Stability Investigation of GFEM and SGFEM Applying the inf-sup Test

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Abstract: The stability issue of the numerical solutions of two-dimensional elasticity problems analyzed through the Generalized/Stable Finite Element Method (GFEM/SGFEM) is hereby addressed. An Inf-Sup Test for the GFEM/SGFEM is proposed aiming the numerical evaluation of the Babuška-Brezzi Condition. Both the GFEM and the SGFEM preserve the conventional structure of the Finite Element Method (FEM). In addition, the resource of the so called nodal enrichment available by those methods conceptually enlarges the approximation bases of the conventional FEM, without the need to introduce new nodal points in the domain. In this work, polynomial functions are exclusively explored for enrichment. The essential difference between SGFEM and GFEM is precisely the previous modification of the enrichment function aiming to improve the conditioning of the solving system. However, even in the SGFEM numerical stability problems can be induced when polynomial enrichment is arbitrarily explored. Therefore, focusing the goal of efficiently exploiting nodal enrichment possibilities, the present investigation on the solvability and convergence of the GFEM/SGFEM numerical responses for plane problems was conceived. Two reference problems in linear elasticity are selected for conducting the investigation, being both discretized by several meshes of four node quadrilateral elements and three node linear triangle elements. In each problem, the issue of solvability is treated through the computation of the condition number of the stiffness matrix. The issue of convergence is then addressed through a numerical evaluation of the Babuška-Brezzi Condition - Inf-Sup Test. The results obtained from the numerical analyzes done have validated the numerical inf-sup test as a consistent tool for investigation of the stability aspects of the numerical solutions, as well as for pointing out good combinations of polynomial functions to be used for nodal enrichment in the GFEM/SGFEM.

Keywords: Generalized Finite Element Method, Stable Generalized Finite Element Method, Babuška-Brezzi condition, Inf-Sup Test, Condition Number

INTRODUCTION

The Generalized Finite Element Method (GFEM) and the Stable Generalized Finite Element Method (SGFEM) are unconventional Finite Element Method (FEM) formulations that are obtained basically exploring good features of both the conventional form of the FEM (finite element mesh and nodal shape functions) and Meshless Methods (nodal enrichment), Duarte, Babuška and Oden (2000) and Babuška and Banerjee (2012).

Essentially, the nodal enrichment technique allows augmenting the FEM approximation space without increasing the node number of a given finite element. However, linear dependences are the main drawback presented by the technique especially when polynomial nodal enrichment is explored. In particular, linear dependences strongly contribute to the GFEM/SGFEM stiffness matrix ill conditioning, therefore affecting stability and convergence of the results. It is noteworthy that there is a conceptual difference between the nodal enrichment methodologies applied to GFEM and SGFEM, Gupta et al. (2013) and Lins (2015). Actually, the SGFEM nodal enrichment strategy was precisely conceived for reducing the influence of the nodal enrichment process on the stiffness matrix ill conditioning.

On the other hand, Babuška-Brezzi condition (or inf-sup condition), independently proposed by Babuška (1971, 1973) and Brezzi (1974), is a sufficient condition that can be verified numerically for controlling stability and convergence of numerical approximations obtained with the FEM. Such condition encompasses a whole class of problems variationally expressed through bilinear and linear forms, such as the linear elasticity problems.

In this work, the numerical evaluation of the so called Babuška-Brezzi inf-sup condition applied to GFEM/SGFEM is addressed. The adopted approach is based on the procedures suggested in Babuška (1996), Chapelle and Bathe (1993). The numerical evaluation of the inf-sup test aims to investigate mainly the polynomial nodal enrichment strategies that can be explored in both GFEM and SGFEM. For further details about the GFEM/SGFEM methodological structure to analyze linear elasticity problems, see Gupta et al. (2013).

On what follows, the Babuška-Brezzi Condition (inf-sup) applied to GFEM and SGFEM is described. Next, we highlight the numerical simulations performed. Finally, the main conclusions are summarized.

BB INF-SUP CONDITION APPLIED TO GFEM / SGFEM

On the numerical evaluation of factor λ

The Babuška-Brezzi Condition (inf-sup) condition is a necessary and sufficient condition for convergence of FEM numerical approximations. It is valid for a whole class of problems characterized by certain linear operators (bilinear forms). However, depending on the FEM adopted approach the analytical evaluation of such condition is not an easy task. Therefore, only a numerical estimation of it remains as a feasible alternative. A numerical verification of the inf-sup condition addressed to a hybrid-mixed approach for the GFEM was originally presented in Góis (2009) and Góis and Proença (2013). The contribution hereby given extends such an investigation for encompassing both the displacement approaches of GFEM and SGFEM.

