Influence of Boundary Layer on the Effective Modulus of 3-1 Longitudinally Porous Elastic Solid

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Abstract: The evaluation of the effective properties of nonhomogeneous solids using analytical methods is, in general, based on the assumption that these solids have infinite dimensions. Here, we investigate the influence of the number of holes and boundary layer of a solid with finite dimensions on the determination of these properties. We use the Asymptotic Homogenization Method (AHM) to determine the effective shear modulus c_{44}^{ef} of an elastic solid with infinite dimensions containing a uniform and periodic distribution of circular cylindrical holes. We also use the Finite Element Method (FEM) to determine this modulus in the case of a solid with finite dimensions containing the same uniform distribution of cylindrical holes away from its boundary. Near the boundary, we consider a layer of material, which is usually left in the fabrication process of samples. Both solids have the same elastic properties and are subjected to similar anti-plane shear loadings. For the finite medium, when the number of holes increases and the boundary layer thickness is fixed, or, when the boundary layer thickness decreases and the number of holes is fixed we note the convergence of numerical solutions to the analytical solution via AHM. For example, graphs of c_{44}^{ef} versus void volume fraction show very good agreement between analytical results obtained via AHM and numerical results obtained via FEM. This investigation represents an ongoing effort of the research group to obtain the effective moduli of elastic solids using analytical, computational, and experimental methods.

Keywords: linear elasticity, asymptotic homogenization method, finite element method, effective modulus, boundary layer.

INTRODUCTION

The study of the behavior of nonhomogeneous solids requires the determination of their effective properties by means of analytical methods, computational methods or a combination of both. In the case of analytical methods, we generally assume that the solid has infinite dimensions. Since the samples used in laboratory experiments have finite dimensions, it is of interest to verify whether the effective properties obtained analytically are in good agreement with the effective properties obtained experimentally. A step towards this goal is to simulate these experiments numerically.

The aim of this research is then to develop a reliable and computationally efficient method of analysis suitable for predicting the response of elastic solids and the influence of their boundaries on the corresponding effective moduli. Here, we employ the Asymptotic Homogenization Method (AHM) and the Finite Element Method (FEM) to determine the effective properties of an elastic solid containing a uniform distribution of circular cylindrical holes centered in unit cells having hexagonal cross-sections. In order to apply the AHM, we consider that the microstructure of the solid consists of two phases distributed periodically over a domain that has infinite dimensions. This distribution of phases allows us to expand the solution of the related equilibrium problem in terms of an asymptotic series and obtain local problems. The solutions of these local problems can be calculated analytically for solids with simple microstructures or numerically for solids with complex microstructures. These local solutions are then used in the calculation of the effective moduli of the elastic solid, which depend upon physical and geometrical properties of its phases.

By using closed form expressions obtained by Bravo-Castillero et al. (2009), we calculate analytically the effective shear elastic modulus c_{44}^{ef} of the solid. The procedure used in this calculation can be used to evaluate the other effective elastic moduli of the solid. In addition, the procedure may also be used to evaluate the elastic moduli of a solid with unit cells of different cross sections, such as the square cross section.

Next, we simulate numerically the laboratory experiments using FEM and we use the results of these simulations to evaluate the effective elastic modulus mentioned above. We then compare this numerical evaluation with the corresponding analytical evaluation obtained through AHM. Recall from above that the solid has infinite dimensions in the AHM analysis and finite dimensions in the numerical simulation. The numerical experiment consists of the anti-plane shear of a cylindrical sample with rectangular cross section containing the same uniform distribution of unit cells of the

infinite solid away from the boundary of the solid. Near the boundary, we consider a layer of material and investigate the influence of this layer on the evaluation of the effective modulus. The cross-section of the solid is illustrated in Fig. 1. The material of this sample is the same linear elastic and isotropic material considered in the analytical approach.

This investigation represents an ongoing effort of the research group to obtain the effective moduli of elastic solids analytically and numerically, and it will be useful to evaluate the influence of the boundary layer in 3D printed samples for mechanical tests. In additive manufacturing, this layer is comprised of several shell perimeters, which are among the main control factors on the mechanical properties of samples fabricated in, for instance, polylactic acid (PLA). According to Lanzotti et al. (2015), other important control factors are the layer thickness of deposition and the infill orientation of each layer. The high variability of results observed by these authors together with a shortage of existing literature concerning the impact of these factors on the mechanical properties of samples made by additive manufacturing has motivated this work.

