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Phase Field and Fractional Derivatives to Model Damage, Fracture and Fatigue in Viscoelastic Materials

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Abstract. This work concerns on the development of a formulation based on phase field theory and fractional derivatives for modeling damage, fracture and fatigue in materials with viscoelastic behavior. Phase field models have been a trend in describing damage, fracture and fatigue due to the regularization of sharp crack surfaces using continuum scalar fields. On the other hand, fractional derivatives are able to model hereditary phenomena with long memory, as in the case of viscoelastic materials subject to stress over time. The coupling of these two theories, fractional derivatives and phase fields, produces a promising and novel framework to predict failure in different materials. A comparison with other formulation found in the literature is presented and some preliminary results are shown. In the present paper we just outline the main arguments. The technical details will appear in a future work.
Keywords: Phase field model. Fractional derivatives. Viscoelastic materials. Damage, Fracture and Fatigue.

INTRODUCTION

Recently, the phase field theory has been considered to describe damage, fracture and fatigue in several materials (Schänzel, 2015; Fabrizio, 2014). Although substantial work has been done, some features of damage and fracture are of tricky description. In particular, the modeling of these phenomena for viscoelastic materials is a subject still under-explored. Many works on viscoelastic behavior analysis have been done (Cai et al., 2016; Abir et al., 2018; Serra et al., 2018). However, few works have been dedicated to analyze the time-dependent behavior of viscoelastic materials under damage and fatigue effects.

In this work we propose a thermodinamically consistent model to describe damage, fracture and fatigue in materials with viscoelastic behavior. Viscoelasticity is considered using fractional derivatives. Such subject has been studied substantially by many authors [e.g., Schimidt and Gaul (2006) and Caputo and Fabrizio (2016)] and results in fractional-order differential stress-strain constitutive equation which are interesting to describing phenomena with long memory (Mainardi, 2012). Damage, fracture and fatigue description are considered following the mathematical development described by Boldrini et al. (2016).

The proposed framework is grounded in the couple of the phase fields theory with fractional derivatives and complementary considerations of continuum mechanics. This methodology produces a system of partial differential equations that describe the motion, damage, temperature and fatigue in materials with viscoelastic behavior. We present a simulation and compare it with a previous result found in the literature.

In the present paper we just outline the main arguments. The technical details will appear in a future work.

GENERAL GOVERNING EQUATIONS

Primarily, consider a body in the reference configuration $\Omega_0 \subset \mathbb{R}^3$ with Lagrangian coordinates denoted by $\boldsymbol{p}$ and an arbitrary regular subdomain $\mathcal{D} \subset \Omega_0$. The thermodynamical state of the body is described by: mass density ρ_0, that must satisfy the principle of conservation of mass; dynamical variables representing displacement and velocity vector fields, respectively denoted by $\boldsymbol{u}$ and $\boldsymbol{v}$, whose equations will be obtained by the principle of virtual power (PVP); and the specific density of the internal energy by unit of mass e_0, whose equation is obtained by the first principle of thermodynamic (balance of energy). The main variable related to damage is the volumetric fraction of damage described by φ. This variable lies in the interval $[0,1]$; $\varphi = 0$ is associated with virgin material and $\varphi = 1$ with fractured material. In the context of this work, the damage is considered a dynamic variable with corresponding equation obtained by the PVP. The fatigue variable $\mathcal{F}$ is an internal variable (does not produce microscopic energy) and the governing equation is derived by the second principle of thermodynamics.

Based on the principle of conservation of mass, first principle of thermodynamics, and the PVP, the basic governing

equations are summarily given by

$$\dot{\boldsymbol{u}} = \boldsymbol{v} \tag{1.a}$$

$$\rho_0 \dot{\boldsymbol{v}} = \mathrm{div}_{\boldsymbol{p}} (\boldsymbol{F}\boldsymbol{S}) + \rho_0 \boldsymbol{f_0}, \tag{1.b}$$

$$0 = \mathrm{div}_{\boldsymbol{p}} (\boldsymbol{h_0}) - b_0 + \rho_0 a, \tag{1.c}$$

