PERTURBATIVE ANALYSIS OF THE STEADY STATE MULTI-GROUP MULTI-LAYER NEUTRON DIFFUSION EQUATION IN CARTESIAN GEOMETRY BY FICTITIOUS BORDERS POWER METHOD

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Abstract. *In this paper presents a perturbative analysis in the solution of one-dimensional steady state multi-layer multi-group neutron diffusion equation in cartesian geometry by Fictitious Borders Power Method. The equation is solved applying the iterative power method that consists in solve the neutron diffusion equation for each iteration in which the source term is always updated by neutron flux on the previous iteration. This iterative process of source is held until a determined stop criterion for the convergence of the solution. For each new iteration is added new terms which becomes very laborious. To overcome this problem is proposed the reconstruction of the neutron flux by polynomial interpolation. The solution remains in a standard form for all iterations. However, when modeled for large dimensions, in the interpolation arise Vandermonde's arrays which are almost singulars. To eliminate this singularity the domain is subdivided in R fictitious regions and solved the neutron diffusion equation for each region. The arbitrary constants arising from solution of the homogeneous problem are found applying boundary conditions, flux and current density continuity in the interfaces. To analyze the sensitivity of the nuclear parameters in the convergence and behavior of the solution is introduced a perturbation in each parameter of same magnitude order using a random fluctuation multiplied by a constant. The results obtained are compared with benchmark results present in the literature.*

Keywords*: neutron diffusion equation, power method, fictitious borders, perturbative analysis.*

1. INTRODUCTION

In nuclear power reactors, there is a balance between the number of neutrons produced by fission and number consumed by absorption, scattering or leakage by contour. A central problem in the development of a nuclear reactor is the calculation of the size and composition system required to ensure this balance. Calculation for conditions criticality are loaded using the neutron transport theory or the diffusion theory. Both involve the calculation global from an eigenvalue problem, which provides the effective multiplication factor (K_{eff}), defined as the dominant eigenvalue, and the distribution of the neutron flux during the core lifetime. Then, the problem to find the effective multiplication factor and the distribution of the neutron flux is solve the eigenvalue problem using transport or diffusion theory (Stacey, 2001; Sekimoto, 2007).

These equations are deterministic and can only be used to estimate average values of the neutron density and power level. However, the nuclear parameter are calculated stochastically by Monte Carlo method and may undergo vary randomly. At high power levels, the random behavior is negligible but at low power levels, such as at start-up (Hayes, 2005) random fluctuations can be significant. Thus it is essential that these parameters calculated accurately to have the effective factor multiplication within an acceptable safety range, since it is one main factors for licensing nuclear power plant.

In this sense we propose to solve the neutron diffusion equation in cartesian geometry by Fictitious Borders Power Method (FBPM) in cartesian geometry analyzing the sensitivity of the nuclear parameters in the convergence and behavior of the solution. Thereunto is introduced a perturbation in each parameter of same magnitude order, using a random fluctuation multiplied by a constant.

2. FICTITIOUS BORDERS POWER METHOD FOR THE SOLUTION OF THE PROPOSED EQUATION

The one-dimensional steady state multi-group neutron diffusion equation without an external source, which describes the balance between losses and gains neutrons, it is given according Lamarsh (1966):

$$-D_g^{(r)}\frac{d^2}{dx^2}\varphi_g^{(r)}(x)+\Sigma_{Rg}^{(r)}\varphi_g^{(r)}(x)=\frac{1}{K_{eff}}\chi_g\sum_{g'=1}^{G}\nu\,\Sigma_{fg'}^{(r)}\varphi_{g'}^{(r)}(x)+\sum_{g'=1;g'\neq g}^{G}\Sigma_{sg'g}^{(r)}\varphi_{g'}^{(r)}(x),\qquad(1)$$

where $0 \le x \le L$, L is the length of the slab; r are the regions; g are energy group, g = 1, 2, …, G; $D_g^{(r)}$ is the diffusion coefficient of the energy group g and region r; $\phi_g^{(r)}$ is the neutron scalar flux of the energy group g and region r; K_{eff} is the effective multiplication factor; $\Sigma_{Rg}^{(r)}$ is the removal cross section of the energy group g and region r; $\Sigma_{fg}^{(r)}$ is the fission cross section of the energy group g and region r; $\Sigma_{sg'g}^{(r)}$ is the scattering cross section from energy group g to g' of the region r; χ_g is the integrated fission spectrum of the energy group g and ν is the average number of neutrons emitted by fission.

