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Dynamics of a linear oscillator in the presence of stochastic stiffness using Itô formulation

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Abstract: In many engineering applications, there are uncertainties associated to system parameters and the statistics of system response are often of interest. These uncertainties are usually modeled as random variables that do not vary in time. This work focus on a different approach, where the random variables varies within time through a stochastic process. This work proposes the introduction of a different stochastic variable into the internal frequency of a single degree-of-freedom system. The stochastic variable is modeled by a stochastic process called Brownian Bridge. The system is also subjected to an harmonic excitation and the equations of motion are expressed using Itô stochastic differential equations. The system of equations is numerically integrated using Euler-Maruyama method. Finally, the system response is analyzed for one simulation in time and frequency domain and the statistics of the response are analyzed for a wide variety of excitation frequencies. It is concluded that the influence of uncertainties depends on the difference between excitation frequency and the natural frequency of the system.

Keywords: stochastic,Ito differential equation,vibration,dynamics,Brownian bridge

INTRODUCTION

Most of times, it is not possible to precisely predict a system response due to uncertainties associated to the system's characteristics. Because of this, the statistics of the response are of interest for many engineering applications. These uncertainties are usually accounted for by using techniques such as Monte-Carlo, FORM, SORM, Importance Sampling and so on. These techniques suppose that the random variables change between simulations but it remains constant during each simulation, i.e. they do not change with time. This work applies a different technique that permits the random variable to vary within time. This technique includes stochastic differential equations in system's description to generate stochastic processes that can be used to model either a system parameter or an external force.

The response of a variety of dynamical systems was analyzed by Wu and Cai (2004) when subjected to random excitations that are assumed to be stationary processes. The excitation is generated using Itô stochastic differential equation and Monte Carlo simulations are carried out in order to obtain the statistics of the transient and stationary properties. Xu et al. (2014) developed a stochastic averaging technique for an energy harvesting system to analytically evaluate the mean-square electric voltage and mean power output. The system is described by the Itô differential equation and the base excitation is modeled as a white noise. The analytical results were validated using Monte Carlo simulations. Tsuchida and Kimura (2016) proposed an equivalent non-Gaussian excitation to obtain response moments up to the 4th order. In this method, the diffusion coefficient in Itô differential equation is expressed by a quadratic polynomial and the moment equations can be expressed in a closed form. Numerical examples show that the method can well represent the moments up to 4th order. On the other hand, Poulin and Flierl (2008) introduced stochastic variables on internal frequencies instead of on external forces. The system of equations are numerically integrated and the stability of dynamical system was evaluated by determining the Lyapunov exponents for wide range of parameters. Cataldo and Soize (2017 and 2018) proposed a stochastic model to generate jitter based on a one-mass model for the dynamics of vocal folds. The stiffness of each vocal fold was considered to be a stochastic process and the system was modeled through Itô stochastic differential equations.

This work is part of an ongoing project to consider uncertainties in a rotating drill-string while drilling. In this manuscript, we propose the introduction of a different stochastic variable into the internal frequency of a single degree-of-freedom system. The stochastic variable is modeled by a stochastic process called Brownian Bridge. The main characteristics of this process are that the first moment is zero for all the time and the second moment increases with the distance from the limits of time interval. The motivation of such modeling is based on engineering applications in which the uncertainty in system parameters is higher in the middle of the operation. The equations of motion are expressed using Itô stochastic differential equations and they are numerically integrated using Euler-Maruyama method. Finally, the system response is analyzed for one simulation in time and frequency domain and the statistics of the response are analyzed for a wide variety of excitation frequencies.

MATHEMATICAL MODEL

A linear damped oscillator is considered. The dynamics of this oscillator can be described by the deterministic equation:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \;, \tag{1}$$

where m is the mass, c is the viscous damping coefficient, k is the stiffness and f(t) is the force applied. In this manuscript, the stiffness is considered to be stochastic, giving rise to a new stochastic equation:

$$m\ddot{X}(t) + c\dot{X}(t) + K(t)X(t) = f(t) \;, \tag{2}$$

The capital letter indicates that the variable is stochastic, i.e. $K(t)$ is the stochastic stiffness, $X(t)$ is the stochastic displacement of the mass and so on. The system is illustrated in Fig. 1.