$$\rho_0 \dot{e}_0 = -\mathrm{div}_{\boldsymbol{p}} (\boldsymbol{q_0}) + \rho_0 r_0 + \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} + b_0 \dot{\varphi} + \boldsymbol{h_0} \cdot \nabla_{\boldsymbol{p}} \dot{\varphi}, \tag{1.d}$$

$$\dot{\mathcal{F}} = Z, \tag{1.e}$$

where, the dot denotes the material derivative, $\boldsymbol{F}$ is the tensor gradient of deformation, $\boldsymbol{S}$ is the second Piola- Kirchoff stress tensor, $\boldsymbol{f_0}$ is the body force vector field per unit of mass, $\boldsymbol{h_0}$ is the energy flux associated to the spatial variation of a unit of φ in a unit of time, b_0 is the volume density of energy exchanged by variation of a unity of φ in a unit of time, a represents the external effects (for example, irradiation effects), $\boldsymbol{q_0}$ is the heat flux vector field, r_0 is the specific heat source density and $\boldsymbol{C}$ is the right Cauchy-Green strain tensor. We stress that (1.e) is a constitutive differential relation for the evaluation of the fatigue variable $\mathcal{F}$. This constitutive relation is not given a priori, but its expression Z must be determined by using the entropy condition. The notations $\mathrm{div}_{\boldsymbol{p}}$ and $\nabla_{\boldsymbol{p}}$ correspond respectively to the divergent and the the gradient operators in the Lagrangian configuration. Additionally, the second principle of thermodynamics (entropy inequality) written for the free energy of Helmholtz, $\psi = e_0 - \theta\eta_0$ (where η_0 is the specific entropy density and θ is the the absolute temperature), leads to

$$-\rho_0 \left(\dot{\psi} + \dot{\theta}\eta_0\right) + \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} + b_0 \dot{\varphi} + \boldsymbol{h_0} \cdot \nabla_{\boldsymbol{p}} \dot{\varphi} - \frac{1}{\theta}\boldsymbol{q_0} \cdot \nabla_{\boldsymbol{p}} \theta + \theta \mathrm{div}_{\boldsymbol{p}} (\boldsymbol{k}) - \rho_0 w \geq 0, \tag{2}$$

where the flux $\boldsymbol{k}$ and the quantity w are both corrective entropy terms associated with the physical processes in the evolution of the the phase fields. We impose that the process of microscopic evolution does not produces flux of entropy. Moreover, the entropy production due to microscopic evolution cannot decrease (Boldrini et al., 2016).

The set of Eqs. (1) and the inequality (2) constitute the general expressions for the model considered in this work. The constitutive relations must be found by considering the previous aspects.

CONSTITUVE EQUATIONS

The approach here considered is general and allows the use of distinct free-energy potentials, ψ, and their associated pseudo-potentials of dissipation, ψ_d, as suggested in Boldrini et al. (2016). The free-energy potential $\psi = \psi(\Gamma)$ is defined in terms of the independent variable set $\Gamma = (\theta, \varphi, \mathcal{F}, \nabla_{\boldsymbol{p}}\theta, \nabla_{\boldsymbol{p}}\varphi, \nabla_{\boldsymbol{p}}\mathcal{F}, \boldsymbol{C})$.

The variables $\boldsymbol{S}$, b_0, $\boldsymbol{h_0}$ and $\boldsymbol{q_0}$ are split into dissipative (irreversible) and non-dissipative (reversible) parts. The first part, indicated by the superscript $(\cdot)^r$, represents the nondissipative, or reversible, contributions. The second part, indicated by the superscript $(\cdot)^{ir}$, represents their dissipative, or irreversible, contributions.

Following Frémond (2013), it is assumed that $\boldsymbol{q_0^r} = \boldsymbol{0}$ (heat flux is purely dissipative) and $\boldsymbol{h_0^{ir}} = \boldsymbol{0}$. By taking these aspects into account and applying the chain rule for $\dot{\psi}$ in expression (2), we obtain

$$-\rho_0 (\eta_0 + \partial_\theta \psi)\dot{\theta} + \left(-\rho_0 \partial_\varphi \psi + b_0^{(ir)} + b^{(r)}\right)\dot{\varphi} - \left(\rho_0 \partial_{\nabla_{\boldsymbol{p}}\varphi}\psi - \boldsymbol{h_0^{(ir)}}\right)\nabla_{\boldsymbol{p}}\varphi + \left(\frac{1}{2}\boldsymbol{S}^{(ir)} + \frac{1}{2}\boldsymbol{S}^{(r)} - \rho_0 \partial_{\boldsymbol{C}}\psi\right) : \dot{\boldsymbol{C}}$$

