The displacement fitting technique to assess Stress Intensity Factors in three-dimensional problems using the Boundary Element Method

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Abstract: This work presents an extension of the Displacement Fitting Technique (DFT) to assess Stress Intensity Factors (SIF) in three-dimensional linear elastic fracture mechanics problems using the Boundary Element Method (BEM). The displacement fitting scheme developed in the present study accounts for higher order terms of the asymptotic near crack front William´s solution. Thus, it results more accurate than first order displacement based SIF assessment techniques. Besides, BEM formulations are well-known to be accurate for the evaluation of physical quantities with high gradients, such as the near crack front displacement fields. Due to the lack of domain discretizations provided by the BEM, the number and location of the internal near crack front points for displacement evaluation can be arbitrarily chosen. In the present work, the accuracy of the SIF results is studied with respect to the number and location of these evaluations points. Two three-dimensional crack problems with both analytical and numerical SIF solutions were utilized to study the performance of the DFT.

Keywords: Stress Intensity Factors, Displacement fitting technique, Boundary Element Method

INTRODUCTION

The structural analysis of cracked components by means of linear elastic fracture mechanics can be usually reduced to the evaluation of Stress Intensity Factors (SIF). The SIF can be assessed experimentally, theoretically or numerically. They are used for example, to predict the crack growth rates and directions, and whether a crack will propagate or not under a given load. The use of numerical methods to assess the SIF has become very popular due to the widely range of problems that can be treated with this approach. The standard Finite Element Method (FEM), the Generalized Finite Element Method (GFEM) and the Boundary Element Method (BEM) have become very popular for the analysis of fracture mechanics in solids. In the context of linear elastic fracture mechanics, several computational strategies have been proposed for calculating SIF values at the crack front using these numerical methods.

SIF extraction methods based on energy release rate concepts include the J-integral [Rice 1968], the Interaction Integral Method [Moran & Shih 1987; Walters et al. 2005], the Contour Integral Method [Stern 1976; Szabo & Babuška 1988] and the Cutoff Function Method [Szabo & Babuška 1988]. A review of method for calculation energy release rates can be found in [Li et al. 1985]. These methods are accurate since theoretically they converge at the same rate as the strain energy [Babuška & Miller 1984]. Assumptions about the crack geometry in three-dimensional problems are usually adopted in their numerical implementation [Gonzalez-Albuixech et al. 2013]. This in general leads to a loss of accuracy [Gonzalez-Albuixech et al. 2013]. Another class of extraction methods is based on the asymptotic expansion of the elasticity solution in the neighborhood of a crack. This includes the Displacement Correlation Technique and the Stress Correlation Technique [Chan et al. 1970; Ingraffea & Manu 1980; Banks-Sills & Sherman 1986; Rigby & Aliabadi 1998; Banks-Sills et al. 2005; Nejati et al. 2015]. These methods have been used for decades to extract SIF from FEM and BEM solutions [Chan et al. 1970; Ingraffea & Manu 1980; Banks-Sills & Sherman 1986; Bezerra & Medeiros 2002; Ortiz & Cisilino 2006; Balderrama et al. 2006; Fu et al. 2012; Gonzalez-Albuixech et al. 2013; Gonzalez et al. 2015; Gupta et al. 2017]. More recently they have been adapted to extract SIF from GFEM solutions [Minnebo et al. 2010; Gupta & Duarte 2018; Schätzer & Fries 2016].

In this work, the Displacement Fitting Technique developed by Gonzalez et al. (2015) is extended to calculate the SIF in three-dimensional crack problems. This scheme is implemented in the context of the BEM in order to exploit its advantages related to the lack of domain discretizations and high gradient internal fields numerical evaluation. Two simple examples are presented for which the SIF are evaluated with the DFI from the internal displacement BEM solutions. The results are compared against available analytical and numerical solutions.