Let be considered a domain Ω of a plane linear elasticity problem discretized through a finite element mesh. A set of polynomial functions is adopted for GFEM/SGFEM nodal enrichments. The Babuška-Brezzi (inf-sup) condition is verified is one can find a scalar λ_n such that:

$$\lambda_n = \inf_u \sup_{\delta u} \frac{\mathbf{u}^T \mathbf{K} \delta \mathbf{u}}{\left(\mathbf{u}^T \mathbf{X} \mathbf{u}\right)^{\frac{1}{2}} \left(\delta \mathbf{u}^T \mathbf{Y} \delta \mathbf{u}\right)^{\frac{1}{2}}} \geq \lambda > 0 \tag{1}$$

where $\mathbf{K}$ is the GFEM/SGFEM stiffness matrix. In Eq. (1), $\mathbf{X}$ and $\mathbf{Y}$ are symmetric positive-definite matrices obtained from norms, respectively through the real and virtual displacement vectors $\mathbf{u}$ $\left(\|\mathbf{u}\|^2 = \left(\mathbf{u}^T \mathbf{X} \mathbf{u}\right) = \int_\Omega \mathbf{u}^2 d\Omega\right)$ and $\delta \mathbf{u}$ $\left(\|\delta \mathbf{u}\|^2 = \left(\delta \mathbf{u}^T \mathbf{Y} \delta \mathbf{u}\right) = \int_\Omega \delta \mathbf{u}^2 d\Omega\right)$. Moreover, λ is the reference stability constant and λ_n is the stability constant estimative in correspondence to the adopted discretization.

Still, in Babuška (1996), it is proved that λ_n^2 is the lowest eigenvalue of the generalized eigenvalue problem:

$$\mathbf{K}^T \mathbf{Y}^{-1} \mathbf{K} \mathbf{u} = \mu \mathbf{X} \mathbf{u} \tag{2}$$

where μ are an eigenvalues. Thus, for a given formulation, physical constants and finite dimensional spaces, the value of λ_n is equal to $\sqrt{\mu_{\min}}$.

Regarding to the $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{K}$ determination, for each enrichment condition of the GFEM/SGFEM, it is only necessary to expand the initial displacements approximate bases in the GFEM/SGFEM methodology.

It is important to point out that the application of the inf-sup condition to investigate stability of a finite element approximation is an almost impractical task, since it would require its evaluation considering a great number of varied examples belonging to a given class of problems. Nevertheless, in this work, aiming to perform numerically an "inf-sup test" indicative of stability of the GFEM/SGFEM solutions, we choose, instead, two distinct problems as representative of a large number of the linear elasticity class.

To perform the numerical analysis hereby described, a methodology similar to the one suggested in Chapelle and Bathe (1993) was adapted to the problems discretized by the GFEM/SGFEM. In short, a sequence of meshes of quadrilateral and triangular elements whose classical FEM shape functions were enriched by polynomial functions was considered. For each mesh, the value of $\lambda_n = \sqrt{\mu_{\min}}$ was computed. In each problem, an indicative of stability of the approximations provided by the quadrilateral and triangular elements of GFEM/SGFEM was obtained if the sequence of $\sqrt{\mu_{\min}}$ do not tend to zero in correspondence to the number of degrees of freedom increasing. In parallel to the determination of the $\sqrt{\mu_{\min}}$, the condition number of the matrix $\mathbf{K}$ was also evaluated for the sequence of meshes, aiming to sound the solvability aspect of the GFEM/SGFEM discrete systems.

The stability in both problems tested, once verified, can be interpreted as a good indication of the robustness of the element approximations.

NUMERICAL RESULTS

This item presents the numerical results related to the "inf-sup test" application to assess the GFEM/SGFEM 2-D elements stability. In addition to the stability aspects, some results regarding to stress fields representation and to the condition number of the stiffness matrix **K** are presented.

As mentioned before, the "inf-sup" test can be very time consuming due to the requirement of a great number of problems in a class, as well variations on the boundary conditions and sequence of meshes used in the discretization. However, it is hereby assumed that even considering a reduced number of representative problems, the inf-sup test may provide stability indications of the adopted approximations of the GFEM/SGFEM method.

Thus, the "inf-sup test" methodology is applied considering two plane problems with very different mechanical behavior and conditions. The first example (a - tension plate) has an expected smooth response, while the second one (b – cracked plate) has a more inhomogeneous response due to the existence of a geometrical discontinuity ('crack'), see Fig.1.
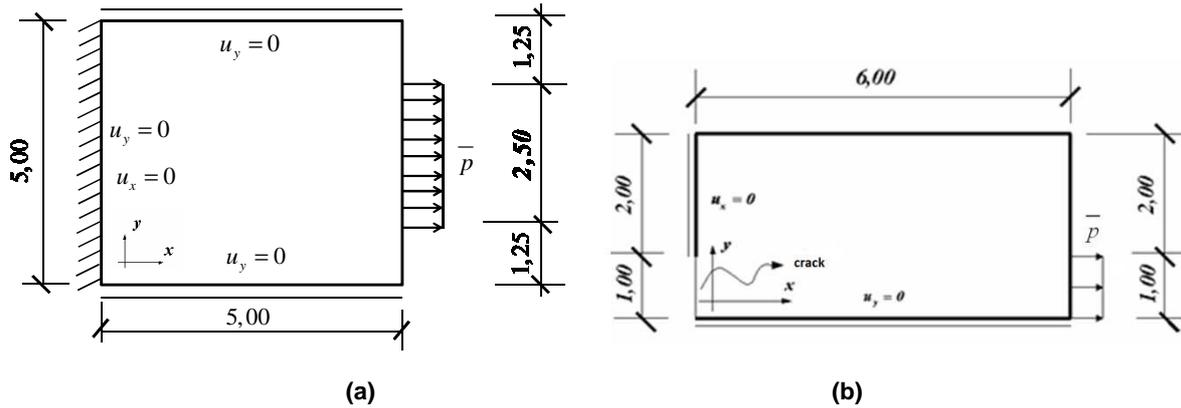

(a)   **(b)**

Figure 1- (a) Tension plate (b) double symmetry of the central cracked plate

For simplicity, units for length and strength were not adopted. In addition, the Young's modulus and the Poisson's coefficient were respectively adopted the following values: $E = 1000$ and $\nu = 0,30$.