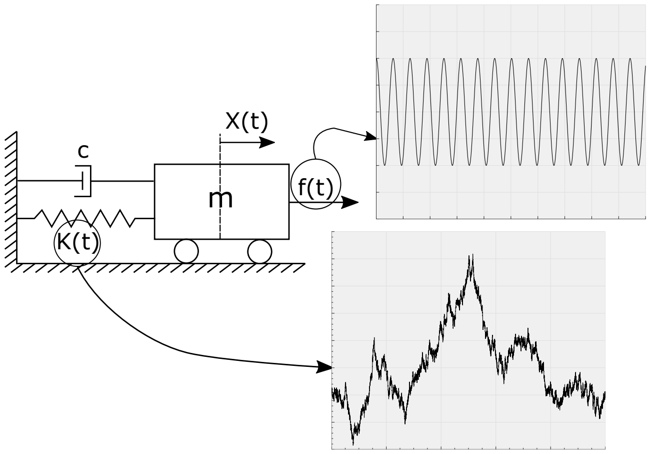


Figure 1: Illustration of the system.

Note that the stiffness now depends upon time t. The force $f(t)$ is considered to be harmonic such that $f(t) = a\cos(\omega t)$, where a and ω are constants. The stochastic model for the stiffness is based on the work of Cataldo and Soize (2017 and 2018) and it is constructed such that the following properties hold:

(i) For all $t > 0$, $0 < k_0 \leq K(t)$ where k_0 is a time-independent positive constant.

(ii) K(t) is a stochastic process with mean value $\bar{k} = E[K(t)] > k_0 > 0$ and finite variance $E[K^2(t)] < \infty$ for all t.

(iii) At the origin ($t = 0$) and at the end time $t = T$, the stiffness values are $K(t=0) = K(t=T) = \bar{k}$. Consequently, $E[K^2(0)] = E[K^2(T)] = 0$.

(iv) The second moment is maximum at $t = T/2$.

In order to respect these properties, the random stiffness is written as

$$K(t) = k_0 + (\bar{k} - k_0)(\bar{z} + Z(t))^2, \tag{3}$$

in which $\bar{z}$ is a parameter such that $E[(\bar{z}+Z(t))^2] = 1$ and $E[(\bar{z}+Z(t))^4] < \infty$. The stochastic process $Z(t)$ is constructed as a Brownian Bridge, which is described by the Itô stochastic differential equation expressed in Eq. 4 with mean $E[Z(t)] = 0$ and $E[Z^2(t)] = c^2 t(T-t)/T$:

$$dZ = \frac{Z(t)}{T-t}dt + \sigma dW, \quad 0 \leq t \leq T, \tag{4}$$

where W is a Brownian motion. In order to satisfy the constraint that $E[(\bar{z}+Z(t))^2] = 1$, the parameter $\bar{z}$ is calculated as

$$E[(\bar{z}+Z(t))^2] = E[\bar{z}^2] + 2E[\bar{z}Z(t)] + E[Z^2(t)] = 1 \rightarrow \bar{z}^2 = 1 - \sigma^2 \frac{t(T-t)}{T}, \tag{5}$$

As $\max(t(T-t)/T) = T/4$, the equation above yields the constraint

$$\sigma < \frac{2}{\sqrt{T}}. \tag{6}$$

Using the state space $[X_1 \ X_2 \ X_3]^T = [X \ \dot{X} \ Z]^T$ in Eqs. 2 and 4, the following system of equations is obtained for $0 \leq t \leq T$:

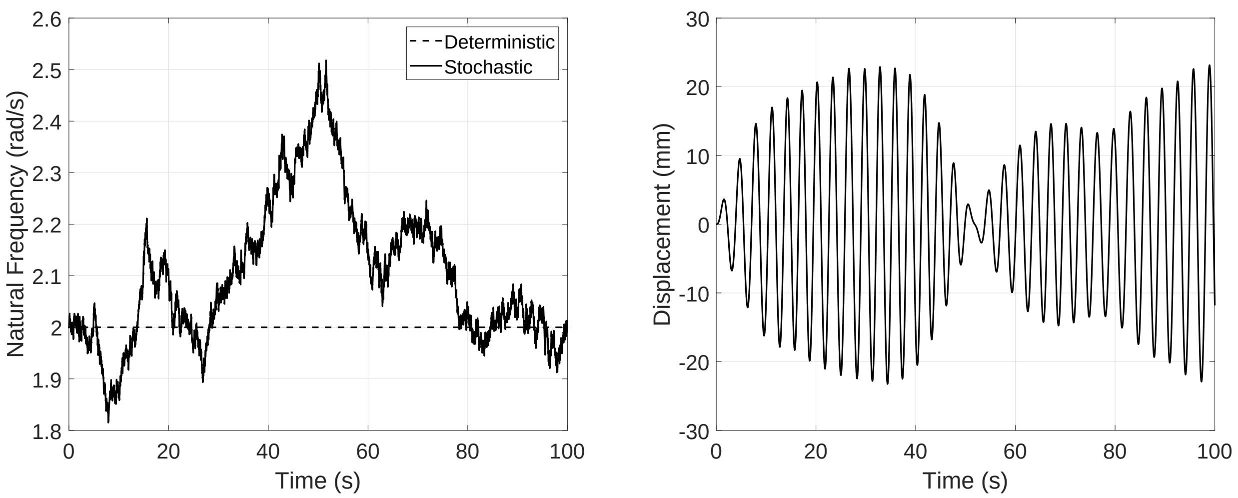
$$\begin{aligned} dX_1 &= X_2 \, dt, \\ dX_2 &= \frac{1}{m} \left(f(t) - cX_2 - K(t) X_1 \right) dt, \\ dX_3 &= \frac{-X_3}{T-t} dt + \sigma dW, \end{aligned} \tag{7}$$

where $K(t)$ depends on X_3 according to Eq. 3.

RESULTS

For the numerical simulations, the following parameters are used: m = 1 kg, c = 0.2 N.s/m, $\bar{k}$ = 4 kN/m; k_0 = 1 kN/m, a = 10 N; σ = 0.0288. The total time of simulation is 100 seconds. The results presented below were obtained using Euler-Maruyama method with time step $dt = 2^{-10}$.

One simulation of Eq. 7 is shown in Fig. 2 for $T = 100$ s. In Fig.2a, the natural frequency of the system is plotted against time. The natural frequency (ω_n) is obtained by $\omega_n = \sqrt{K(t)/m}$, where the stiffness is a stochastic process for the stochastic case. In Fig. 2b, the stochastic response is shown for an excitation frequency of 2 rad/s. Although the deterministic response would give a constant amplitude over time after the transient response, in its stochastic counterpart the amplitude varies during simulation due to the variation of system natural frequency.



(a) (b)

Figure 2: (a) Natural frequency over time and; (b) Displacement over time.

In Fig. 3, the difference between deterministic and stochastic results can be noticed in the frequency domain for one simulation. The frequency response is obtained by taking the Fourier transform of system displacement depicted in Fig. 2b. The deterministic result presents almost all the energy concentrated in the excitation frequency, that is also the natural frequency in this case. For the stochastic results, there is a greater contribution of other frequencies induced by the random variation of stiffness. There is a noise over all frequency range and uncertainty about the peak value.

Next, we analyze the statistics of the response in Fig. 4 through the second central moment of the displacement. The second central moment gives a measure of variability and will make possible the conclusions related to the amplitude of vibration. To make it possible, two different time intervals are considered: T1 = $\{t \in [40, 40+6\pi]\}$ and T2 = $\{t \in [80, 80+6\pi]\}$, where 6π is the time for the system to make 6 complete cycles in 2 rad/s. The time intervals were selected in a way to represent the mid of simulation, where variation due to Brownian motion is maximum, and the end of simulation,