THE DISPLACEMENT FITTING TECHNIQUE

The Displacement Fitting Technique (DFT), recently presented by Gonzalez et al. (2015), is extended for three-dimensional problems in the following. In three-dimensions, a near crack front point can be written in terms of local crack front coordinates, i.e., local Cartesian coordinates (x', y', z') or local spherical coordinates (r, θ, φ), which have its origin defined in a crack front point O', as illustrated in Fig.1a.
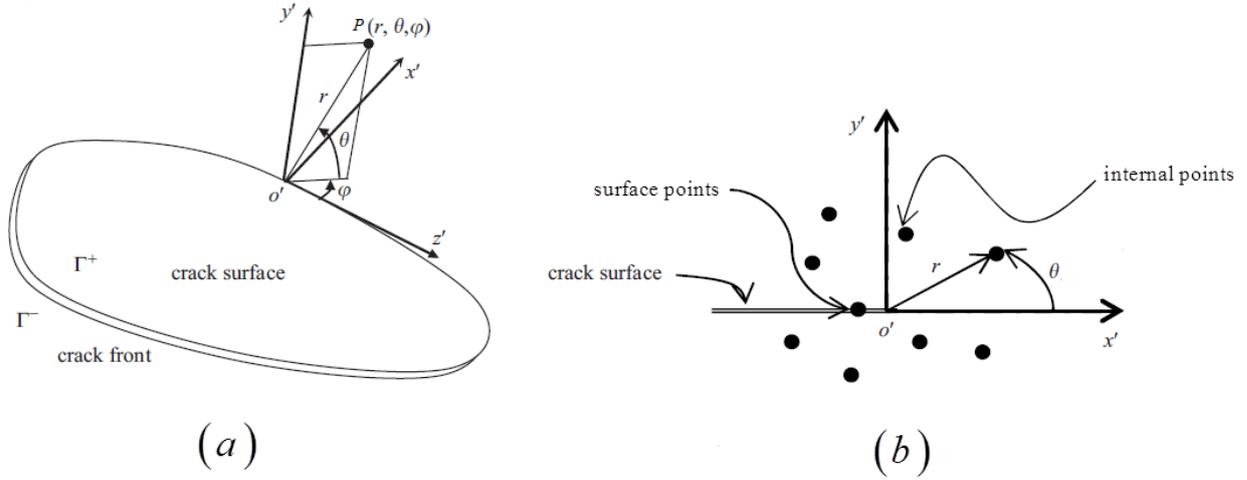
Figure 1 – (a) Local crack front coordinates (b) Plane crack front coordinates for $\varphi = 0$.

The William´s asymptotic solution for the near crack front displacement fields are presented in Eq. 1 in terms of a polar coordinate system (r, θ) with origin defined at the point O', which is defined for the plane $\varphi = 0$, as illustrated in Fig. 1b.

$$
\begin{Bmatrix} u_{x'}(r,\theta) \\ u_{y'}(r,\theta) \\ u_{yz}(r,\theta) \end{Bmatrix} = \frac{(1+v)}{E\sqrt{2\pi}} K_I \sqrt{r} \begin{Bmatrix} f_{1I}(\theta) \\ f_{2I}(\theta) \\ 0 \end{Bmatrix} + \frac{(1+v)}{E\sqrt{2\pi}} K_{II} \sqrt{r} \begin{Bmatrix} f_{1II}(\theta) \\ f_{2II}(\theta) \\ 0 \end{Bmatrix}
$$

$$
+ \frac{(1+v)}{E\sqrt{2\pi}} K_{III} \sqrt{r} \begin{Bmatrix} 0 \\ 0 \\ f_{3III}(\theta) \end{Bmatrix} + r \begin{Bmatrix} g_1(\theta) \\ g_2(\theta) \\ g_3(\theta) \end{Bmatrix} + O(r^{3/2})
$$

(1)

The functions $f_{iI}(\theta)$, $f_{iII}(\theta)$ and $f_{iIII}(\theta)$, are well-known from the fracture mechanics literature. The functions $g_i(\theta)$, which multiply the terms of order $O(r)$ in Eq. (1) are not treated explicitly. For a given fixed point $(\bar{r}, \bar{\theta})$, which may be either a surface or an internal point, as illustrated in Fig. 1b, these functions may be understood as unknown constants, i.e., $g_i(\bar{\theta}) = c_i$, which must be solved for as discussed in the following. Considering the first two terms $O(\sqrt{r})$ and $O(r)$ of the William´s asymptotic solution, the near crack front displacements can be written, for the fixed point $(\bar{r}, \bar{\theta})$, as follows:

$$
\begin{Bmatrix} u_{x'}(\bar{r},\bar{\theta}) \\ u_{y'}(\bar{r},\bar{\theta}) \\ u_{z'}(\bar{r},\bar{\theta}) \end{Bmatrix} \approx \frac{(1+v)}{E\sqrt{2\pi}} \begin{bmatrix} \sqrt{r}f_{1I}(\bar{\theta}) & \sqrt{r}f_{1II}(\bar{\theta}) & 0 & \bar{r} & 0 & 0 \\ \sqrt{r}f_{2I}(\bar{\theta}) & \sqrt{r}f_{2II}(\bar{\theta}) & 0 & 0 & \bar{r} & 0 \\ 0 & 0 & \sqrt{r}f_{3III}(\bar{\theta}) & 0 & 0 & \bar{r} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix}
$$

(2)