The problem (a) in the Fig. 1 is a square plate of 5 units side length, and tensioned by a partially uniform distributed force, $\bar{p} = 10$, along the right vertical border. The left vertical side of the plate is fixed ($u_x = u_y = 0$) and the horizontal borders have restricted vertical displacements ($u_y = 0$).

The problem (b) is a rectangular plate with central crack aligned with the direction y. Since the plate has double symmetry (of geometry and applied forces), only a quarter of the plate is considered in the analysis. In this example, a partially uniform distributed force $\bar{p} = 10$ applied on the right vertical side, see Fig. 1 – (b).
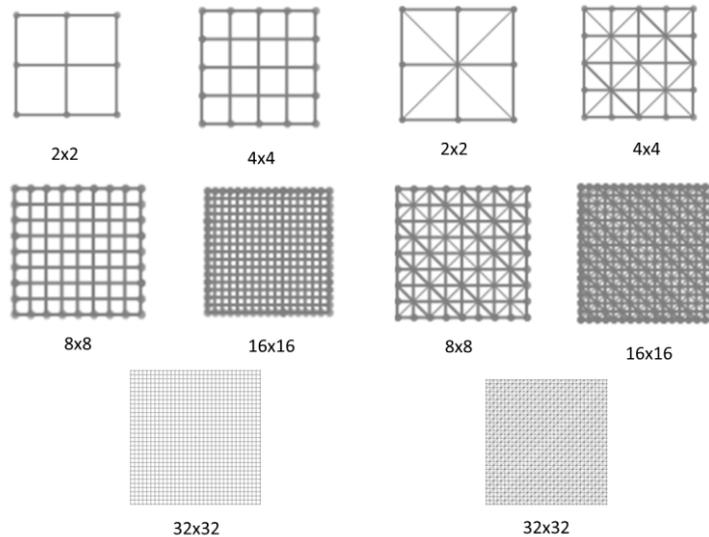

Figure 2 - Sequence of regular meshes - quadrilateral and triangular elements - problem (a)

Chapelle and Bathe (1993) suggest that for a problem class a numerical "inf-sup" test can be evaluated using a sequence of meshes. At least three refinement levels are recommended to predict whether λ_n it will be lower limited by a positive constant, therefore indicating test verification. Following these recommendations for the "inf-sup" test application to the GFEM/SGFEM, the quadrilateral four-node and triangular three-node finite element sequences of meshes shown in Fig. 2 and Fig. 3 were considered for the problems (a) and (b), respectively. To the problem (a), the mesh sequence is denoted as (2x2, 4x4, 8x8, 16x16 and 32x32) according to the number of elements in the border directions. Similarly, for the problem (b), cracked plate, the mesh sequence is refered as (3x3, 6x6, 12x12, 24x24 and 48x48).

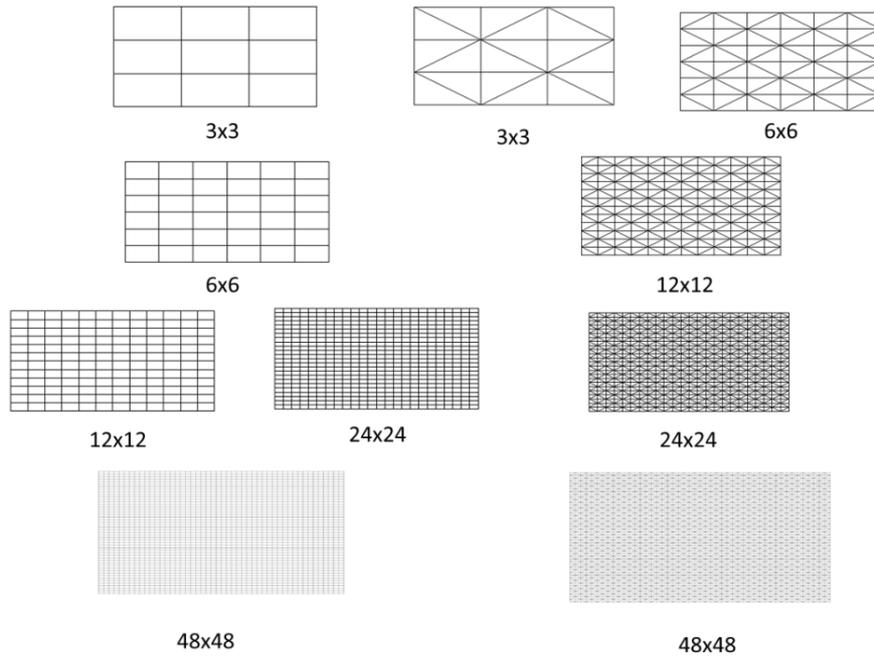As already mentioned, polynomial functions were adopted to enrich the displacement field.

Figure 3 - Regular mesh sequence - quadrilateral and triangular elements - problem (b)

After applying the essential boundary conditions the matrices $\mathbf{X}, \mathbf{Y}$ and $\mathbf{K}$ were assembled and the value of λ_n (inf-sup) computed for each one of the regular meshes shown in Fig. 2 and Fig. 3. The results obtained are hereby represented trough $\log\left(1/N\right) \times \log(\lambda_n)$ plots (where N is for the number of degrees of freedom) Once the curve $\log\left(1/N\right) \times \log(\lambda_n)$ converges asymptotically to a certain value $\lambda_n > 0$, in both problems, it is concluded that the GFEM/SGFEM quadrilateral and triangular elements satisfy the "inf-sup" test, therefore appointing to the solution stability.