First, we formulate the anti-plane shear problem for linear elastic isotropic media with infinite and finite dimensions containing uniform and periodic distributions of circular cylindrical holes and use AHM and FEM, respectively, to determine the effective shear modulus c_{44}^{ef}. We then compare the results obtained via AHM and via FEM by taking into account the concentration of holes and the boundary layer thickness of the solid with finite dimensions. Finally, in Conclusions, we present the final remarks of this work.

PROBLEM STATEMENT

Analytical Method – Here, we consider identical circular cylindrical holes periodically distributed in an isotropic homogeneous medium, which is made of a linear and isotropic elastic material with infinite dimensions. The circular cylindrical holes are centered in unit cells with hexagonal cross sections. The effective properties are calculated by means of the AHM. The relevant aspects of these calculations can be found in Bravo-Castillero et al. (2009). For the problem at hand, which is to calculate analytically the effective shear elastic modulus in the longitudinal direction, c_{44}^{ef}, of the elastic solid, only the *Local Problem* $_{13}L$ over the local domain Y, which corresponds to the solid part of the unit cell illustrated on the right-hand side of Fig. 1, must be specified at the outset. This problem consists of finding a function $U : Y \rightarrow \mathbb{R}$, harmonic and of zero average in Y, that satisfies the system of equations

$$\Delta U = 0 \quad \text{in } Y, \tag{1}$$

$$U_{,1}n_1 + U_{,2}n_2 = -n_1 \quad \text{on } \Gamma, \tag{2}$$

$$\langle U \rangle = 0, \tag{3}$$

where Δ is the two-dimensional Laplacian in Y, Γ corresponds to the boundary of the hole in the cell, and $\langle U \rangle \triangleq \int_Y U(\mathbf{y})d\mathbf{y}/|Y|$, with $|Y|$ being the volume of Y. The solution of this problem can be found in Aguiar, Prado and da Silva (2018) and yields the formula

$$c_{44}^{ef} = c_{44}\left(1 - 2A_2 K^0\right), \tag{4}$$

where $A_2 = 2\pi R^2/\sqrt{3}$ is the porous volume fraction of the hexagonal cell, $K^0 = \dfrac{1}{1 + A_2 - \mathbf{V}_p^T \mathbf{M}_p^{-1} \tilde{\mathbf{V}}_p}$, and $\mathbf{V}_p = \mathbf{V}_p(\omega_s)$, $\mathbf{M}_p = \mathbf{M}_p(m_{ts})$, $\tilde{\mathbf{V}}_p = \tilde{\mathbf{V}}_p(\tilde{\omega}_t)$ are vectors and matrices of infinite dimensions. The arguments of these vector- and matrix-valued functions are given by $\omega_s = R^{12s}\eta_{(1\ 6s-1)}$, $m_{ts} = \delta_{(6t-1\ 6s-1)} - R^{12s}\sum_{i=1}^{\infty} R^{12i}\eta_{(6t-1\ 6i+1)}\eta_{(6i+1\ 6s-1)}$, $\tilde{\omega}_t = \eta_{(6t-1\ 1)}$, $t, s = 1, 2, 3, \ldots$, where R is the radius of the hole in the unit cell and, for $k, l = 1, 3, \ldots$, $\delta_{(kl)}$ is the Kronecker delta and $\eta_{(kl)}$ is an infinite series defined in the reference above.

Computational Method – Here, the material of the sample is isotropic and linearly elastic with the same elastic modulus adopted in the analytical approach for the matrix with infinite dimensions. The governing differential equation and the boundary conditions at the walls of the holes have the same form of Eq. (1) and Eq. (2), respectively.