Usually the neutron diffusion equations are written with two energy group, so the Eq. (1) is rewritten as follows:

$$
\begin{aligned}
&-D_1^{(r)}\frac{d^2}{dx^2}\varphi_1^{(r)}(x)+\Sigma_{R1}^{(r)}\varphi_1^{(r)}(x)=\frac{1}{K_{eff}}S\\
&-D_2^{(r)}\frac{d^2}{dx^2}\varphi_2^{(r)}(x)+\Sigma_{R2}^{(r)}\varphi_2^{(r)}(x)=\Sigma_{s12}^{(r)}\varphi_1^{(r)}(x),
\end{aligned}
\tag{2}
$$

where $S=\nu\Sigma_{f1}^{(r)}\varphi_1^{(r)}(x)+\nu\Sigma_{f2}^{(r)}\varphi_2^{(r)}(x)$.

The equation described in (2) are subject to boundary conditions given by:

$$\alpha\varphi_g(x)+\beta\frac{d\varphi_g(x)}{dx}=0 \tag{3}$$

where $g=1:G$ and $|\alpha|+|\beta|>0$. In multi-layer problems are used on interfaces flux and current density continuity respectively, given by:

$$\varphi_g^{(r)}(x)=\varphi_g^{(r+1)}(x) \text{ and } -D_g^{(r)}\frac{d\varphi_g^{(r)}(x)}{dx}=-D_g^{(r+1)}\frac{d\varphi_g^{(r+1)}(x)}{dx}. \tag{4}$$

The Eq. (2) is a eigenvalue (K_{eff}) and eigenvector ($\phi_g^{(r)}$) dominant problem . For solve it's used the Power Method. This iterative method is presented by Duderstadt and Hamilton (1976) and following steps:

i. Initial estimate for source $S^{[1]}$ and $K_{eff}^{[1]}$;
ii. Solve the Eq. (2) with boundary conditions (3);
iii. Update the value of new $K_{eff}^{[i+1]}$, according to $K_{eff}^{[i+1]}=K_{eff}^{[i]}\dfrac{\int_0^L S^{[i+1]}dx}{\int_0^L S^{[i]}dx}$;
iv. Stop criterion: $\dfrac{\left|K_{eff}^{[i+1]}-K_{eff}^{[i]}\right|}{\left|K_{eff}^{[i+1]}\right|}<\varepsilon_1$ and $\dfrac{\left|\int_0^L S^{[i+1]}-S^{[i]}dx\right|}{\left|\int_0^L S^{[i+1]}dx\right|}<\varepsilon_2$;

where ε_1 and ε_2 are small constants prescribed for the stop criterion.

The Power Method applied in this form becomes very laborious, because in every new iteration, the number of terms in neutron flux expression increase and it don't have a recurrence standard form. Just know that is a combination of sines and cosines hyperbolic functions with a polynomial function. An alternative to overcome this problem is to reconstruct the neutron flux for each new iteration in standard polynomial function. For this, we propose to interpolate the points found in a polynomial of the same degree in all iterations.

However, when are modeled problem of large dimensions, in the interpolation arise Vandermonde's arrays which are almost singulars. To eliminate this singularity, the domain is subdivided in *R* regions (Fig. 1). As these sub-regions are not part of the proposed problem, they are called of fictitious regions and should have careful to divide the domain real to coincide with the fictitious borders.