With respect to the enrichment possibilities, the polynomial combinations shown in Tab. 1 were adopted. It is emphasized that for the task of brevity the whole set of numerical tests performed will not be detailed.

Table 01 – Polynomial enrichment approximation functions until second degree

Monomials	Polynomial Enrichment GFEM	Polynomial Enrichment SGFEM
x		
y		
x^2	$x+y+xy+x^2+y^2$	x^2+y^2
y^2		
xy		

The numerical simulations were runned in a Fortran language toolkit. Conventional gaussian quadrature rules (Hammer to the triangular elements) were adopted for the necessary numerical integrations that define $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{K}$ matrices. The perturbation procedure (error tolerance of 10^{-15} and perturbation of 10^{-8}), suggested by Strouboulis, Babuška and Copps (2000) was also adopted for the solution of the scaled GFEM/SGFEM system of equations.

Finally, according to Szabo and Babuška (1991), in the case of FEM if an asymptotic value $\lambda_n > 0$ is found, then it is guaranteed that the condition number of matrix $\mathbf{K}$ will not become very high with the mesh refinement. This confirms the Babuška-Brezzi (inf-sup) condition as necessary and sufficient for convergence of the numerical solution by FEM. In this work, the validity of such an statement for the GFEM/SGFEM was also checked. The condition number of scaled matrix $\mathbf{K}$ was computed using a Fortran's library subroutine.

Tensioned panel - problem (a)

The results of the GFEM/SGFEM "inf-sup test" for the problem (a) are highlighted in Fig. 4. A complete polynomial of second degree (x+y+x^2+y^2+xy) was adopted for GFEM enrichment and applied to the whole set of nodes in the meshes (excluding nodes with essential boundary conditions imposed). However, for the SGFEM the monomials x and y are useless, once the method modifies the enrichment functions by discounting the linear interpolation of its nodal values in each cloud (set of elements sharing a common node).

From Fig. 4, one can be see that regardless the element used it was possible to compute a clear asymptotic positive value $\left(\lambda_n > 0 \right)$ considering the GFEM/SGFEM results, therefore appointing to the stability of the approximations tested.
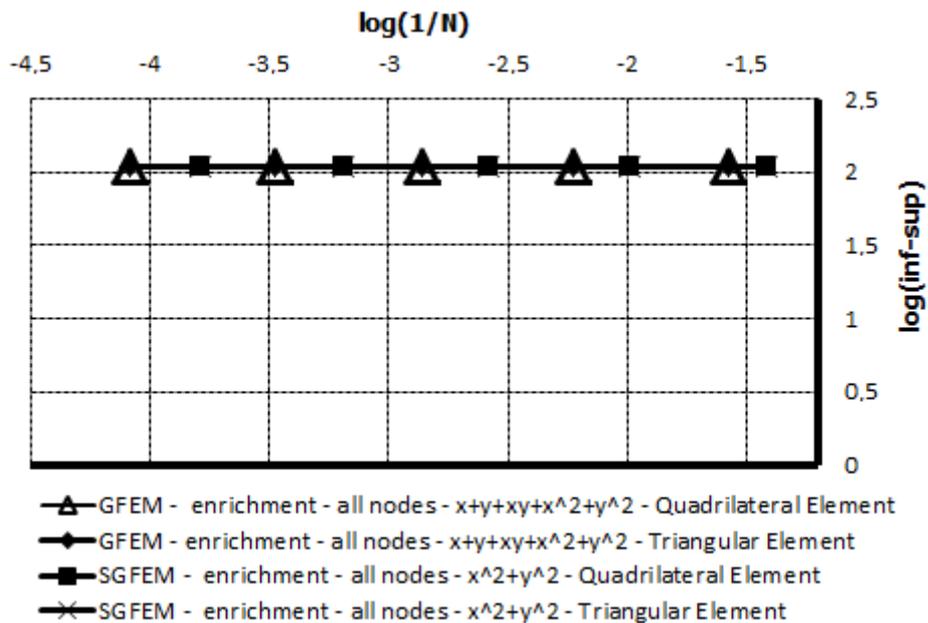

Figure 4 - Problem (a) - results of the "inf-sup test" - GFEM/SGFEM

In Fig. 5, for instance, it is shown that the enrichment of the SGFEM quadrilateral element trough the x^2+y^2 $\left(\log\left(1/N \right) = -2.59 \right)$ function provide excellent stress fields $\left(\sigma_x \right)$ distributions for the problem (a). This kind of answer is then expected once the "inf-sup" test has delivered an asymptotic value $\lambda_n > 0$ to such approximation of the SGFEM quadrilateral element .