$$-\rho_0 \partial_{\nabla_{\boldsymbol{p}}\theta}\psi \nabla_{\boldsymbol{p}}\dot{\theta} - \rho_0 \partial_{\mathcal{F}}\psi \mathcal{Z} - \rho_0 \partial_{\nabla_{\boldsymbol{p}}\mathcal{F}}\psi \nabla_{\boldsymbol{p}} \cdot \mathcal{Z} - \frac{1}{\theta}\boldsymbol{q_0^{(ir)}} \cdot \nabla_{\boldsymbol{p}}\theta + \theta \mathrm{div}_p k - \rho_0 w \geq 0. \tag{3}$$

At this stage, the reversible terms are chosen in the last inequality such that they do not contribute to produce entropy increase, for all admissible processes. Therefore,

$$-\rho_0 (\eta_0 + \partial_\theta \psi)\dot{\theta} - \rho_0 \partial_{\nabla_{\boldsymbol{p}}\theta}\psi \nabla_{\boldsymbol{p}}\dot{\theta} + \left(-\rho_0 \partial_\varphi \psi + b_0^{(r)}\right)\dot{\varphi} - \left(\rho_0 \partial_{\nabla_{\boldsymbol{p}}\varphi}\psi - \boldsymbol{h_0^{(r)}}\right)\nabla_{\boldsymbol{p}}\dot{\varphi} + \left(\frac{1}{2}\boldsymbol{S}^{(r)} - \rho_0 \partial_{\boldsymbol{C}}\psi\right) : \dot{\boldsymbol{C}} = 0. \tag{4}$$

The variables $\dot{\theta}$ and $\nabla_{\boldsymbol{p}}\dot{\theta}$ in the above equation have linear dependencies. Then

$$\eta_0 = -\partial_\theta \psi, \tag{5.a}$$

$$\partial_{\nabla_{\boldsymbol{p}}\theta}\psi = 0. \tag{5.b}$$

Additionally, we impose

$$b_0^{(r)} = \rho_0 \partial_\varphi \psi, \tag{6.a}$$

$$\boldsymbol{h_0^{(r)}} = \boldsymbol{h_0} = \rho_0 \partial_{\nabla_{\boldsymbol{p}}\varphi} \psi, \tag{6.b}$$

$$\boldsymbol{S}^{(r)} = 2\rho_0 \partial_{\boldsymbol{C}} \psi. \tag{6.c}$$

By using Eqs. (5), (6) and considering $\boldsymbol{k} = \frac{\rho_0}{\theta} \partial_{\nabla_{\boldsymbol{p}}\mathcal{F}} \psi \mathcal{Z}$ and $w = 0$, inequality (3) can be rewritten as

$$\frac{b_0^{(ir)}}{\theta}\dot{\varphi} + \frac{1}{2\theta}\boldsymbol{S}^{(ir)} : \dot{\boldsymbol{C}} - \frac{q_0^{(ir)}}{\theta^2} \cdot \nabla_p \theta - \mathcal{Z}\xi \geq 0, \tag{7}$$

where

$$\xi = \frac{1}{\theta}\rho \partial_{\mathcal{F}} \psi - \mathrm{div}_{\boldsymbol{p}} \left(\frac{\rho_0}{\theta} \partial_{\nabla_{\boldsymbol{p}}\mathcal{F}} \psi \right). \tag{8}$$

