

Stress intensity factors evaluation using an enriched dual boundary element method formulation

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Abstract: In this paper, an enriched formulation of the boundary element method is applied to directly evaluate the stress intensity factors (SIFs) of plane crack problems. A novel shifted enrichment crack tip function is used to preserve the meaning of physical displacements for the nodal parameters. Only two additional degrees of freedom, associated to the SIFs, are introduced per crack tip. To accommodate these new parameters, a crack tying constrain is used. Some numerical examples are presented to demonstrate the accuracy of the direct method. The results are compared with the answers obtained with the J-integral (indirect method) and with those available in the literature.

Keywords: *Stress intensity factors, linear-elastic fracture mechanics, enriched formulation, extended dual boundary element method*

INTRODUCTION

In the conventional numerical methods applied to fracture problems, the polynomial shape functions are used to represent the distribution of the mechanical fields near the crack tip. However, the behaviour of such fields is highly non-polynomial, leading to inaccuracies in the responses obtained in this region and, consequently, affecting the precision of the Stress Intensity Factors (SIFs). Several strategies have been proposed to overcome this numerical deficiency, among which the enriched formulations stand out. This approach consists of augmenting the conventional polynomial approximation with additional functions containing the behaviour from the solution space of the problem.

The enrichment formulations are well established in the Finite Element Method (FEM) framework, especially with the concept of the enrichment of the partition of unit (PU). Using the PU enrichment, Belytschko and Black (1999) obtained accurate responses for the elastic fields at the vicinity of the crack fields with coarse discretization and also modelled the displacement discontinuity along the crack surfaces without the need of remeshing. This numerical strategy is known as eXtended Finite Element Method (XFEM) and is widely applied to solve crack problems. One of the drawbacks of the XFEM is the ill conditioning of the system of equations that may occur because of the amount of Degrees of Freedom (DOF) introduced by the enrichment.

Recently, Simpson and Trevelyan (2011a) introduced the PU enrichment in the Boundary Element Method (BEM) framework to improve the mechanical responses near the crack tip. Because of the similarities with the XFEM, the method was named as the eXtended Boundary Element Method (XBEM). The supplementary equations to accommodate the introduced DOF were provided by extra collocation points. A drawback in the conditioning of the system with the increase of the DOF was also observed in this version of the XBEM. To reduce the amount of DOF and the system ill conditioning, Simpson and Trevelyan (2011b) adopted an enriched element approximation similar to that of Benzley (1974). This approach introduced only two additional DOF per crack tip for plane problems. An independent boundary integral equation derived from the linear-elastic fracture mechanics (LEFM) was used to solve for the extra parameters. Moreover, the SIFs were obtained indirectly from the J-integral technique, which is computationally costly in the BEM framework since the kernels related to the internal points defining the integration path must be evaluated.

To obtain the SIFs directly from the solution of the system of equations with the XBEM approach, Alatawi and Trevelyan (2015) proposed a crack tip tying constrain as additional equations to accommodate the extra DOF. This imposed condition aimed to enforce the displacement continuity at the crack tip, which is not guaranteed by the use of discontinuous elements in the ordinary BEM approach. Consequently, since the same condition observed in the displacement asymptotic fields of the LEFM is imposed, the additional parameters are the approximate values of the SIFs for the investigated problem.

In this paper, the XBEM is applied to evaluate the SIFs in two-dimensional problems involving isotropic materials. A new enrichment function, based on a shifted form of that proposed by Benzley (1974), is used to preserve the meaning of physical displacements for the nodal parameters. The crack tip tying constrain is used to accommodate the

additional DOF. The SIFs are computed directly from the system of equations with the XBEM and the results are compared with those obtained indirectly with the J-integral technique and with the responses available in the reference works.

LINEAR ELASTIC FRACTURE MECHANICS

Asymptotic expansions

Concerning homogeneous and isotropic materials, Williams (1957) determined the asymptotic expansion for the stress components near a crack tip for the Griffith problems. It was found that the stress components present a singularity of the order $\rho^{-0.5}$ and they are fully defined by the SIFs. For plane problems, the stress components referred to the local Cartesian coordinate system (Fig. 1) are given by the following expressions:

$$\sigma_{11} = \frac{1}{\sqrt{2\pi\rho}} \left\{ K_I \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] - K_{II} \sin\left(\frac{\theta}{2}\right) \left[2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] \right\} \quad (1)$$

$$\sigma_{22} = \frac{1}{\sqrt{2\pi\rho}} \left\{ K_I \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + K_{II} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right\} \quad (2)$$

$$\sigma_{12} = \frac{1}{\sqrt{2\pi\rho}} \left\{ K_I \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + K_{II} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \right\} \quad (3)$$

where K_I and K_{II} are, respectively, the mode-I and mode-II SIFs and ρ and θ are depicted in Fig. 1.