As predictable, the enrichment coupled with mesh refinement has induced an exponential growing in the condition number of the GFEM matrix $\mathbf{K}$, (see Fig. 6). On the other hand the SGFEM nodal enrichment proposal, as already mentioned, was conceived precisely to provide a matrix $\mathbf{K}$ conditioning similar to the FEM. Note that in Fig. 6 the condition number of the matrix $\mathbf{K}$ obtained by SGFEM was much smaller than that of GFEM and about the same order of the FEM. It is worth of mention that once conditioning is a proprety related to the solvability of the linear system, an improvement towards a good conditioning of the matrix $\mathbf{K}$ does not necessarily implies in stability of the problem. Actually, the stability of the GFEM and SGFEM elements, within the limits of the investigation hereby conducted, will only be minimally guaranteed if the "inf-sup" test provides an asymptotic value $\lambda_n > 0$ for the selected problems (a) and (b).
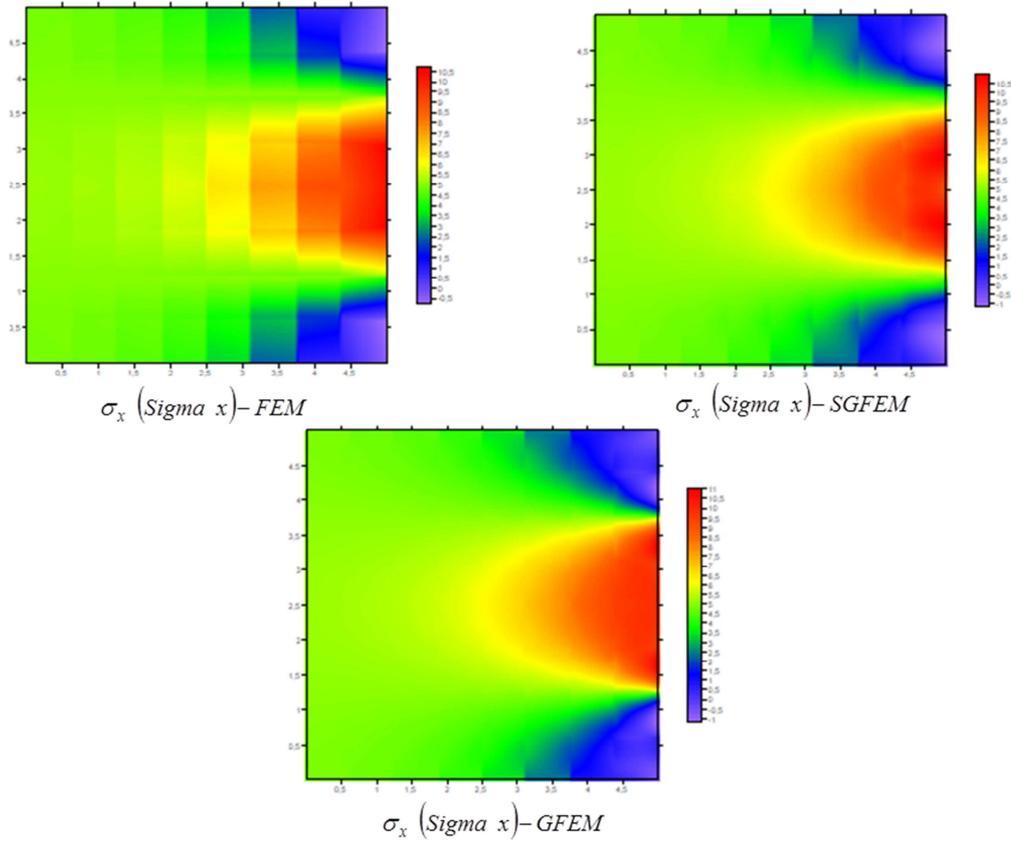
$\sigma_x \left(Sigma\ x \right) - FEM$

$\sigma_x \left(Sigma\ x \right) - SGFEM$

$\sigma_x \left(Sigma\ x \right) - GFEM$

Figure 5 – Problem (a) – stresses - 8x8 mesh of quadrilateral elements – FEM – $\left(\log\left(\frac{1}{N} \right) = -2.21 \right)$, GFEM – enrichment - x+y+xy+x^2+y^2 - $\left(\log\left(\frac{1}{N} \right) = -2.86 \right)$ and SGFEM – enrichment - x^2+y^2 - $\left(\log\left(\frac{1}{N} \right) = -2.59 \right)$

- FEM - Quadrilateral Element
- FEM - Triangula Element
- GFEM - enrichment - all nodes - x+y+xy+x^2+y^2 - Quadrilateral Element
- GFEM - enrichment - all nodes - x+y+xy+x^2+y^2 - Triangular Element
- SGFEM - enrichment - all nodes - x^2+y^2 - Quadrilateral Element
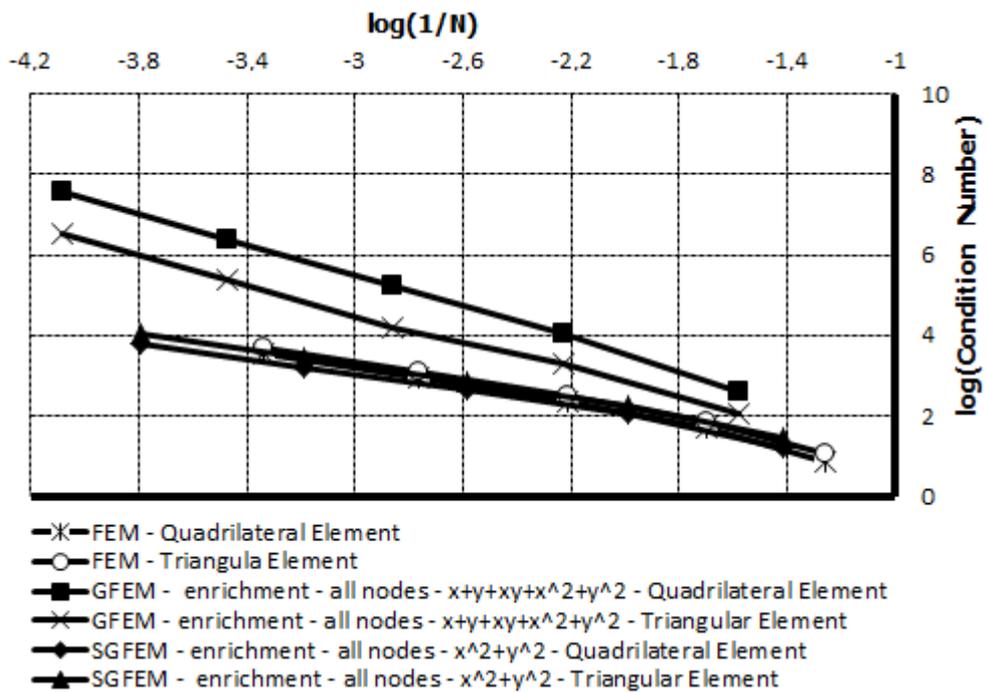- SGFEM - enrichment - all nodes - x^2+y^2 - Triangular Element