Next, we must find expressions for $b_0^{(ir)}$, $\boldsymbol{S}^{(ir)}$, $q_0^{(ir)}$ and $\mathcal{Z}$ such the inequality (7) is satisfied. Moreover, we want to include viscoelaticity in the formulation and for this, fractional derivatives will be used. A related approach to consider viscoelasticity was used by Fabrizio (2014), but in his work viscoelasticity was considered as a term in the free energy potential. The formulation proposed in the present work is different because viscoelasticity is considered in the pseudo potential of dissipation, since this behavior is time dependent and dissipative. The pseudo potential of dissipation ψ_d, used to satisfy inequality (7) is split in two parts, ψ_{d_1} and ψ_{d_2}. This is done in order to separate the irreversible part of the second Piola stress tensor as $\boldsymbol{S}^{(ir)} = \boldsymbol{S}^{(ir)}_{class} + \boldsymbol{S}^{(ir)}_{frac}$, where the subscripts *class* and $frac$ correspond to the classical part of the tensor and the fractional counterpart, respectively. Taking this in account, inequality (7) is rewritten as

$$\frac{b_0^{(ir)}}{\theta}\dot{\varphi} + \frac{1}{2\theta}\boldsymbol{S}^{(ir)}_{class} : \dot{\boldsymbol{C}} + \frac{1}{2\theta}\boldsymbol{S}^{(ir)}_{frac} : \dot{\boldsymbol{C}} - \frac{q_0^{(ir)}}{\theta^2} \cdot \nabla_p \theta - \mathcal{Z}\xi \geq 0. \tag{9}$$

The classical part of the pseudo-potential of dissipation is considered in ψ_{d_1}. Such functional is defined for the variables $\psi_{d_1} = \psi_{d_1}(\dot{\varphi}, \dot{\boldsymbol{C}}, \nabla_{\boldsymbol{p}}\theta, \xi, \tilde{\Gamma}) \geq 0$, where $\tilde{\Gamma} = (\theta, \varphi, \mathcal{F}, \nabla_{\boldsymbol{p}}\varphi, \nabla_{\boldsymbol{p}}\mathcal{F}, \boldsymbol{C})$, and $\psi_{d_1}(0,0,0,0,\tilde{\Gamma}) = 0$ being continuous and convex with respect to the variables $\dot{\varphi}, \dot{\boldsymbol{C}}, \nabla_{\boldsymbol{p}}\theta$ and ξ. A choice for ψ_{d_1} is

$$\psi_{d_1}(\dot{\varphi}, \nabla_{\boldsymbol{p}}\theta, \xi, \tilde{\Gamma}) = \frac{\tilde{\lambda}(\tilde{\Gamma})}{2}|\dot{\varphi}|^2 + \frac{\tilde{b}(\tilde{\Gamma})}{2}|\dot{\boldsymbol{C}}|^2 + \frac{\tilde{c}(\tilde{\Gamma})}{2}\nabla_{\boldsymbol{p}}\theta \cdot \boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\theta + \frac{\tilde{F}(\tilde{\Gamma})}{2}|\xi|^2, \tag{10}$$

where the coefficients are nonnegative and $\tilde{\Gamma} = (\theta, \varphi, \mathcal{F}, \nabla_{\boldsymbol{p}}\varphi, \nabla_{\boldsymbol{p}}\mathcal{F})$ (Boldrini et al., 2016). We use Eq. (10) to satisfy part of the inequality (9)

$$\partial_{\dot{\varphi}}\psi_{d_1} = \tilde{\lambda}(\tilde{\Gamma})\dot{\varphi} = \frac{b_0^{(ir)}}{\theta}, \tag{11}$$

$$\partial_{\nabla_{\boldsymbol{p}}\theta}\psi_{d_1} = \tilde{c}(\tilde{\Gamma})\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\theta = \frac{\boldsymbol{q_0^{(ir)}}}{\theta^2}, \tag{12}$$

$$\partial_{\xi}\psi_{d_1} = \tilde{F}(\tilde{\Gamma})\xi = -Z, \tag{13}$$

$$\partial_{\dot{\boldsymbol{C}}}\psi_{d_1} = \tilde{b}(\tilde{\Gamma})\dot{\boldsymbol{C}} = \frac{\boldsymbol{S}^{(ir)}_{class}}{2\theta}. \tag{14}$$