Observe from Fig. 1 that M_C corresponds to the solid part of the sample, which has dimensions $L \times H$. Near the boundary, there is a layer of material of thickness a separating the holes from the boundary. On the external boundary of the sample, we impose zero displacement on the left-hand side, $\partial_1 M_C$, zero tractions on the lower, $\partial_2 M_C$, and upper,

$\partial_4 M_C$, sides, and displacement $\bar{\mathbf{u}}$ on the right-hand side $\partial_3 M_C$, which has magnitude $\bar{u}$ and is applied normal to the $x_1 x_2$ - plane. Also, $\mathbf{n}$ is the outward normal vector to the external boundary.
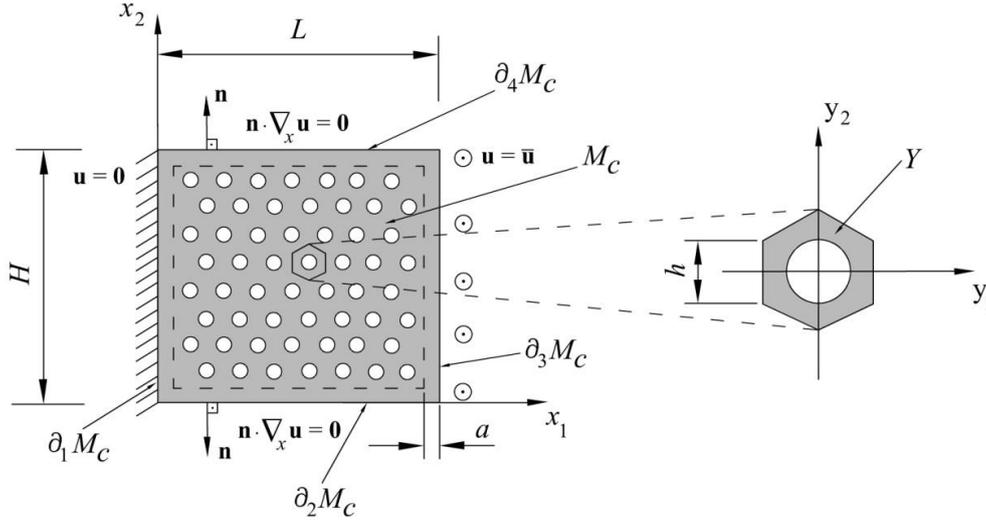

Figure 1. Cross section of the cylindrical sample together with boundary conditions and the unit cell of hexagonal shape.

To solve numerically the equilibrium problem described above, we observe that this problem is analogous to a linear steady state heat conduction problem, which consists of finding the temperature field $T : M_C \rightarrow \mathbb{R}$ that satisfies

$$\Delta_x T = 0 \quad \text{in} \ M_C, \tag{5}$$

$$\mathbf{n} \cdot \nabla_x T = 0 \quad \text{on} \ \Omega_C \cup \partial_2 M_C \cup \partial_4 M_C, \tag{6}$$

$$T = 0 \ \text{on} \ \partial_1 M_C, \ T = \bar{T} \ \text{on} \ \partial_3 M_C, \tag{7}$$

where Δ_x and ∇_x are, respectively, the Laplacian and the gradient operators defined in M_C, Ω_C is the union of all the contours of the holes, and ∂M_C is the external boundary of M_C. We use the finite element commercial package COMSOL 4.4® to obtain approximate solutions of Eq. (5)-Eq. (7). The temperature field T is approximated by quadratic triangular finite elements.

Once an approximate solution of the thermal problem is obtained for a given discretization, we associate the temperature $\bar{T}$ with the displacement $\bar{u}$ and the conductivity k with the elastic modulus c_{44}^{ef}. The effective elastic modulus c_{44}^{ef} can then be determined from the expression

$$c_{44}^{ef} = \frac{\int_{\partial_3 M_C} \mathbf{n} \cdot \nabla_x T \, dx_2}{\bar{T}/L}, \tag{8}$$

where $\int_{\partial_3 M_C} \mathbf{n} \cdot \nabla_x T \, dx_2$ is the heat flux on $\partial_3 M_C$.

RESULTS AND DISCUSSION

Here, we present and discuss some results of this investigation, which represents an ongoing effort of the research group to obtain the effective moduli c_{44}^{ef} of elastic solids analytically and numerically. Below, we consider $c_{44} = 1$ GPa as the elastic modulus of the solid part of both the finite and the infinite media.

