness, approximately the same behaviour of the first case could be notice. Starting from a ($\delta = 1/80$) and increasing this thickness, the instability transition increases for a larger range of outer jet velocity. However, if this thickness increases at a certain value, no instability transition is found ($\delta = 1/20$).

6. CONCLUSION

In the present work, an investigation of the instability transition was conducted in jets varying some control parameters. A new novel provided by Alves and Hirata (2016) was performed where the zero group velocity condition was applied directly to the differential dispersion relation. A case was analysed comparing with the MatrixFoarming approach, where good agreement was found, showing the region where the absolute instability begins. Two more cases were performed and the results showed that even increasing the wall between the jets or the momentum thickness of the jets, the transition from convective to absolute occurs for some outer jet velocities. However, if we keep increasing the sizes, the effect of the interaction decreases and the jet goes back to the convectively unstable scenario.

7. ACKNOWLEDGEMENTS

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