For a fixed direction $\bar{\theta}$ in the plane $\varphi = 0$, chosen a given finite number of near crack front points $\bar{r}_i$, $i = 1 \dots N$, it is possible to write a system of equations from Eq. 2, as follows:

$$
\begin{Bmatrix}
u_{x'}(\bar{r}_1,\bar{\theta}) \\
u_{y'}(\bar{r}_1,\bar{\theta}) \\
u_{z'}(\bar{r}_1,\bar{\theta}) \\
\vdots \\
u_{x'}(\bar{r}_N,\bar{\theta}) \\
u_{y'}(\bar{r}_N,\bar{\theta}) \\
u_{z''}(\bar{r}_N,\bar{\theta})
\end{Bmatrix}
\approx
\frac{(1+\nu)}{E\sqrt{2\pi}}
\begin{bmatrix}
\sqrt{\bar{r}_1}\,f_{1I}(\bar{\theta}) & \sqrt{\bar{r}_1}\,f_{1II}(\bar{\theta}) & 0 & \bar{r}_1 & 0 & 0 \\
\sqrt{\bar{r}_1}\,f_{2I}(\bar{\theta}) & \sqrt{\bar{r}_1}\,f_{2II}(\bar{\theta}) & 0 & 0 & \bar{r}_1 & 0 \\
0 & 0 & \sqrt{\bar{r}_1}\,f_{3III}(\bar{\theta}) & 0 & 0 & \bar{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sqrt{\bar{r}_N}\,f_{1I}(\bar{\theta}) & \sqrt{\bar{r}_N}\,f_{1II}(\bar{\theta}) & 0 & \bar{r}_N & 0 & 0 \\
\sqrt{\bar{r}_N}\,f_{2I}(\bar{\theta}) & \sqrt{\bar{r}_N}\,f_{2II}(\bar{\theta}) & 0 & 0 & \bar{r}_N & 0 \\
0 & 0 & \sqrt{\bar{r}_N}\,f_{3III}(\bar{\theta}) & 0 & 0 & \bar{r}_N
\end{bmatrix}
\begin{Bmatrix}
K_I \\
K_{II} \\
K_{III} \\
c_1 \\
c_2 \\
c_3
\end{Bmatrix}
\qquad (3)
$$

Eq. 3 written in a more compact matrix form results:

$$
\underset{(3N\,\mathrm{x}\,1)}{\bar{\mathbf{u}}} \approx \frac{(1+\nu)}{E\sqrt{2\pi}}\, \underset{(3N\,\mathrm{x}\,6)}{\mathbf{M}}
\begin{Bmatrix}
K_I \\
K_{II} \\
K_{III} \\
c_1 \\
c_2 \\
c_3
\end{Bmatrix}
\qquad (4)
$$

The unknown of the system of equations are the values of K_I, K_{II}, K_{III}, $c_1 = g_1(\bar{\theta})$, $c_2 = g_2(\bar{\theta})$ and $c_3 = g_3(\bar{\theta})$ of the near crack front displacement solution, i.e., Eq. 1, for the fixed direction $\bar{\theta}$. Thus, the SIF solutions K_I, K_{II} and K_{III} for the crack front point O' are obtained solving Eq. 4. Although it is a non-square system of equations, it is linear. Thus, the SIF solutions can be retrieved using the pseudo inverse concept:

$$
\begin{Bmatrix}
K_I \\
K_{II} \\
K_{III} \\
c_1 \\
c_2 \\
c_3
\end{Bmatrix}
\approx \frac{E\sqrt{2\pi}}{(1+\nu)}\left[\left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T\right]\bar{\mathbf{u}}
\qquad (5)
$$

Considering that the above solution can be constructed for any arbitrarily fixed $\bar{\theta}$ direction, the SIF solution can be retrieved from a completely arbitrary near crack front point distribution in the plane $\varphi = 0$. This fitting between numerical and analytical near crack front displacement field solution is suitable to BEM formulations where the internal solution for the displacements at any internal point is easily and accurate calculated in a post-processing routine, by resorting to the displacement boundary integral equation, which is a singular boundary integral equation. Even though the present fitting technique require computing a matrix inversion, it is still efficient since the number of crack front points where the SIF must be evaluated in small for single crack problems. It is worth mentioning that if stresses, strains and displacements derivatives are accurate computed in any internal point (this is not an easy task in finite element computations), the J-integral is straightforward calculated using numerical quadrature. However, to compute stresses, strains and displacements derivatives in the context of the BEM, it is required resorting to hypersingular boundary integral equation in a post-processing routine. The hypersingular boundary integral equations are computationally more expansive to be evaluated than the singular boundary integral equations required to evaluate the displacements, especially if none regularization technique is applied to the almost-singular integral. Thus, the coupling of BEM and the displacement fitting technique may be considered an interesting and efficient approach for single crack problems.

NUMERICAL RESULTS

Two applications were considered in order to evaluate the accuracy of the DFT to assess the SIF in three-dimensional linear elastic fracture mechanics problems using the BEM. The first application addresses a mode I crack problem with known analytical solution for the SIF. Thus, only the numerical results of were evaluated. The second application addresses a mixed mode crack problem, for which the numerical and experimental SIF reference solutions are available from the literature. This last application allows evaluating the accuracy of the DFF to assess all SIF numerical results.