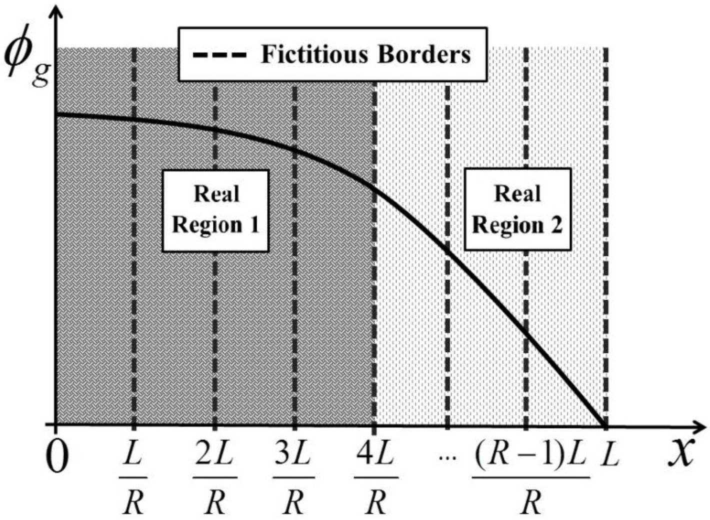

Figure 1. Representation of the real and fictitious regions.

This quantity *R* should be just sufficient for that the inversion matrix be not singular. A number larger than required, slows down the process computational. The last step to eliminate the singularity is displace all fictitious regions to the first region (Fig. 2).

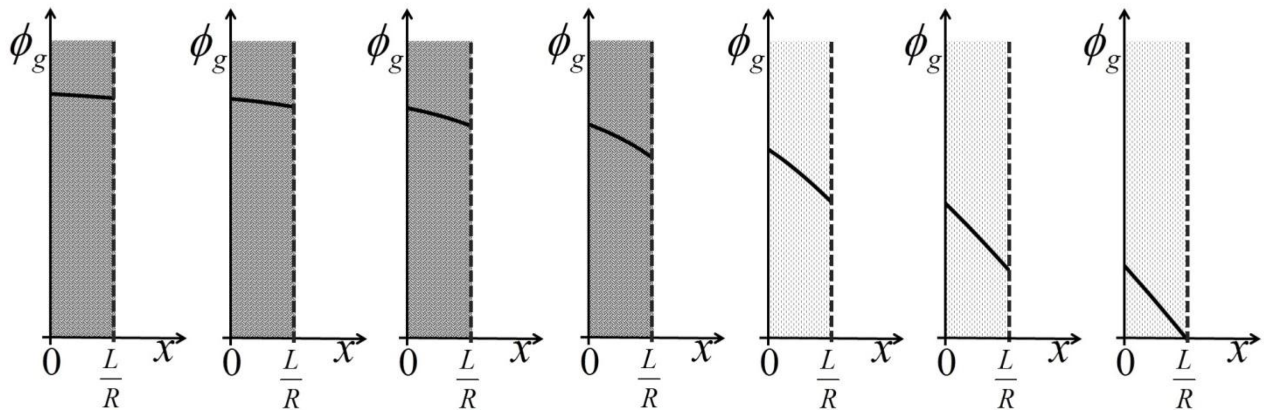

Figure 2. Representation of the regions displace for first region.

After overcome the singularity for each subdomain the points are interpolated. Note that the maximum order polynomial depends on the number of points of each interval. If there are *n* points, the maximum order may be *n+1*. Thus, the neutron flux to each energy group and fictitious regions assumes the standard form given by:

$$\varphi_{Rg} = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \cdots + c_n x^{n-1}, \tag{5}$$

where c are arbitrary constants.

Next, the neutron diffusion equation is solved for each region. Once the problem to be solved is an Ordinary Differential Equation (ODE) 2nd order, generate *2R* unknowns constant for each energy group. Applying the boundary conditions, flux and current density continuity at the interfaces given by (3) and (4) is obtained a coupled linear system of *2R* equations and *2R* unknowns for each energy group. Solving the linear system, the new neutron flux expression for each fictitious region and energy group is found.