Figure 6 - Problem (a) - matrix condition number - FEM/GFEM/SGFEM

Another important observation related to problem (a) is that the "inf-sup" tests of the GFEM elements have pointed out to stable results for the complete set of enrichment functions presented in Tab. 1. Hence, it was always possible to identify an asymptotic value $\lambda_n > 0$.

On the other hand, regarding the SGFEM approximations, the combinations with stable responses were the ones combining monomials x^2 and y^2.

Cracked Panel – problem (b)

To the problem (b) a similar sequence of numerical experiments established for problem (a) was done, aiming to identify which enrichment combinations would provide asymptotic values $\lambda_n > 0$ for the meshes of Fig. 3.

In Fig. 7 the results of the "inf-sup test" computed for GFEM triangular element and SGFEM elements are shown. Note that just with the GFEM triangular element an asymptotic value $\lambda_n > 0$ was not clearly identified, thus characterizing the instability of these approximations. Similar conclusion, not depicted in Fig.7, has been achieved for GFEM quadrilateral element enriched with x+y+xy+x^2+y^2. Nevertheless, the SGFEM results have shown a more defined tendency appointing to stability.
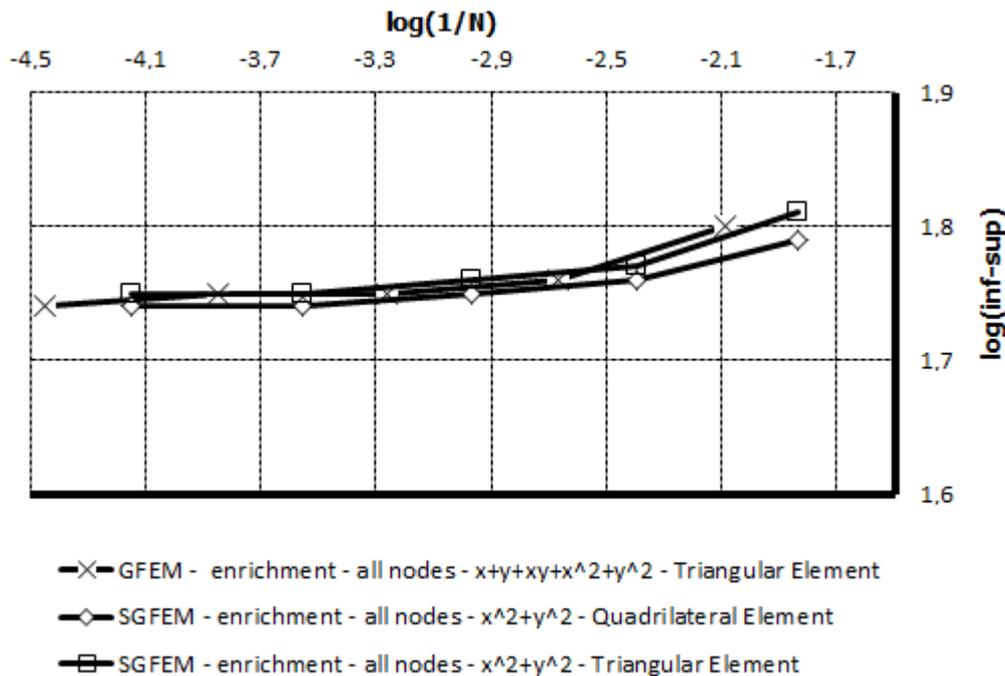

—✕— GFEM - enrichment - all nodes - x+y+xy+x^2+y^2 - Triangular Element

—◇— SGFEM - enrichment - all nodes - x^2+y^2 - Quadrilateral Element

—☐— SGFEM - enrichment - all nodes - x^2+y^2 - Triangular Element

Figure 7 - Problem (b) - results of the "inf-sup test" - GFEM/SGFEM

Figure 8 reveals that the (x+y+x^2+y^2+xy) polynomial enrichment causes a significantly increase of the condition number of the matrix $\mathbf{K}$, in particular for the GFEM quadrilateral element. Such drawback may disturb the inf-sup evaluation.

On the other hand, for the SGFEM in the analysis of problem (b), stability was demonstrated through the combinations of the monomials x^2 and y^2. Moreover, the SGFEM condition number is of the same order as the FEM one.