Mode I crack problem

The first application concerns the problem of a stressed prism with an edge crack, as illustrated in Fig. 2. The applied stress was equal to the unity, $\sigma = 1$. The dimensions of the problem are characterized by the crack length

$a = 2.5$ (length unit) and by the ratios illustrated in Fig. 2. The elastic constants adopted for the linear elastic isotropic material were $E = 1000$ (stress unit) and $\nu = 0.3$.

$$T/a = 2$$
$$W/a = 3$$
$$H/a = 3.5$$
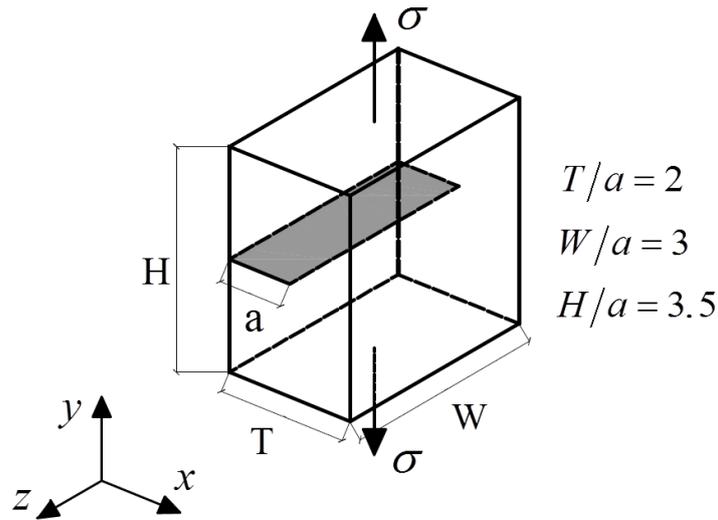
Figure 2 – Stressed prism with an edge crack.

The prism was discretized with three different boundary element meshes, as illustrated in Fig. 3. The meshes Mesh 1, Mesh 2 and Mesh 3 resulted, respectively, 344, 1142 and 1718 quadrilateral boundary elements of linear order of approximation and 695, 1591 and 3895 collocation points. It is worth to mention that the crack surfaces were discretized with discontinuous boundary elements to ensure the existence of the hypersingular integrals involved in the DBEM integral equations.

Mesh 1   Mesh 2   Mesh 3
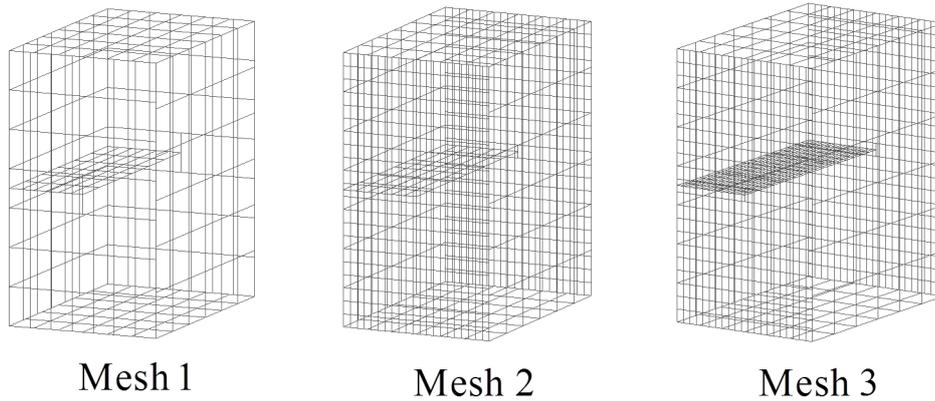
Figure 3 – Meshes utilized into the discretization of the problem

The SIF assessment through the DFT was performed utilizing 7 internal points in the plane $\varphi = 0$ for the displacement evaluation. These points were equally distributed in a circular path around the crack front with arbitrary radius r, in the interval of the arc $\theta = \left[-3\pi/4, \, 3\pi/4 \right]$. The crack is solicited in mode I. Thus, only the K_I SIF result was studied in this preliminary example. The analytical solution of K_I is well-known for the plane strain problem (Murakami, 1987). To study the accuracy of the SIF assessment, the numerical K_I results at the center of the crack front obtained with the conventional Displacement Correlation Technique (DCT) and with the DFT are compared with the analytical solution in Fig. 4. The K_I results are presented as a function of the dimensionless extraction distance, r/a, where a is the crack length.
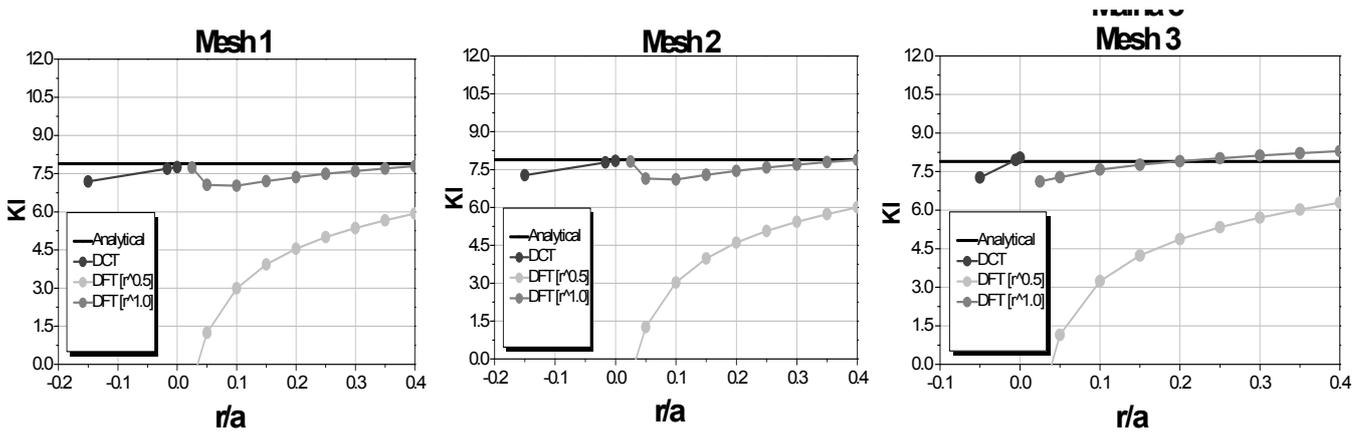
Figure 4 – SIF results