In order to verify if the numerical solutions correspond to analytical prediction obtained via AHM, in which an infinite periodic medium can be replaced by an equivalent homogenized medium, we present a sequence of solutions obtained via FEM. First, we hold the boundary layer thickness fixed, $a = 0.0001$ m, to investigate the influence of the number of holes of a solid with finite dimensions on the determination of the effective elastic modulus c_{44}^{ef}. In addition, we consider that the rectangular cross sections of the elastic samples have the fixed dimensions $L \times H = 0.1 \times 0.1 \ \text{m}^2$. In Fig. 2 we

show the effective moduli c_{44}^{ef} plotted against the void volume fraction A_2 for decreasing values of the hexagon side dimension h in the set $\{L/2, L/4, L/8, L/16, L/32\}$ [m]. The corresponding curves were obtained via FEM by using Eq. (8). We also show a curve obtained via AHM by using Eq. (4). Note that the curves obtained via FEM converge to the curve obtained by AHM as $h \to 0$, even at a high void volume fraction A_2.
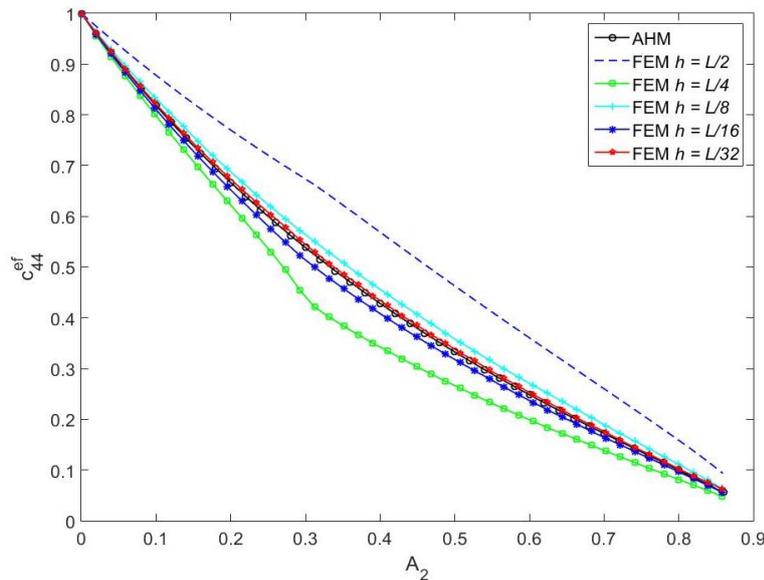

Figure 2 – Effective elastic modulus c_{44}^{ef} versus void volume fraction A_2 .

After, we compare the best numerical curve presented in the Fig. 2 with the best curve, FEM F4 , obtained by Aguiar, Prado and da Silva (2018). For this case, the authors considered elastic samples with rectangular cross sections having the fixed dimensions $L \times H = 0.6397 \times 0.5925$ m^2 and the thickness of the boundary layer $a = 0.0031$ m. While those authors varying the domain, here we have it fixed. Thus, in Fig. 3 we show again the curves obtained via AHM and via FEM with $h = L/32$, which for convenience we will call *FEM h = L/32* as *FEM F3# h = L/32* . Each numerical curve was obtained by using different approaches. Observe from this figure that all the curves are very close to each other. This result allows us to choose $h = L/32$ as a representative parameter of number of holes in a fixed domain and next vary the boundary layer thickness a.
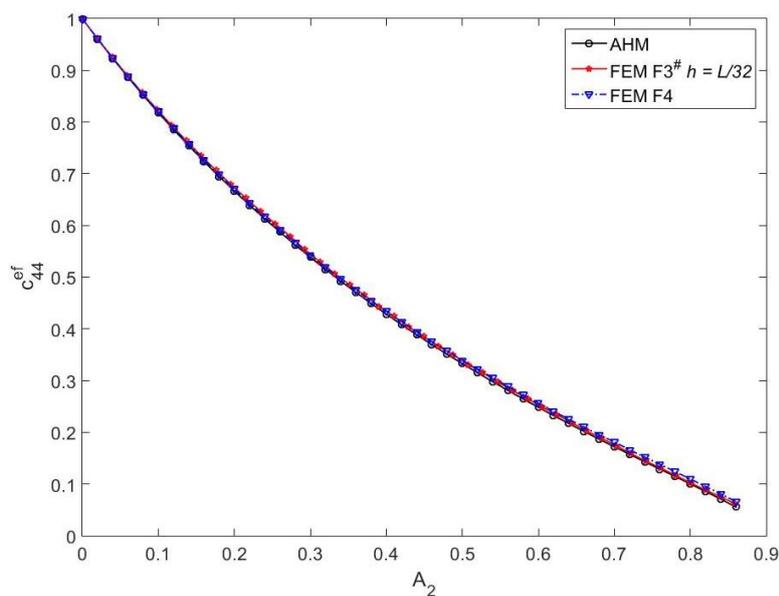

Figure 3 – Effective elastic modulus c_{44}^{ef} versus void volume fraction A_2 using results of Fig. 2 to compare with the corresponding curve FEM F4 from Aguiar, Prado and da Silva (2018).