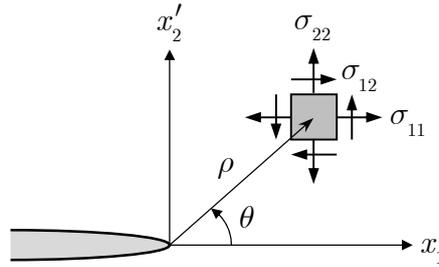


Figure 1 – Stress components near the crack tip.

The displacement components near the crack tip can also be related to the SIFs and are given as follows:

$$u_1 = \frac{1}{2\mu} \sqrt{\frac{\rho}{2\pi}} \left[K_I \cos\left(\frac{\theta}{2}\right) (\kappa - \cos\theta) + K_{II} \sin\left(\frac{\theta}{2}\right) (\kappa + 2 + \cos\theta) \right] \quad (4)$$

$$u_2 = \frac{1}{2\mu} \sqrt{\frac{\rho}{2\pi}} \left[K_I \sin\left(\frac{\theta}{2}\right) (\kappa - \cos\theta) - K_{II} \cos\left(\frac{\theta}{2}\right) (\kappa - 2 + \cos\theta) \right] \quad (5)$$

where $\mu = E/2(1 + \nu)$ is the shear modulus and κ is the Kolosov's constant defined as $(3 - \nu)/(1 + \nu)$ for plane stress and as $3 - 4\nu$ for plane strain. E and ν in these expressions are, respectively, the Young's modulus and the Poisson's ratio of the material.

Indirect evaluation of the SIFs – J-integral

As shown by the asymptotic expansions for the stress and displacements components, the SIFs represent the local behaviour of the elastic fields at the vicinity of the crack tip. However, for linear elastic materials, they can be related to the energy release rate of the body, which is a global parameter. For plane problems, the total energy release rate G is given by the superposition of the energy release rate of each mode of fracture, resulting in:

$$G = G_I + G_{II} \quad (6)$$

Irwin (1957) showed that the energy release rate of each mode is related to the corresponding SIFs as follows:

$$G_M = \frac{K_M^2}{E'} \quad (7)$$

where $M = I, II$ indicates the basic of fracture and $E' = E$ for plane stress and $E' = E/(1 - \nu^2)$ for plane strain.

By combining Eqs. (6) and (7), the following relation is obtained:

$$G = \frac{K_I^2 + K_{II}^2}{E'} \quad (8)$$

Rice (1968) showed that for linear-elastic materials the path-independent J-integral is equivalent to the energy release rate. This integral is evaluated along a path Γ_j enclosing the crack tip and is expressed by:

$$J = \int_{\Gamma_j} (Wn_1 - p_j u_{j,1}) d\Gamma \quad (9)$$

where W is the strain energy density given by $\sigma_{ij} u_{i,j}/2$, p_j are the tractions along the integration path given by $\sigma_{ij} n_i$, u_j are displacement components along Γ_j and n_i are the components of the unit outward normal vector to the path.

Because of the equality between J and G for linear-elastic materials, Eqs. (8) and (9) can be used to evaluate the SIFs for pure-mode fracture problems, i.e., for problems in which one of the SIFs is nil. However, for mixed-mode problems, a mode decoupling strategy must be applied first. In the work reported here, the M-integral technique (Chen and Shield, 1977) is used to perform the mode decomposition. This approach is based on the definition of a conservative integral for two equilibrium states of a linear-elastic body. By defining a state (0) obtained from the superposition of two equilibrium states, denoted as (1) and (2), the following relations between the mechanical fields can be written:

$$\sigma_{ij}^{(0)} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \quad (10)$$

$$u_i^{(0)} = u_i^{(1)} + u_i^{(2)} \quad (11)$$

$$K_M^{(0)} = K_M^{(1)} + K_M^{(2)} \quad (12)$$

The substitution of Eqs. (10) and (11) into the J-integral expression, Eq. (9), written for the problem (0) leads to:

$$J^{(0)} = J^{(1)} + J^{(2)} + M^{(1,2)} \quad (13)$$

in which $J^{(k)}$ corresponds to the J-integral expression assessed with the mechanical fields of problem (k). $M^{(1,2)}$ is defined as the M-integral that represents an interaction integral between the equilibrium states (1) and (2) and is given by:

$$M^{(1,2)} = \int_{\Gamma_j} \left[\frac{\sigma_{ij}^{(1)} u_{i,j}^{(2)} + \sigma_{ij}^{(2)} u_{i,j}^{(1)}}{2} n_1 - (\sigma_{ij}^{(1)} u_{i,1}^{(2)} + \sigma_{ij}^{(2)} u_{i,1}^{(1)}) n_i \right] d\Gamma \quad (14)$$

The J-integral of the state (0) can be related SIFs of states (1) and (2) by combining Eqs. (8) and (12), since $J = G$. The resulting expression is given by:

$$J^{(0)} = \frac{(K_I^{(1)})^2 + (K_{II}^{(1)})^2}{E'} + \frac{(K_I^{(2)})^2 + (K_{II}^{(2)})^2}{E'} + \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \quad (15)$$

Equation (15) can be organized as follows:

$$J^{(0)} = J^{(1)} + J^{(2)} + \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \quad (16)$$

where:

$$J^{(k)} = \frac{(K_I^{(k)})^2 + (K_{II}^{(k)})^2}{E'} \quad (17)$$

represents the $J - K$ relation for state (k) .

By comparing Eqs. (13) and (16), the M-integral can also be written in terms of the interaction between the SIFs of the states (1) and (2) as follows:

$$M^{(1,2)} = \frac{2}{E'} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) \quad (18)$$

Equation (14) together with Eq. (18) allow the determination of the SIFs values of a mixed-mode fracture problem when the problems (1) and (2) are properly chosen. For this purpose, the state (1) is taken as the analysed problem, for which the values of K_I and K_{II} are desired. The state (2) is chosen as an auxiliary solution, with known mechanical fields. The first auxiliary solution, denoted here by the superscript I , is taken as a cracked body subjected to a pure mode-I loading. Therefore:

$$K_I^{(I)} = 1 \quad \text{and} \quad K_{II}^{(I)} = 0 \quad (19)$$

By combining Eqs. (14) and (18), and by applying the conditions of Eq. (19), the value of K_I can be evaluated directly from:

$$K_I = \frac{E'}{2} \int_{\Gamma_j} \left[\frac{\sigma_{ij} u_{i,j}^{(I)} + \sigma_{ij}^{(I)} u_{i,j}}{2} n_1 - \left(\sigma_{ij} u_{i,1}^{(I)} + \sigma_{ij}^{(I)} u_{i,1} \right) n_i \right] d\Gamma \quad (20)$$

in which the fields without the superscript are related to the investigated problem and are obtained from the numerical analysis. $\sigma_{ij}^{(I)}$ and $u_i^{(I)}$ correspond to the asymptotic components of stress and displacement fields determined, respectively, from Eqs. (1)-(3) and Eqs. (4)-(5) after the conditions of Eq. (19) are imposed.

Similarly, the second auxiliary solution, denoted here by the superscript II , is chosen as the problem of a cracked body subjected to a pure mode-II loading. This case is represented by the following conditions:

$$K_I^{(II)} = 0 \quad \text{and} \quad K_{II}^{(II)} = 1 \quad (21)$$

For this situation, the value of K_{II} can be determined after Eqs. (14) and (18) are combined and the conditions of Eq. (21) are imposed:

$$K_{II} = \frac{E'}{2} \int_{\Gamma_j} \left[\frac{\sigma_{ij} u_{i,j}^{(II)} + \sigma_{ij}^{(II)} u_{i,j}}{2} n_1 - \left(\sigma_{ij} u_{i,1}^{(II)} + \sigma_{ij}^{(II)} u_{i,1} \right) n_i \right] d\Gamma \quad (22)$$

where $\sigma_{ij}^{(II)}$ and $u_i^{(II)}$ are the components of stress and displacement obtained, respectively, from Eqs. (1)-(3) and Eqs. (4)-(5) after the conditions of Eq. (21) are prescribed.