For the DCT, the K_I results are obtained from the displacement discontinuities of the two colocation points of the crack front element, which corresponds to evaluation distances $r = r_1$, $r = r_2$, and the result is linear extrapolated for the crack front, i.e., $r = 0$. For the DFT, evaluation distances that range from $r = 0.025a$ to $r = 0.4a$ were adopted. Evaluation distances for points at the crack surfaces were considered to be negative, while evaluation distances for points ahead the crack front were considered to be positive. Two variants of the DFT were studied. The first one, named DFT $[r^0.5]$, is the conventional one, were the near crack front asymptotic displacement expansion is truncated at the terms of order $O(\sqrt{r})$. Thus, matrix reduces to a $3N \times 3$ matrix. The second variant, named DFT $[r^1.0]$, consider both the terms of orders $O(\sqrt{r})$ and $O(r)$ for the near crack front asymptotic displacement expansion, as previously discussed. Notice that the inclusion of the terms of order $O(r)$ of near crack front asymptotic displacement expansion into the DFT system of equations improves the SIF numerical results. Besides, they also become less sensitive to the adopted evaluation distance r. Finally, Fig. 5 illustrates the displacement numerical results obtained with the model Mesh 2. The deformed shape is presented considering the displacements 10 times magnified.
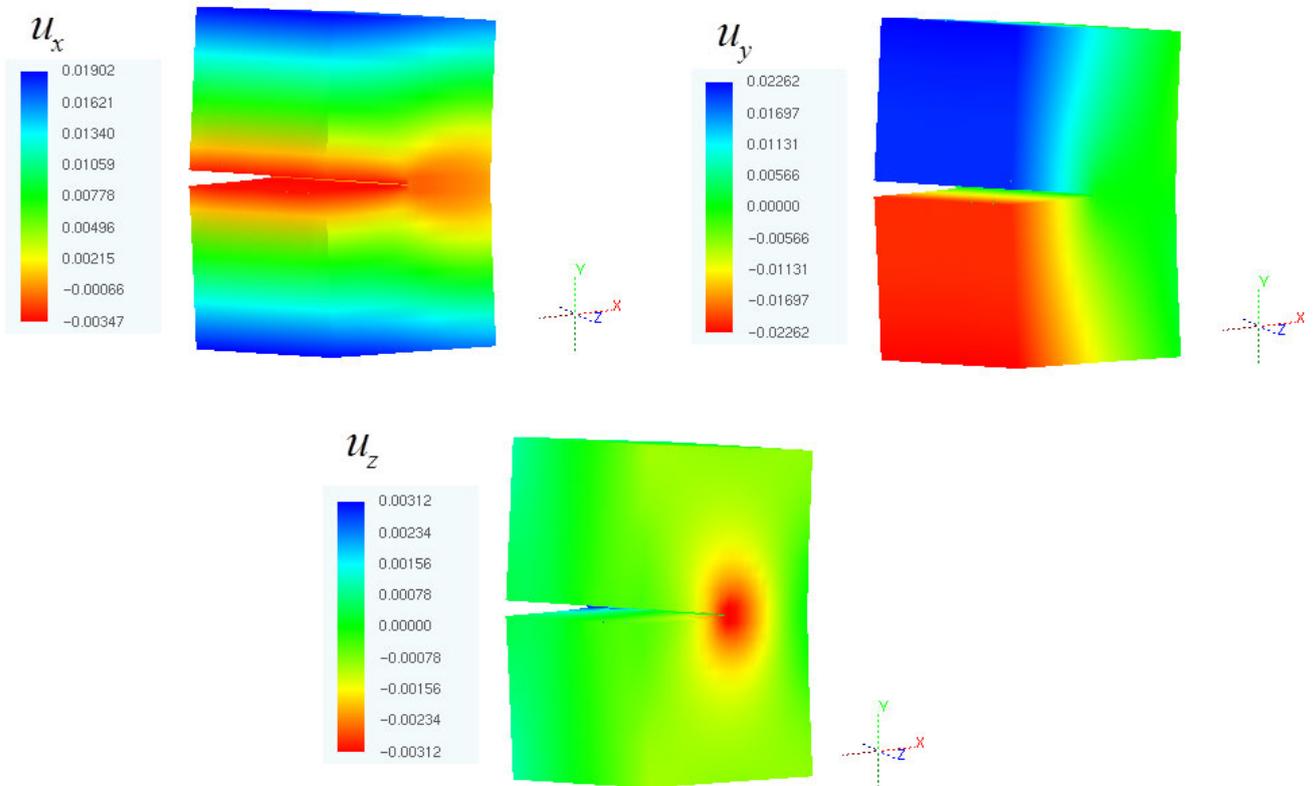