Next, we continue holding the side of the hexagon fixed with length $h = L/32$ and we vary the boundary layer thickness a to investigate the influence of the boundary layer of a solid with finite dimensions on the determination of the effective elastic modulus c_{44}^{ef}.

In Fig. 4 we show graphs of the effective elastic modulus c_{44}^{ef} versus the void volume fraction A_2 obtained via AHM, by means of Eq. (4), and via FEM, using Eq. (8), for the elastic samples with rectangular cross sections having the fixed dimensions $L \times H = 0.1 \times 0.1$ m^2 and decreasing values of thickness a in the set $\{0.01, 0.001, 0.0001, 0.00001\}$ [m]. These values correspond to FEM Fi $^\#$, i=1,...,4 , in the legend of Fig. 4, yielding a *fixed domain sequence*. For comparison purposes, in the same figure we show graphs for $L \times H \in \{0.0756 \times 0.0714, 0.6397 \times 0.5925\}$ [m^2], which are also presented in Aguiar, Prado and da Silva (2018). Here, the thickness of the boundary layer is fixed and given by $a = 0.0031$ m. These dimensions correspond to FEM Fi, i=1,4, in Fig. 4, which is part of a *fixed layer sequence*.

Observe from Fig. 4 that the numerical results obtained from the *fixed domain sequence* (FEM Fi $^\#$, i=1,...,4) converge to the analytical results obtained from AHM as a decreases. The convergence is also observed in the case of the *fixed layer sequence*, in which case the bigger the rectangular cross sectional area the closer the numerical solution is to the analytical solution. In particular, good convergence is observed at all values of void volume fraction shown in Fig. 4 up to the limit where the walls of the holes touch each other. In the case of cells with regular hexagonal cross section, this limit is 0.9069.
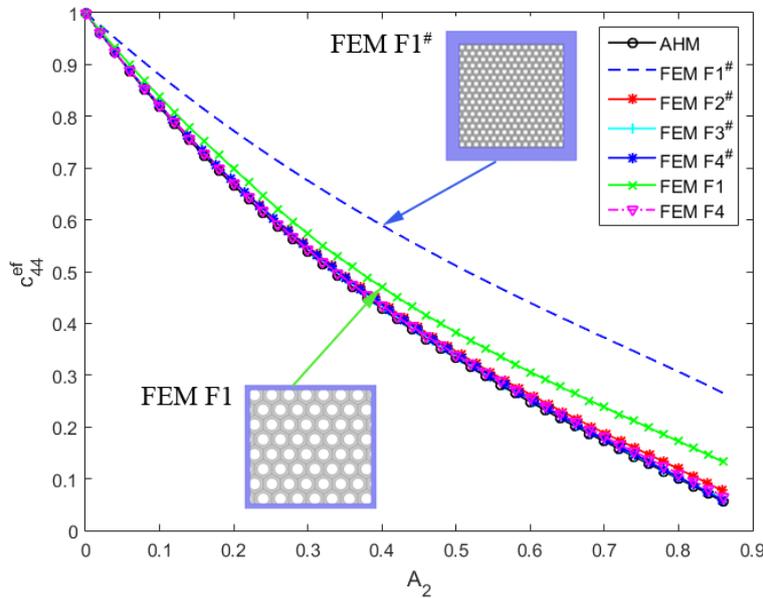

Figure 4 – Curves of effective elastic modulus c_{44}^{ef} versus void volume fraction A_2 . Hexagonal lattice consisting of the fixed layer sequence (Fi) and the fixed domain sequence (Fi $^\#$).

CONCLUSIONS

In this paper we have shown some results concerning the analytical and numerical evaluations of the effective modulus c_{44}^{ef} of linear elastic solids. When the side of the hexagon with length h decreases and the boundary layer thickness a is fixed, or, when a decreases and h is fixed, we observe convergence of numerical solution for a finite medium to the analytical solution via AHM for an infinite medium.

ACKNOWLEDGMENTS

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