3. RESULTS

The Fictitious Borders Power Method (FBPM) was implemented in *Scilab* software on a personal computer with *16 GB* of *Ram* memory, *Core i7* processor and *Windows* operating system. The proposed problem is a benchmark problem of the Argonne National Laboratory presented by Pollard (1977). This problem is one dimensional with 3 regions, domain $0cm \le x \le 240cm$, interfaces at $x = 40cm$ e $x = 200cm$ (where 0 to $40\,cm$ and $200\,cm$ to $240\,cm$ is the fuel region and $40\,cm$ to $200\,cm$ is the reflector region) with Dirichlet boundary conditions $\phi(0) = 0$ and $\phi(240) = 0$. It is considered two energy groups: fast ($g=1$) and thermal ($g=2$) without up-scattering ($\Sigma_{s21}=0$) and nuclear parameters as shown in Tab. 1.

Table 1. Nuclear Parameters.

Regions	D_1	D_2	Σ_{R1}	Σ_{R2}	$\nu\Sigma_{f1}$	$\nu\Sigma_{f2}$	Σ_{s12}
1 and 3	1.5	0.5	0.026	0.18	0.01	0.2	0.015
2	1.0	0.5	0.02	0.08	0.005	0.099	0.01

Other parameters used in FBPM algorithm: $\boldsymbol{\Delta x = 0.25cm}$; the convergence criterion $\varepsilon_1=\varepsilon_2=10^{-10}$ and polynomial interpolation of fourth degree. The numerical results for K_{eff} and power level are compared with benchmark Pollard (1977) by region, shown in Tab. 2.

Table 2. Results for K_{eff} and power level by region.

Reference	K_{eff}	1st region	2nd region	3rd region
Pollard (1977)	0.901560	0.2790	0.4421	0.2790
FBPM	**0.901596**	**0.2788**	**0.4423**	**0.2788**

The result obtained for comparison with the benchmark suggests a good agreement for global calculations in reactor physics. The FBPM spent **41,37s** for obtain converged results, which demonstrates a low computational time for this case test. The behavior of the fast and thermal neutron flux along the domain are shown in Fig. 3.

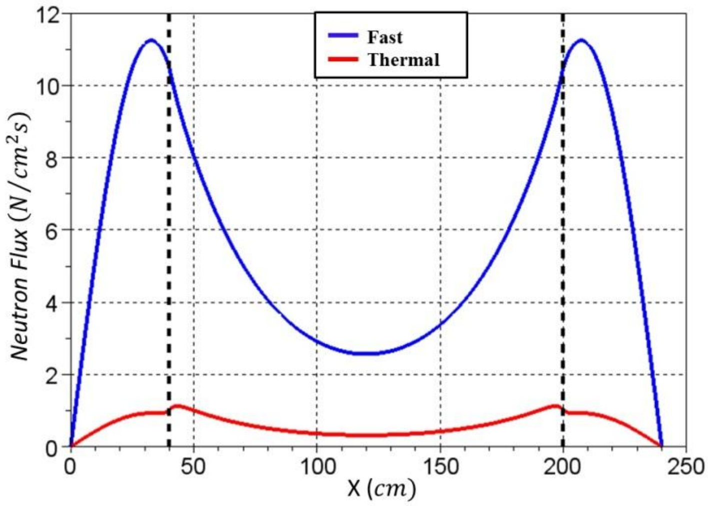

Figure 3. Fast and thermal neutron flux.

The Fig. 3 shows the satisfaction of the boundary conditions, maximum neutron flux around the point $40cm$, minimum neutron flux around the point $120cm$ and an small increase of the thermal neutron flux at the interface between fuel and reflector region due to contribution of fast neutrons that collide with the atoms in reflective region, losing energy (down-scattering) and become thermal neutron.

4. PERTURBATIVE ANALYSIS

In this section, we analyze of the solution by adding a perturbation in the nuclear parameters. The aim idea in this analysis is to determine the interference degree of each parameter in the convergence and solution behavior. Each test was disturbed by a single parameter. The perturbation is inserted adding a random fluctuation in the desired parameter, that in every iteration of the power method assumes a new value.

The magnitude of the perturbation must respect a certain range of values, so that even with the perturbation the solution can converge. If the perturbation overcome this range of values, can occur loss of the nuclear reactor control causing an accident or low energetic generation. An example is shown in Fig. 4, where the $\nu\Sigma_{f2}$ parameter was 10% perturbed.