Figure 5 – Displacements numerical results for the model Mesh 2

From Fig. 5 it is possible to observe poisons effect, which results in the displacements in the z direction. However, due to the symmetry of problem, the crack surfaces displacements are symmetric, ensuring that the relative displacements, i.e., displacement discontinuities, are observed only in the opening direction. Thus, the problem in pure mode I condition.

Mixed mode crack problem

The second application concerns the problem of a three-point bending beam with an initial central notch. Fig. 6 presents the geometry and the boundary conditions of the problem. The beam's dimension are: $L_t = 260mm$, $L = 240mm$, $H = 60mm$ and $t = 10mm$. The length of the initial notch is $a_0 = 20mm$ and it is declined $\beta = 45^0$ with respect to the beam axis, which causes the mixed mode condition.
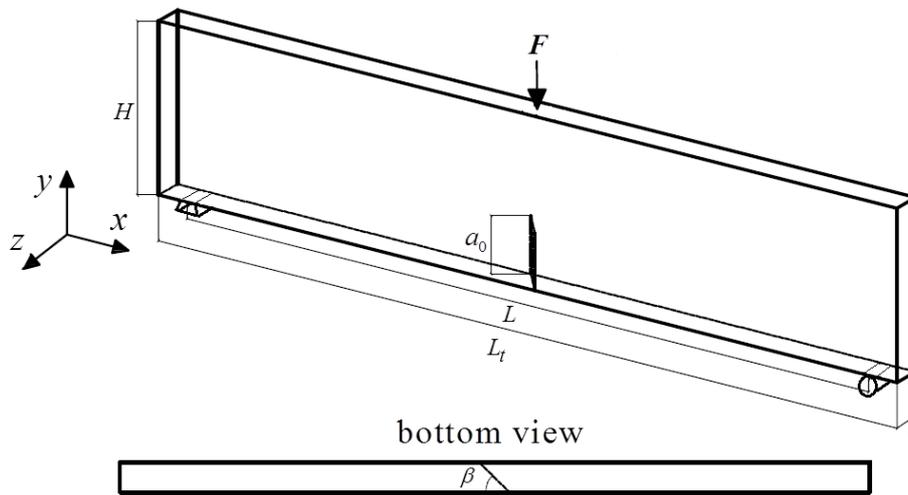

Figure 6 – Three-point bending beam with an initial kinked notch

The three-point bending condition is imposed as a prescribed traction boundary condition over a small area, such that the resultant force is equal to $F = 2kN$. The same problem was analyzed experimentally and numerically by Citarella & Buchholz (2008). The elastic constants of the problem are $E = 70.656\,kN/mm^2$ and $v = 0.34$. The DBEM numerical model created in the present study is composed of 7453 boundary elements of linear order of approximation and resulted in 10012 collocation points. The mesh of the numerical model is illustrated in Fig. 7. The crack surface refinement is evident. Thus, accurate SIF results are expected.
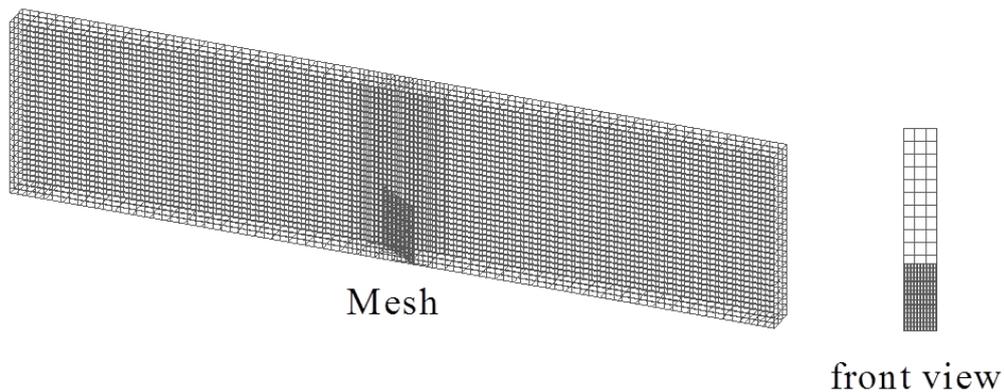