As a final remark, in Fig. 9 the stress $\left(\sigma_x\right)$ distributions for the problem (b) are shown. Endorsed by the "inf-sup" test, selective enrichement was applied to 39 nodes close to the crack tip of the GFEM/SGFEM quadrilateral elements. Specifically, when (x+y+x^2+y^2+xy) polynomial selective enrichment was applied to GFEM quadrilateral elements $\left(\log\left(1/N\right)=-3.21\right)$, it is noticed a more distinguished stress concentration compared to the FEM $\left(\log\left(1/N\right)=-3.10\right)$ stress $\left(\sigma_x\right)$ answer.

log(1/N)

FEM - Quadrilateral Element
FEM - Triangular Element
GFEM - enrichment - all nodes - x+y+xy+x^2+y^2 - Quadrilateral Element
GFEM - enrichment - all nodes - x+y+xy+x^2+y^2 - Triangular Element
SGFEM - enrichment - all nodes - x^2+y^2 - Quadrilateral Element
SGFEM - enrichment - all nodes - x^2+y^2 - Triangular Element
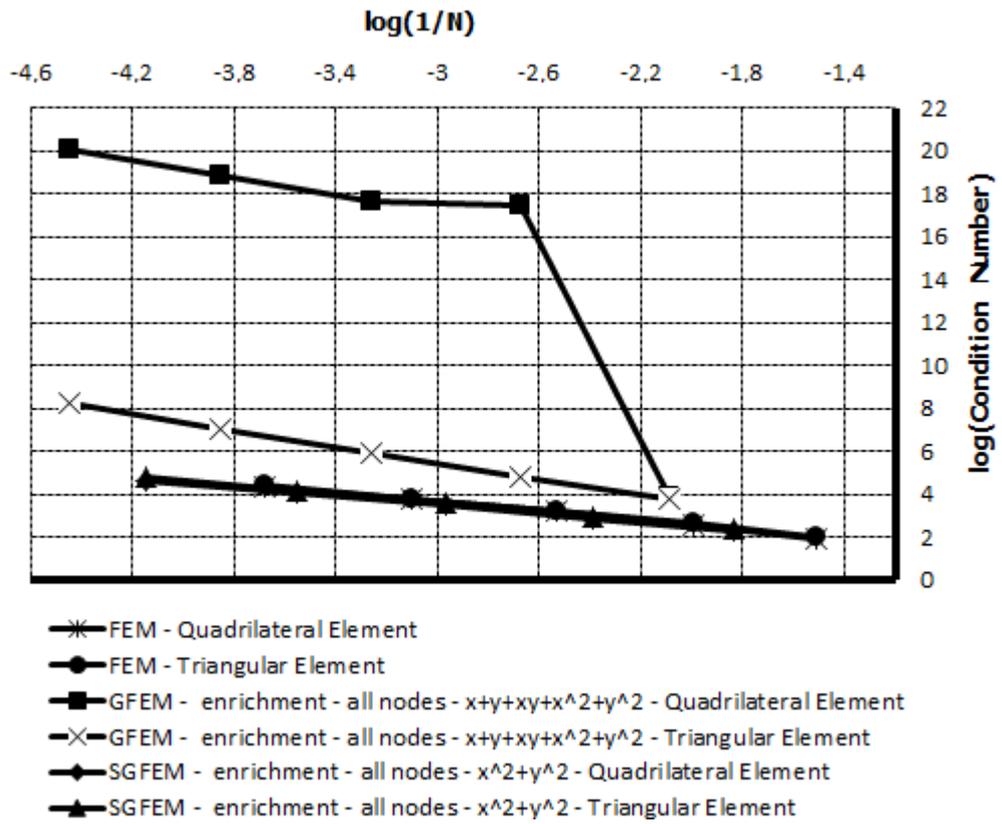
Figure 8 - Problem (b) - matrix condition number - FEM/GFEM/SGFEM