BOUNDARY ELEMENT METHOD

Dual boundary element formulation

In this study, the dual boundary element formulation is used to determine the mechanical response for the crack problems. In this numerical technique, two integral equations are applied to obtain the system of linear equations. The first, known as the Displacement Boundary Integral Equation (DBIE), is given by:

$$c_{ij}(s)u_j(s) + c_{ij}(\bar{s})u_j(\bar{s}) + \int_{\Gamma} P_{ij}^*(s,f)u_j(f)d\Gamma = \int_{\Gamma} U_{ij}^*(s,f)p_j(f)d\Gamma \quad (23)$$

where s denotes the source point and f represents the field point along the boundary Γ . U_{ij}^* and P_{ij}^* are, respectively, the displacement and traction fundamental solutions, which are functions of the distance r between the points s and f . u_j and p_j are the displacement and traction components. c_{ij} is the free term, which is equal to $\delta_{ij}/2$ if s is placed at smooth boundary, with δ_{ij} representing the Kronecker delta. The point \bar{s} refers to a potential source point at the same position of s but placed at a different surface. This condition occurs for corresponding source points at opposite crack surfaces. The term relative to \bar{s} is nil on Eq. (23) if s does not have a corresponding point.

The second integral equation used in the dual formulation is known as the Traction Boundary Integral Equation (TBIE). Assuming s positioned at a smooth boundary, the TBIE is expressed as follows:

$$\frac{1}{2}[p_j(s) - p_j(\bar{s})] + n_k(s) \int_{\Gamma} S_{ijk}^*(s, f) u_i(f) d\Gamma = n_k(s) \int_{\Gamma} D_{ijk}^*(s, f) p_i(f) d\Gamma \quad (24)$$

where n_k are the components of the unit outward normal vector at the source point and D_{ijk}^* and S_{ijk}^* are fundamental solutions obtained from U_{ij}^* and P_{ij}^* derivatives, respectively.

With Eqs. (23) and (24), the system of equations provided by the dual BEM can be assembled by applying the collocation method. In this process, the boundary Γ is subdivided into isoparametric elements, in which high-order polynomials are available for approximating both the geometry and the mechanical fields. The DBIE is used for collocation on nodes placed at the external boundary and at the upper crack surface, whereas the TBIE is used for collocation on nodes positioned at the lower crack surface. Thus, these nodes become the source point s of their respective boundary integral equation. It is worth mentioning that the TBIE requires the continuity of the displacement derivatives at the collocation points, which is guaranteed with discontinuous elements. In such elements, the collocation points do not coincide with the end nodes but are positioned inside the element. Therefore, this type of element is used along the crack surfaces.

After the subdivision of the boundary into elements, the discretized form of the DBIE is expressed as follows:

$$c_{ij}(s)u_j(s) + c_{ij}(\bar{s})u_j(\bar{s}) + \sum_{e=1}^{N_e} \sum_{m=1}^n P_{ij}^{em} u_j^{em} = \sum_{e=1}^{N_e} \sum_{m=1}^n U_{ij}^{em} p_j^{em} \quad (25)$$

where N_e is the number of elements into the boundary mesh, n is the number of nodes of the element and:

$$P_{ij}^{em} = \int_{-1}^{+1} P_{ij}^*(s, f(\xi)) \phi^{em}(\xi) J^e(\xi) d\xi \quad (26)$$

$$U_{ij}^{em} = \int_{-1}^{+1} U_{ij}^*(s, f(\xi)) \phi^{em}(\xi) J^e(\xi) d\xi \quad (27)$$

in which ϕ^{em} is the shape function associated to the m -th node of element e and J^e is the Jacobian of transformation from the local coordinate system ξ to the global coordinate system $x_1 x_2$.

Analogously, the discretized version of the TBIE is given by:

$$\frac{1}{2}[p_j(s) - p_j(\bar{s})] + n_k(s) \sum_{e=1}^{N_e} \sum_{m=1}^n S_{ijk}^{em} u_i^{em} = n_k(s) \sum_{e=1}^{N_e} \sum_{m=1}^n D_{ijk}^{em} p_i^{em} \quad (28)$$

where:

$$S_{ijk}^{em} = \int_{-1}^{+1} S_{ijk}^*(s, f(\xi)) \phi^{em}(\xi) J^e(\xi) d\xi \quad (29)$$

$$D_{ijk}^{em} = \int_{-1}^{+1} D_{ijk}^*(s, f(\xi)) \phi^{em}(\xi) J^e(\xi) d\xi \quad (30)$$