The fractional viscoelastic part of the pseudo-potential of dissipation is considered in ψ_{d_2}. It also should satisfy the requirements of positiveness and convexity. A choice for ψ_{d_2} that can ensure the positiveness of the term $\frac{1}{2\theta}\boldsymbol{S}^{(ir)}_{frac} : \dot{\boldsymbol{C}}$ and complete the validity of the inequality (9) is

$$\begin{aligned}\psi_{d_2}(\boldsymbol{p},t) \quad = \quad & \frac{G(\varphi)}{(2\theta)(2\rho_0)\Gamma(1-\alpha)}\Big[\frac{[\boldsymbol{C}(\boldsymbol{p},t)-\boldsymbol{C}(\boldsymbol{p},0])) : \boldsymbol{A} : [\boldsymbol{C}(\boldsymbol{p},t)-\boldsymbol{p},0)]}{t^\alpha} \\ & + \int_0^t \frac{[\boldsymbol{C}(\boldsymbol{p},t-s)-\boldsymbol{C}(\boldsymbol{p},t])) : \boldsymbol{A} : [\boldsymbol{C}(\boldsymbol{p},t-s)-\boldsymbol{p},t)]}{s^{1+\alpha}} ds\Big],\end{aligned} \tag{15}$$

where $G(\varphi) > 0$ is a degradation function, Γ is the gamma function, and $\boldsymbol{A}$ is a forth order symmetric tensor related to the physical properties of the material, with the property $A_{ijkl} = A_{klij}$. This will generate a $\boldsymbol{S}_{frac}^{(ir)}$ with intermediates between the elastic and viscous behavior. Taking it into account, we define

$$\boldsymbol{S}_{frac} = G(\varphi)\boldsymbol{A} : {}_0\mathrm{D}_{t\boldsymbol{C}}^{\alpha}\boldsymbol{C}, \tag{16}$$

where ${}_a\mathrm{D}_{t\boldsymbol{C}}^{\alpha}$ denotes the Caputo fractional derivative of $\boldsymbol{C}$ in the interval $[0,t]$ and α is the fractional derivative order.

Equations (6.c), (14) and (16) lead to the following stress equation

$$\boldsymbol{S} = \underbrace{\overbrace{2\rho_0\partial_{\boldsymbol{C}}\psi}^{\boldsymbol{S}^{(r)}} + \overbrace{2\theta\partial_{\dot{\boldsymbol{C}}}\psi_{d_1}}^{\boldsymbol{S}_{class}^{(ir)}}}_{\text{classical stress}} + \underbrace{\overbrace{\boldsymbol{A} : {}_0\mathrm{D}_{t\boldsymbol{C}}^{\alpha}\boldsymbol{C}}^{\boldsymbol{S}_{frac}^{(ir)}}}_{\text{fractional stress}}, \tag{17}$$

In the unidimensional case, the formulation presented above and the split of the stress $\boldsymbol{S}$ in Eq. (17) can be related with the fractional Kelvin-Voigt rheological model as shown in the Fig. 1. This viscoelastic model consists in a spring and a spring-pot (fractional rheological element represented by the rhombus in the Fig. 1) in parallel. The spring represents the elastic contribution and the spring-pot represents the viscoelastic counterpart (Mainardi, 2010).

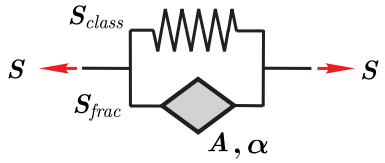

Figure 1 – Representation of the fractional Kelvin-Voigt viscoelastic model in the unidimensional case.

FREE ENERGY

The free energy ψ is decomposed in three parts: elastic distortions ψ_e, purely thermal efects ψ_θ and damage and fatigue contributions $\mathcal{I}$. Then volumetric density of free energy is given by

$$\rho_0\psi(\theta,\varphi,\nabla_{\boldsymbol{p}}\varphi,\boldsymbol{C}) = (1-\varphi)^2\psi_e(\boldsymbol{C}) + \psi_\theta(\theta) + \mathcal{I}(\varphi,\nabla_{\boldsymbol{p}}\varphi,\mathcal{F}). \tag{18}$$

The usual elastic energy for the compressible Neo-Hookean material (Bonet and Wood, 2008) is

$$\psi_e(\boldsymbol{C}) = \frac{\mu}{2}(\mathrm{tr}(\boldsymbol{C}-3) - \mu\ln((\det(\boldsymbol{C}))^{\frac{1}{2}} + \frac{\lambda}{2}(\ln(\det(\boldsymbol{C}))^{\frac{1}{2}})^2,$$

where μ and λ are material parameters. The choice for Neo-Hookean material is due to the possibility of modeling large strain.