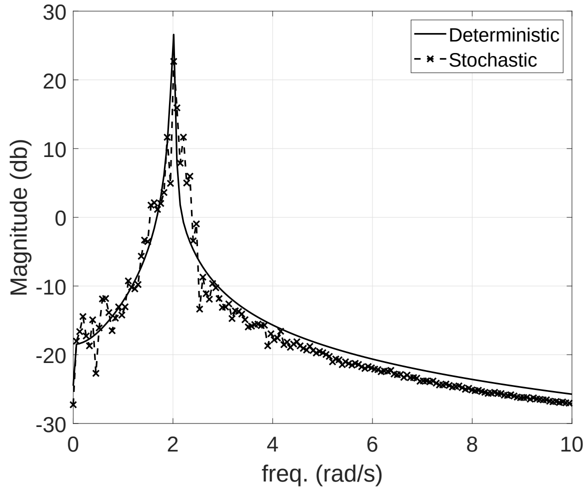


Figure 3: Frequency response obtained by the fourier transform of system displacement.

where the variation is minimum. The second central moment of a system displacement under harmonic excitation for a given time interval is directly related to the amplitude of vibration. To calculate the second central moment, 2000 Monte Carlo simulations are performed and the displacement of all simulations are put together for each time interval and the sample variance of such vector is taken.

In Fig. 4, the second central moment ($E[(X-E[X])^2]$) is plotted against excitation frequency (ω). At first, the deterministic result shows that the second central moment is higher at 2 rad/s, because it is the deterministic natural frequency of the system. Further, as expected, the second central moment rapidly decreases as the excitation frequency differs from 2 rad/s. In the region near deterministic natural frequency, the second central moments of the stochastic results are lower. It occurs because the variation in the natural frequency causes the frequency of excitation not to coincide exactly with the natural frequency, assuming values away from it during simulation. Therefore, the vibration amplitude is lower. In T1, the variation is maximum and thus the amplitude will be lower than in T2. For excitation frequencies away from deterministic natural frequency, the second central moment on T1 is higher because the variation of natural frequency makes it assume values near the excitation frequency during simulation, even though the mean natural frequency (i.e. deterministic natural frequency) is away from excitation frequency.

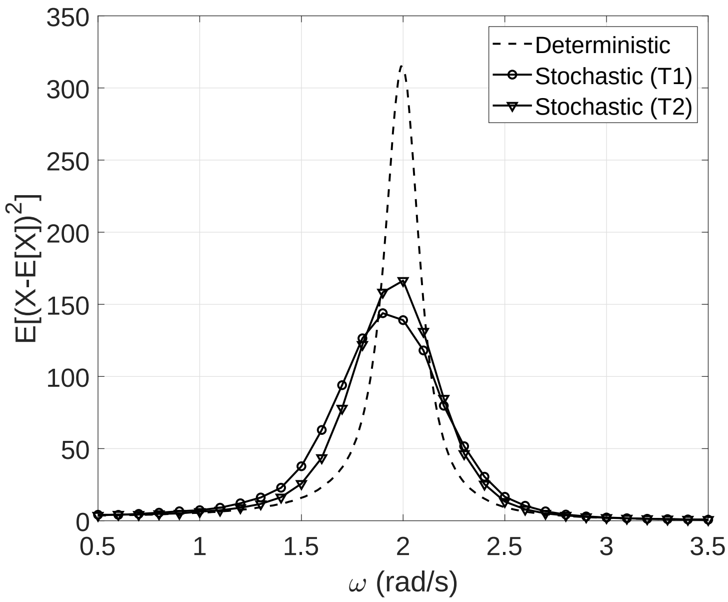


Figure 4: Variation of system displacement for two time intervals in stochastic response and one time interval in deterministic response.

CONCLUSIONS

In this manuscript, a 1-DOF linear oscillator was analyzed under uncertainties on stiffness, that were modeled as stochastic processes that varies in time. The system of equations was derived using Itô stochastic differential equations and the numerical results were obtained using Euler-Maruyama method. At first, a single simulation is analyzed and we conclude that the uncertainties in stiffness results in a variation of the natural frequency that causes amplitude to change over time when the system is excited with the frequency equal to the natural frequency. The frequency response presented some noise near the excitation frequency that was related to the random stiffness. Next, the statistics of the response were assessed using the second central moment calculated over 2 time intervals for 2000 Monte Carlo simulations. We conclude that the uncertainties in natural frequency helps to reduce vibration amplitude when system is excited near natural frequency, but increases vibration amplitude when excitation frequency is away from natural frequency.

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