After the collocation process using Eqs. (25) and (28), the resulting system of equations can be expressed in the matrix notation as follows:

$$\mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{p} \quad (31)$$

where \mathbf{H} is a $2N \times 2N$ matrix with the influence coefficients obtained from the free term of the DBIE and from the kernels containing P_{ij}^* (Eq. (26)) and S_{ijk}^* (Eq. (29)). \mathbf{G} is a $2N \times 2N$ matrix with the influence coefficients obtained from the free term of the TBIE and from the kernels U_{ij}^* (Eq. (27)) and D_{ijk}^* (Eq. (30)). \mathbf{u} and \mathbf{p} are $2N$ vectors with, respectively, the displacement and traction components of the boundary discretization and N is the number of collocation points into the boundary mesh. A solution for the mechanical problem is obtained from Eq. (31) after the known boundary conditions are imposed.

Extended boundary element method

In the XBEM approach used here, the displacement approximation over a crack elements e enriched by a tip λ is augmented with functions related to the asymptotic fields of the LEFM. This enriched approximation is given as follows:

$$u_j^e(\xi) = \sum_{m=1}^n u_j^{em} \phi^{em}(\xi) + \tilde{K}_I^\lambda R_{jk}^\lambda \sum_{m=1}^n \phi^{em}(\xi) \left(\psi_{Ik}^\lambda(\xi) - \psi_{Ik}^\lambda(\xi_m) \right) + \tilde{K}_{II}^\lambda R_{jk}^\lambda \sum_{m=1}^n \phi^{em}(\xi) \left(\psi_{IIk}^\lambda(\xi) - \psi_{IIk}^\lambda(\xi_m) \right) \quad (32)$$

The first term of the right-hand side of Eq. (32) is the ordinary polynomial approximation that allows the displacement approximation to capture the rigid body motion. It is composed by summation of the nodal parameters u_j^{em} multiplied by the polynomial shape functions ϕ^{em} assessed at the non-dimensional coordinate ξ . The summation is carried out over the number of nodes n of the element e . The second and the third terms on the right-hand side of Eq. (32), associated to the two additional parameters \tilde{K}_I^λ and \tilde{K}_{II}^λ , are responsible for representing the LEFM asymptotic behaviour for the displacements (Eqs. (4)-(5)), improving the accuracy of the mechanical response at the vicinity of the tip. These enrichment terms are composed by the polynomial shape function ϕ^{em} evaluated at the non-dimensional coordinate ξ multiplied by the difference between the enrichment function at the coordinate ξ and the enrichment function assessed at the m -th node. This shifted enrichment is zero at the nodes of the enriched elements and, consequently, preserves the meaning of physical displacement for the nodal parameters u_j^{em} . The enrichment functions ψ_{Mk} , with $M = I, II$ indicating the mode of fracture, are obtained from Eqs. (4)-(5) and are expressed as follows:

$$\begin{bmatrix} \psi_{I1} & \psi_{II1} \\ \psi_{I2} & \psi_{II2} \end{bmatrix} = \frac{1}{2\mu} \sqrt{\frac{\rho}{2\pi}} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right)(\kappa - \cos\theta) & \sin\left(\frac{\theta}{2}\right)(\kappa + 2 + \cos\theta) \\ \sin\left(\frac{\theta}{2}\right)(\kappa - \cos\theta) & -\cos\left(\frac{\theta}{2}\right)(\kappa - 2 + \cos\theta) \end{bmatrix} \quad (33)$$

Since the enrichment functions are related to crack tip local coordinate system, the rotation matrix defined by the components R_{jk} is introduced in Eq. (32) for a transformation from the local to the global coordinate system. The rotation matrix is given by:

$$\mathbf{R} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \quad (34)$$

where ω is the angle between the local and global coordinates systems.

Because of the enrichment terms of the displacement approximation shown by Eq. (32), the discretized form of the DBIE is expanded as follows:

$$c_{ij}(s)u_j(s) + c_{ij}(\bar{s})u_j(\bar{s}) + \sum_{e=1}^{N_e} \sum_{m=1}^n P_{ij}^{em} u_j^{em} + \tilde{K}_I^\lambda \sum_{e=1}^{N_e} \bar{P}_{iI}^{e\lambda} + \tilde{K}_{II}^\lambda \sum_{e=1}^{N_e} \bar{P}_{iII}^{e\lambda} = \sum_{e=1}^{N_e} \sum_{m=1}^n U_{ij}^{em} p_j^{em} \quad (35)$$

where N_e^λ is the number of elements enriched by tip λ and:

$$\bar{P}_{iM}^{e\lambda} = \int_{-1}^{+1} P_{ij}^*(s, f(\xi)) R_{jk}^\lambda J^e(\xi) \sum_{m=1}^n \left[\phi^{em}(\xi) \left(\psi_{Mk}^\lambda(\xi) - \psi_{Mk}^\lambda(\xi_m) \right) \right] d\xi \quad (36)$$

in which $M = I, II$ indicates the mode of fracture.