$\sigma_x \left(Sigma\ x\right) - FEM$

$\sigma_x \left(Sigma\ x\right) - SGFEM$

$\sigma_x \left(Sigma\ x\right) - GFEM$
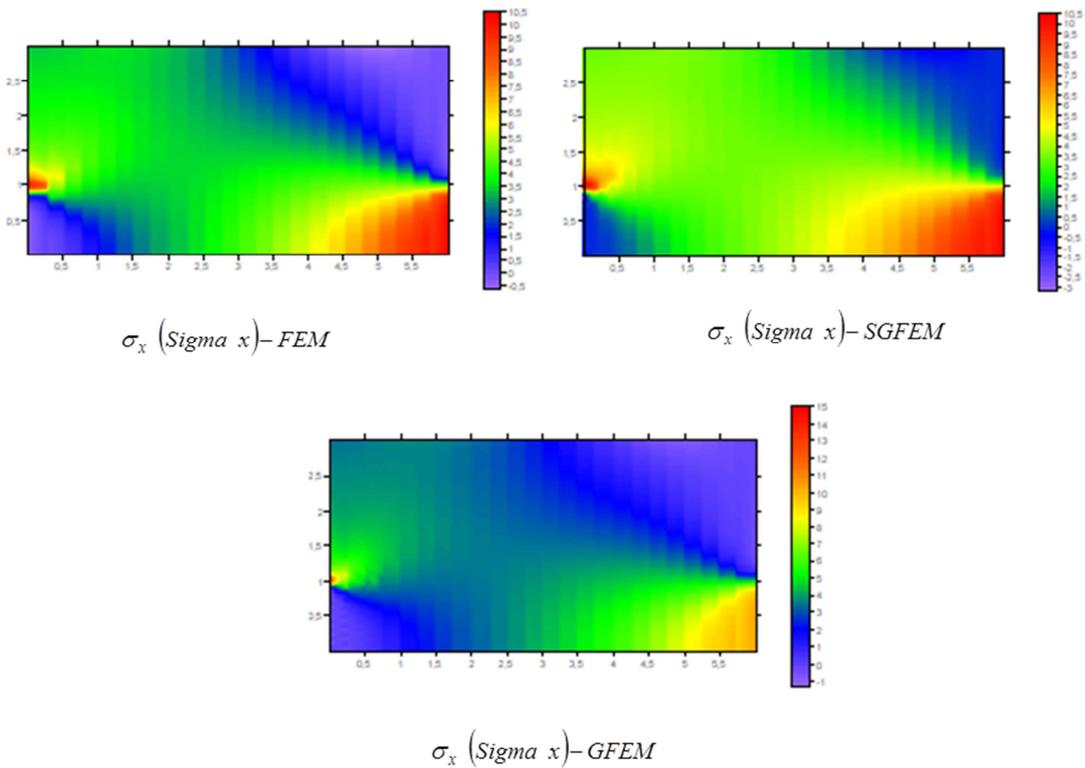
Figure 9 – Problem (b) – stresses - 12x12 mesh of quadrilateral elements – FEM - $\left(\log\left(1/N\right) = -3.10\right)$, GFEM – selective enrichment - x+y+xy+x^2+y^2 - $\left(\log\left(1/N\right) = -3.21\right)$ and SGFEM – selective enrichment - x^2+y^2 - $\left(\log\left(1/N\right) = -3.15\right)$

CONCLUSIONS

The issue of stability of the GFEM/SGFEM approximations provided by quadrilateral four-nodes and triangular three-nodes elements was hereby addressed. A numerical evaluation of the Babuška-Brezzi (inf-sup) condition called the "inf-sup test" was adjusted to both methods. Two reference problems, a tensioned panel and a cracked panel were selected for the numerical experiments.

The reference problems were selected due to its very distinct features of the respective stress responses. The tensioned panel has a stress concentration in the region next to the distributed load application (vertical right border). The cracked panel, according to classical linear solution, has a strong stress concentration next to the crack tip.

 Polynomial functions spanned by the monomials x, y, x^2, y^2 and xy were selected as regular enrichment functions. These functions were applied to the whole set of nodes of five meshes - 2x2, 4x4, 8x8, 16x16 and 32x32 – for the tensioned panel and five meshes - 3x3, 6x6, 12x12, 24x24 and 48x48 – for the cracked panel. The meshes were composed by quadrilateral and triangular elements.

The condition numbers of the matrices $\mathbf{K}$ of the linear systems associated to both problems were also computed as a complementary information to the "inf-sup" test conclusions. Specifically, these numbers have confirmed the previous expectance that the conditioning of the matrix $\mathbf{K}$ by SGFEM should present superior quality than that of GFEM. The improvement of the matrix $\mathbf{K}$ conditioning, proposed by the SGFEM, is indeed a favourable feature for solvability, however it does not implies necessarily on numerical stability, i.e., "inf-sup test" verification.

It is noteworthy that the "inf-sup test" has clarified some questions related to the enrichment strategies explored in the numerical experiments done, leading to the following conclusions:

- The polynomial enrichment is not appropriate to the GFEM. Actually, the "inf-sup" test did not exhibit an asymptotic value $\lambda_n > 0$ when considering both problems analyzed here. Moreover, for this kind of enrichment, the GFEM matrix $\mathbf{K}$ has undertaken an extremely ill conditioning when compared to the conditioning of the classical MEF. At large, convergence problems of the numerical solutions have appeared at all times that the Babuška-Brezzi (inf-sup) condition has failed;

- However, polynomial enrichment of the approximation can be applied selectively, i.e., considering a limit number of nodes. In the selective enrichment condition it was possible to evidence an asymptotic value $\lambda_n > 0$ for the GFEM. For the SGFEM, in the selective enrichment condition with the functions spanned by basis functions x^2 and y^2 it was possible to obtain an asymptotic value $\lambda_n > 0$, as well. Moreover, the solutions for selective enrichment have revealed representative values of the stress and displacements fields and straing energy in the problems (a) and (b). Nonewithstanding, another point worth of mentioning is on the choice of nodal enrichment density (number of enriched nodes) in the selective enrichment process. It is very difficult to identify precisely an enrichment density that yields $\lambda_n > 0$ and coherent results in terms of the displacement, stress fields representation and the strain energy.

Regarding to the extensive use of the selective enrichment, it is indeed an advisable expedient provided by the GFEM/SGFEM that can be applied in small regions of the domain aiming to locally improve the numerical approximation. The choice for such regions depends strongly on the characteristics of the problem and in some cases can be done straightforwardly. For instance, in the cracked panel the crack tip region must be logically selected for enrichment.

Finally, despite the limited number of experiments done the numerical evaluation of the Babuška-Brezzi (inf-sup) condition was crucial for a better understanding of the possibilities for nodal enrichment provided by the GFEM/SGFEM.

ACKNOWLEDGMENTS

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