The thermal part of the free energy is

$$\psi_\theta(\theta) = c_v\theta\ln\theta, \tag{19}$$

where c_v is the heat capacity.

Finally, the damage and fatigue contributions are

$$\mathcal{I}(\varphi,\nabla_{\boldsymbol{p}}\varphi,\mathcal{F},\boldsymbol{C}) \quad = \quad g_c\left(\frac{\gamma}{2}\nabla_{\boldsymbol{p}}\varphi\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\varphi + \frac{1}{\gamma}H(\varphi)\right) + \frac{1}{\gamma}\mathcal{F}H_f(\varphi).$$

The Griffith fracture energy g_c is assumed positive and constant; $\gamma > 0$ is related with the length of the fractured layers. Moreover, $H(\varphi) = \frac{\varphi^2}{2}$ and $H_f = -\varphi$ for $\varphi \in [0,1]$ (Boldrini et al., 2016).

From previous considerations and after some algebraic manipulations, we summarize the governing equations as

$$\dot{\boldsymbol{u}} = \boldsymbol{v} \tag{20.a}$$

$$\dot{\boldsymbol{v}} = \operatorname{div}(\boldsymbol{F}\boldsymbol{S}) + \rho_0 f_0, \tag{20.b}$$

$$\begin{aligned}\boldsymbol{S} = {} & (1-\varphi)^2\left(\mu(1-\boldsymbol{C}^{-1}) + \lambda\ln(\det\boldsymbol{C})^{\frac{1}{2}}\boldsymbol{C}^{-1}\right) - g_c\gamma(\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\varphi)\otimes(\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\varphi) + {}_{-\infty}\mathcal{D}_t^{\alpha}\boldsymbol{C} : \boldsymbol{A} \\ & + 2\theta\tilde{b}(\tilde{\Gamma})\dot{\boldsymbol{C}},\end{aligned} \tag{20.c}$$

$$\begin{aligned}\dot{\varphi} = {} & \frac{1}{\theta\tilde{\lambda}(\tilde{\Gamma})}\operatorname{div}\left(\rho_0 g_c\frac{\gamma}{2}\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\varphi\right) - \frac{2(1-\varphi)}{\theta\tilde{\lambda}(\tilde{\Gamma})}\left[\frac{\lambda}{2}(\ln(\det\boldsymbol{C})^{\frac{1}{2}} - \mu\ln(\det\boldsymbol{C})^{\frac{1}{2}} + \frac{\mu}{2}(tr[\boldsymbol{C}]-3)\right] \\ & - \frac{1}{\gamma\theta\tilde{\lambda}(\tilde{\Gamma})}\left[\frac{g_c}{\gamma}H'(\varphi) + \frac{1}{\gamma}\mathcal{F}H_f'(\varphi)\right],\end{aligned} \tag{20.d}$$

$$\dot{\mathcal{F}} = -\tilde{F}(\tilde{\Gamma})\cdot\frac{1}{\theta}\frac{1}{\gamma}H_f(\varphi), \tag{20.e}$$

$$\dot{\theta} = \frac{1}{c_v}\operatorname{div}[\theta^2\tilde{c}(\tilde{\Gamma})\boldsymbol{C}^{-1}\nabla_{\boldsymbol{p}}\theta] + \frac{\theta}{c_v}\tilde{\lambda}(\tilde{\Gamma})|\dot{\varphi}|^2 + \frac{\tilde{F}(\tilde{\Gamma})}{c_v\gamma^2\theta}H_f^2 + \frac{1}{2c_v}{}_{-\infty}\boldsymbol{A} : {}_0\mathrm{D}_{t\boldsymbol{C}}^{\alpha}\boldsymbol{C} + \frac{\rho_0 r_0}{c_v} + \theta\frac{\tilde{b}(\lambda)}{c_v}|\dot{\boldsymbol{C}}|^2 \tag{20.f}$$

where $\tilde{F}$ is a material parameter. Equations (20.a) and (20.b) give the motion of the body. Equation (20.c) is derived from Eq. (17) and the definitions of ψ and ψ_{d_1}. Equation (20.d) is derived from Eq. (1.c) and Eqs. (20.e) and (20.f) are derived from Eqs. (1.e) and (1.d), respectively.