Similarly, the discretized TBIE considering the enrichment terms is given by:

$$\frac{1}{2} [p_j(s) - p_j(\bar{s})] + n_k(s) \left[\sum_{e=1}^{N_e} \sum_{m=1}^n S_{ijk}^{em} u_i^{em} + \tilde{K}_I^\lambda \sum_{e=1}^{N_e} \bar{S}_{jkI}^{e\lambda} + \tilde{K}_{II}^\lambda \sum_{e=1}^{N_e} \bar{S}_{jkII}^{e\lambda} \right] = n_k(s) \sum_{e=1}^{N_e} \sum_{m=1}^n D_{ijk}^{em} p_i^{em} \quad (37)$$

where:

$$\bar{S}_{jkM}^{e\lambda} = \int_{-1}^{+1} S_{ijk}^*(s, f(\xi)) R_{ip}^\lambda J^e(\xi) \sum_{m=1}^n \left[\phi^{em}(\xi) \left(\psi_{Mp}^\lambda(\xi) - \psi_{Mp}^\lambda(\xi_m) \right) \right] d\xi \quad (38)$$

In this work, the crack tip tying constraint proposed by Alatawi and Trevelyan (2015) is used to accommodate the additional parameters introduced by the crack tip enrichment and, consequently, to recover a square system of equations. This condition imposes the displacement continuity at the crack tip, which is not guaranteed with the use of discontinuous elements for the crack discretization. For this purpose, a set of n_c collocation points at the upper and at the lower crack surfaces are selected to define two auxiliary elements used to extrapolate the displacements to the tip. By imposing the equality between these displacements, the following expression is obtained:

$$\sum_{m=1}^{n_c} u_j^{mU} \phi^{mU} (+1) - \sum_{m=1}^{n_c} u_j^{mL} \phi^{mL} (-1) + \tilde{K}_I^\lambda R_{jk}^\lambda \left(\sum_{m=1}^{n_c} (\phi^{mL} (-1) - \phi^{mU} (+1)) \psi_{Ik}^\lambda(\xi_m) \right) + \tilde{K}_{II}^\lambda R_{jk}^\lambda \left(\sum_{m=1}^{n_c} (\phi^{mL} (-1) - \phi^{mU} (+1)) \psi_{IIk}^\lambda(\xi_m) \right) = 0 \quad (39)$$

where U and L indicate the extrapolation over the upper and lower crack surfaces, respectively.

With the introduction of the enriched approximation and the crack tip tying constraint, the resulting system of equations has the following form:

$$\begin{bmatrix} \mathbf{H} & \mathbf{H}_C \\ & \mathbf{H}_R \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \tilde{\mathbf{K}} \end{Bmatrix} = \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix} \{\mathbf{p}\} \quad (40)$$

By comparing Eq. (40) with Eq. (31), it can be noted that the enrichment introduces additional columns \mathbf{H}_C in the system which are associated to the tip's parameters. Moreover, the crack tip tying constraint given by Eq. (39) is responsible for introducing new rows \mathbf{H}_R to recover a square system of equations. After the solution of Eq. (40), a mechanical response for the body is obtained as well as the additional parameters $\tilde{\mathbf{K}}$. Since the additional equations impose the displacement continuity at the tip, which is the same condition observed in the LEFM expressions, the additional parameters $\tilde{\mathbf{K}}$ represent a good approximation for the SIFs. Thus, the XBEM is able to evaluate the SIFs directly from the solution of the system of equations.