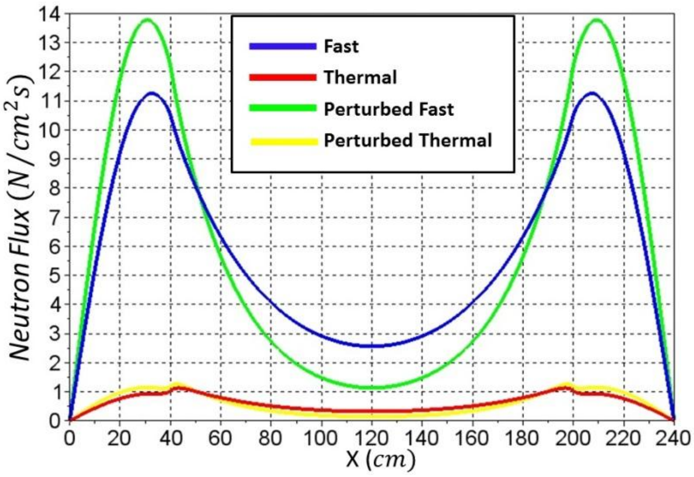

Figure 4. Neutron flux for $\nu\Sigma_{f2}$ 10% perturbed and not perturbed.

In this case test the value of K_{eff} is 0.943018 and the maximum value of the fast neutron flux is 13.773513, while the values not perturbed are 0.901596 and 11.249779 respectively.

A series of tests are done to find the convergence limit so that the perturbation not change the results significantly. In Table 3 shows the maximum percentage that each parameter allows for convergence. The problem was simulated 30 times for each perturbed parameter, with purpose to find the range for K_{eff} and neutron flux behavior (fast and thermal). The Tab. 3 shows the maximum, minimum, average and standard deviation for K_{eff} for 30 times which were simulated.

Table 3. Parameters perturbed and values for K_{eff} .

Parameter	Admitted Percentage	Maximum	Minimum	Average	Standard Deviation
D_1	0.95%	0.9017088	0.8997128	0.901354	0.0005303
D_2	7%	0.9017527	0.9006341	0.901531	0.0002335
Σ_{R1}	0.01%	0.9017380	0.9006053	0.901440	0.0003189
Σ_{R2}	0.01%	0.9016165	0.9002317	0.901346	0.0003486
$\nu\Sigma_{f1}$	0.02%	0.9016333	0.9001131	0.901395	0.0004042
$\nu\Sigma_{f2}$	0.01%	0.9016403	0.8998142	0.901467	0.0003504
Σ_{s12}	0.04%	0.9017428	0.8992533	0.901363	0.0005827

Knowing that the value for K_{eff} without the perturbation is 0.901596, so when analyzing the mean values and standard deviations we can realize a good agreement between them, it is also observed that with exception diffusion coefficients (D_1 and D_2), the other parameters admitted small percentage range. In Fig. 5 show the behavior of the perturbed neutron flux for each parameter and the not perturbed neutron flux.