RESULTS

The developed model is able to consider several effects. Equations (20.a) and (20.b), that describes the body motion, were discretized by using the finite element and the Newmark methods. It was implemented taking into account the stress tensor $\boldsymbol{S}$ of Eq. (20.c) with $\varphi = 0$ and $\tilde{b} = 0$. We compared the dynamical response of a viscoelastic rod subject to a time dependent load, with the formulation proposed by Schmidt and Gaul (2006). Both models were made compatible for comparisons. The same discretization methods are used and modifications in the original Schmidt and Gaul's framework is performed to ensure that it represents the rheological fractional Kelvin-Voigt model. It is important to emphasize that the model proposed in this work was implemented for the bidimensional case, whereas the Schmidt and Gaul's model is unidimensional and adopts a linear constant p multiplied by the fractional derivative to calculate the stress. This constant is similar to $\boldsymbol{A}$ (see Eq. (17)) for the unidimensional case.

The problem considered consists of a rod fixed at one end and subjected to a single step load $F(t) = 100\text{N}$ for $t > 0$ at the free end. The bar rod length is $\ell = 2m$ and the cross section area is $A = \pi(7.5mm)^2$. The density and Poisson ratio are respectively $\rho = 1420kg/m^3$ and $\nu = 0.01$. In both models, the rod was discretized into $n_s = 30$ spatial nodes for the finite element method and the time discretization used a time step of $\Delta t = 4.0\times10^{-5}$s. The total analysis time was 1 second. Some values of α and p were tested and the mean squared errors are shown in Tab.1 for two tensors $\boldsymbol{A} := \boldsymbol{A_1} = \tilde{\lambda}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1} + 2(\tilde{\mu} - \tilde{\lambda}\ln(J))\boldsymbol{I}$, where λ and μ are the Lamé constants; and $\boldsymbol{A} := \boldsymbol{A_2}$ where $\boldsymbol{A_2}$ is a fourth order tensor with $A(1,1,1,1) = p$ and $A(i,j,k,l) = 0$ for all $i,j,k,l \neq 1$. Figure 2 shows the displacement for different values of p in our formulation for the tensor $\boldsymbol{A}_1$. Table 1 shows that the bidimensional model proposed in this work is able the recover the results for the unidimensional model of Schmidt and Gaul (2006) for suitable choices of $\boldsymbol{A}$. Furthermore, the displacements shown in the Fig. 2 agree with similar results found in the literature (Atanacković et al., 2012).

α	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$
0.00794 ($P = 214.6e+6$)	5.9305e-12	3.2411e-13
0.2 ($p = 214.6e+6$)	2.1834e-13	9.0854e-14
0.5 ($p = 214.6e+4$)	4.0709e-12	3.1313e-12
0.5 ($p = 214.6e+6$)	1.0163e-13	8.2084e-14
0.7 ($p = 214.6e+4$)	2.1065e-13	1.5900e-13
0.9 ($p = 214.6e+4$)	1.1267e-13	1.0952e-13

Table 1 – Mean square error in the comparison with Schimidt and Gaul (2006).

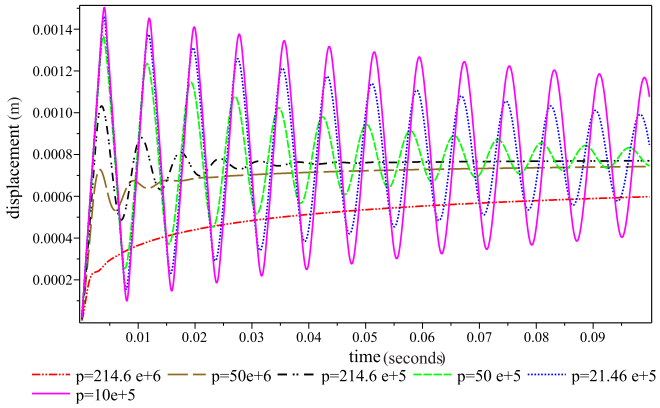

Figure 2 – Displacement for $\alpha = 0.5$ and different values of p.

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