NUMERICAL RESULTS

Example 1:

In this example, the convergence of the enriched BEM formulation is studied, as well as the accuracy of the method for defining the SIFs for crack problems. To perform the analyses, a square plate with an edge crack submitted to a uniform loading σ is considered, as illustrated by Fig. 2. The length of the crack is $a = 0.5w$. Different meshes composed of quadratic elements are used and the SIFs are defined using the unenriched J-integral, the XBEM and the enriched J-integral. In this latter method, the mechanical fields are evaluated considering the enrichment of crack elements and the J-integral is applied to compute the SIFs in a post-processing phase. For the enriched analyses, the elements at the right-half of the crack are enriched by the tip, as indicated in Fig. 2. Moreover, the crack tip tying constraint is defined over three elements.

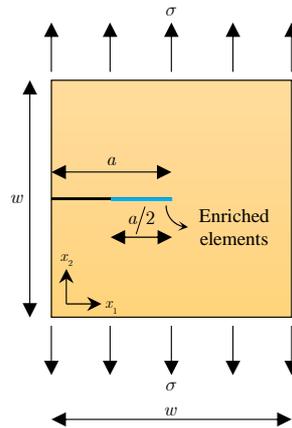


Figure 2 – Square plate containing an edge crack.

Figure 3 presents the convergence of K_I (normalized by $\sigma\sqrt{\pi a}$) with the number of nodes into the boundary element mesh. The results obtained using the unenriched J-integral, the XBEM and the enriched J-integral are compared with the reference value provided by Civelek and Erdogan (1982), which is also shown in Fig. 3. For all methods applied in this work, the mode-I SIF converges to a value near to the reference solution with the mesh refinement. In this example, the XBEM better approximates the reference value when compared with the unenriched J-integral. Moreover, the enrichment is able to improve the responses of the indirect method, with the enriched J-integral giving the results with the lowest relative error, as shown by Fig. 4.

Figure 4 also shows that the errors committed by the XBEM and the enriched J-integral varies around 0.3% and 0.1%, respectively, whereas the errors obtained with unenriched J-integral have a wider range of variation, from 0.3%

to 1.4%. This demonstrate that the enriched formulation, in addition to improving the accuracy, is also capable of improving the rate of convergence for the SIF value.

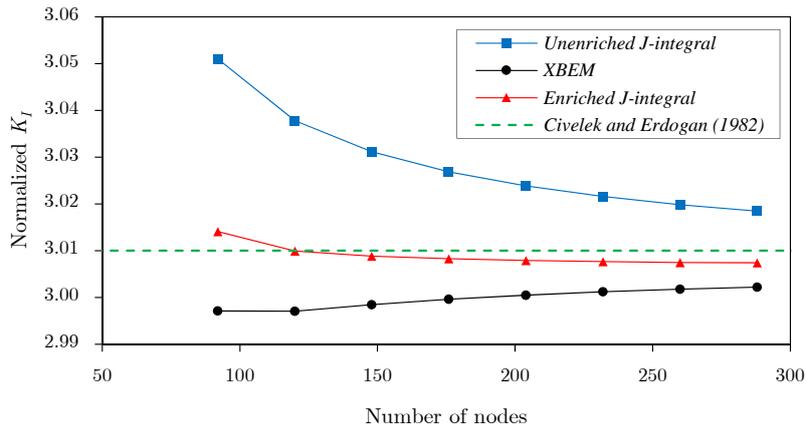


Figure 3 – Variation of the normalized mode-I SIF with the number of nodes into the boundary element mesh.

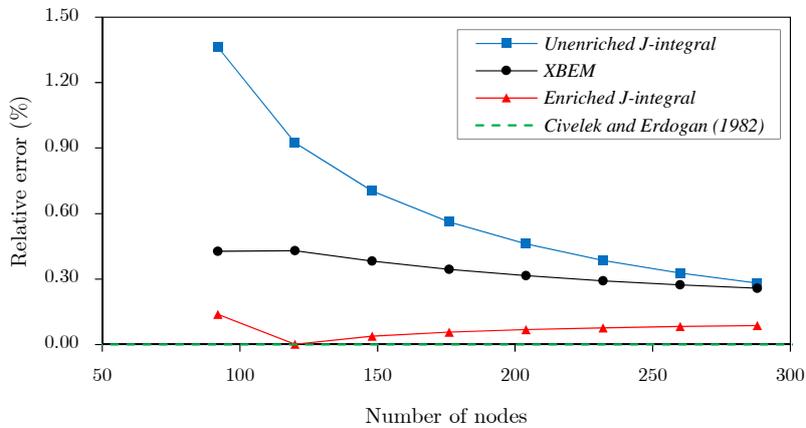


Figure 4 – Variation of the relative error of the mode-I SIF with the number of nodes into the boundary element mesh.