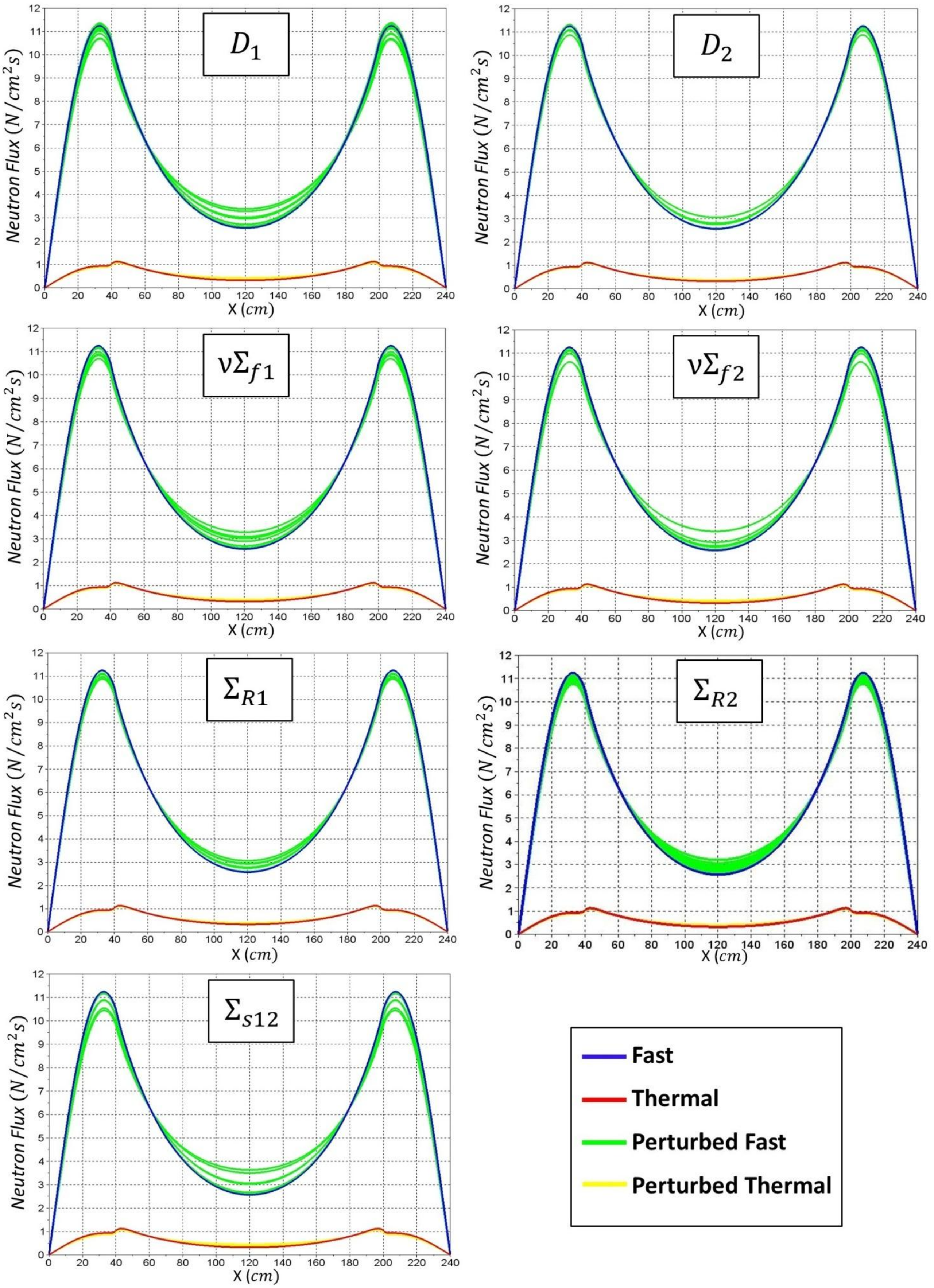

Figure 5. Neutron flux for perturbed and not perturbed parameters

In Fig. 5 note that for all nuclear parameters the neutron flux behave is similarly when added a small perturbation. In fuel region the perturbed neutron flux are below of the not perturbed neutron flux and in reflector region the behavior is reversed. This is best viewed for the fast neutron flux.

5. CONCLUSIONS

There is the need to foresee the neutron density in a nuclear reactor, once it's essential for energy efficiency and safety in operations. The FBPM has been successfully used to evaluate the neutron flux behavior by presenting easy implementation, low cost computational and good agreement with benchmark results present in the literature.

By analyzing of the sensitivity in each nuclear parameter perceive the importance in the precision of their values to not lose the nuclear reactor control and cause an accident or even low energy. The sensitivity analysis show that the diffusion coefficients admit a smaller accuracy when compared with the other parameters. The perturbed neutron flux consider that if there is a small inaccuracy or a constant perturbation in the parameters there is a downward shift in the fuel region and a shift up in reflector region leaving the neutron flux more uniform along of domain.

Remembering that this study is a linear problem, will be that in a nonlinear problem this perturbed flux will remain similar to unperturbed? Or the perturbed flux diverge? In this sense in future works we intend to extend this methodology for solve in more realistic nuclear reactor physics nonlinear multidimensional problems.

6. ACKNOWLEDGEMENTS

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