Example 2:

Figure 5 shows an isotropic plate containing a central slanted crack submitted to a uniform loading. The dimensions of the structure are such that $h/w = 2$ and the crack length is defined as $2a$. The slope of the crack is equal to $\theta = 45^\circ$. In this example, different values for the crack length are considered and the SIFs are evaluated for each configuration directly with the XBEM and indirectly with the unenriched and enriched versions of the J-integral. To perform the numerical analyses, 36 quadratic elements are used to discretize the external boundary and discontinuous elements with the same order of approximation and length equal to $0.1w$ are applied at the crack surfaces.

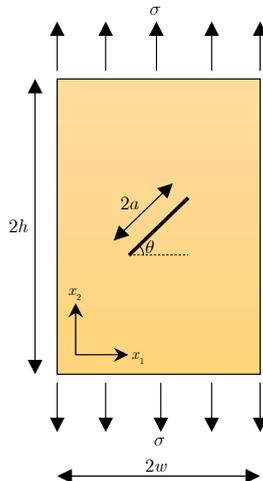


Figure 5 – Isotropic plate containing a central slanted crack.

Figure 6 shows the results obtained for K_I and K_{II} with the three methods used in this work for different crack lengths. The SIFs are normalized by $\sigma\sqrt{\pi a}$. The reference solutions provided by Murakami (1987) are also depicted in Fig. 6. A good agreement is observed among the results determined here and the reference responses. The relative error in the solutions obtained with the XBEM is in order of 0.5% for both SIFs when compared with the answers given by Murakami (1987), which demonstrates the accuracy of the direct approach for analysing mixed-mode problems. In addition, the enriched formulation is also very effective when coupled with the J-integral. In this case, the relative errors are around 0.05% for K_I and 0.2% for K_{II} .

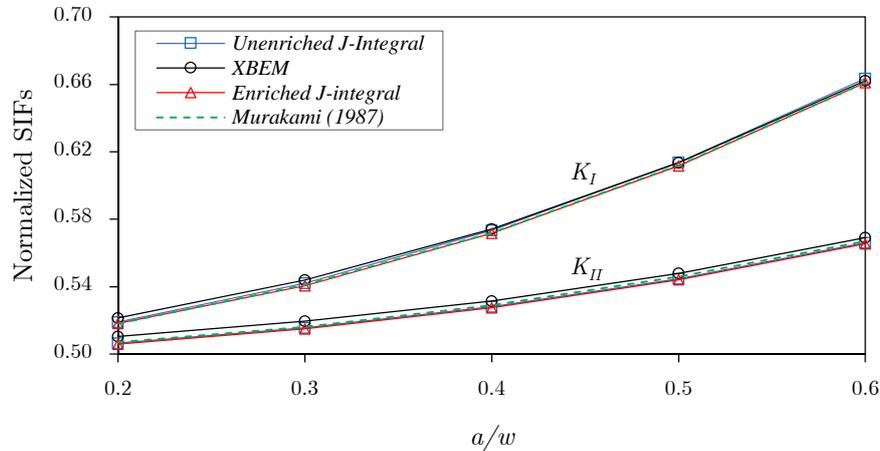


Figure 6 – Normalized mode-I and mode-II SIFs for the slanted crack in an isotropic plate.

CONCLUDING REMARKS

This paper presented an enriched formulation of the DBEM to analyse fracture problems regarding the LEFM. A shifted crack tip function was first used as the enrichment terms for the displacement approximation along the boundary elements. This strategy preserves the physical meaning of displacements for the nodal parameters and, consequently, no post-processing is needed to obtain the real nodal displacements. Besides, this form of enrichment does not affect the contribution of the free term into the \mathbf{H} matrix when the DBIE is applied.

The enriched formulation adds only two DOF per crack tip, independently of the number of enriched elements. These additional parameters correspond to the SIFs after the crack tip tying constrain is imposed to recover a square system of linear equations. Thus, the SIFs can be evaluated directly from solution of the system of equations, avoiding the use of post-processing technique such as the J-integral, which is time consuming in the BEM framework.

The applications demonstrated that the XBEM is able to provide accurate solutions for the SIFs, including for mixed-mode problems. Moreover, the enriched formulation was capable of improving accuracy and the rate of convergence for the SIFs obtained indirectly with the J-integral approach.

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