Figure 7 – Mesh utilized into the discretization of the problem

In this mixed mode example, it was possible to study the K_I, K_{II} and K_{III} SIF responses. The numerical reference solution for this problem was obtained by Citarella & Buchholz (2008) utilizing the commercial boundary element software BEASY, which is based on the DBEM formulation and adopt the J-integral together with techniques to decompose the SIF into the basic modes of fracture. This results are also in agreement with the ones obtained by the same authors utilizing the software IDEAS, which is based on finite elements and adopts the modified crack-closure

integral to assess the SIF. In the present work, both the DET and the developed DFT were adopted to assess the mixed mode SIF. Once again the SIF assessment was performed with the DFT utilizing 7 internal points in the plane $\varphi = 0$ for the displacement evaluation. These points were equally distributed in a circular path around the crack front with arbitrary radius r, in the interval of the arc $\theta = \left[-3\pi/4, \, 3\pi/4\right]$. Evaluation distances equal to $r = 0.1mm$, $r = 0.3mm$ and $r = 0.5mm$ were adopted for the DFT.

The SIF results are present in terms of normalized SIF as follows: $K_i norm = K_i 2H^2 t/\left(3L\sqrt{\pi a_0}\right)$. The SIF computed along the crack front, through the thickness of the beam, with the DET and with the DFT are presented in Fig. 9. The first graphic in Fig. 9 presents the SIF responses obtained with the DEF. The other graphics present the SIF obtained with the DFT for the different displacement evaluation distances. The reference solutions from by Citarella & Buchholz (2008) are also illustrated in all graphics.
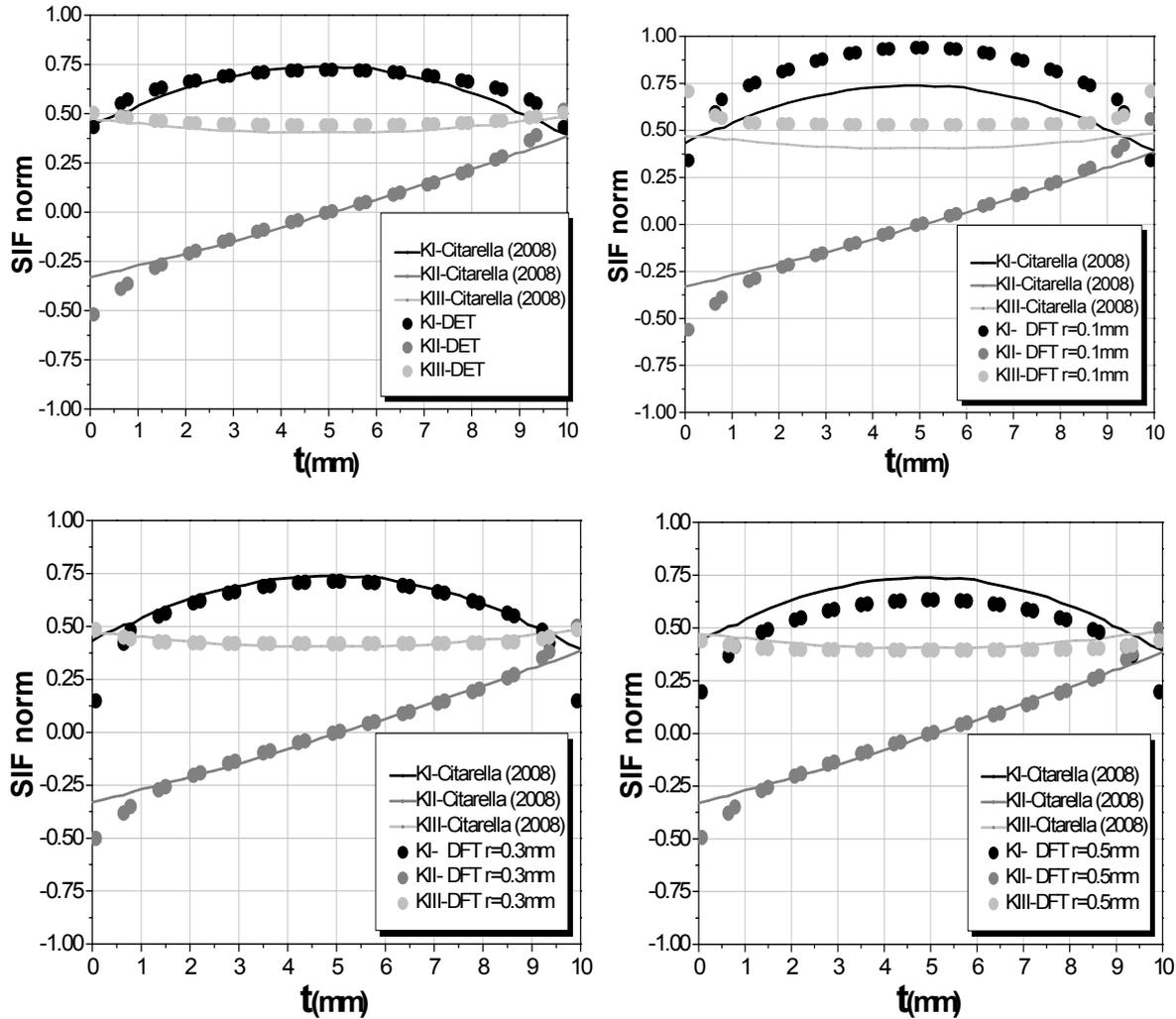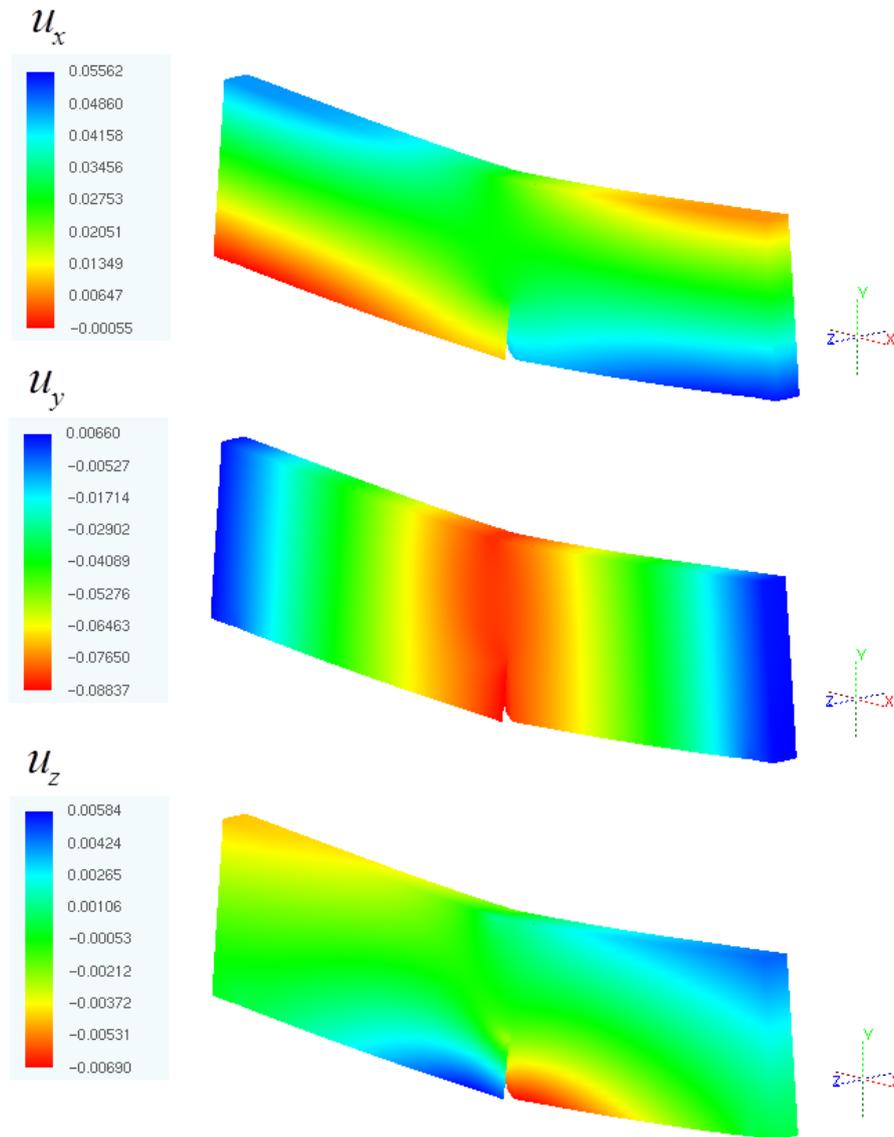

Figure 9 – SIF results along the dimension t of the beam

Front Fig. 9 one observes that the SIF responses obtained with the DET fit quite-well the reference solutions. The DFT technique is also capable of reproducing satisfactory SIF responses. However, the reported DFT responses resulted very sensible with respect to the evaluation distances. Such sensitivity was also observed in the previous application. r. Finally, Fig. 10 illustrates the displacement numerical results obtained in the present work. The deformed shape is presented considering the displacements 100 times magnified.

Figure 10 – Displacements numerical results

Fig. 9 shows that the SIF responses obtained with the DET fit quite-well the reference solutions. The DFT technique is also capable of reproducing satisfactory SIF responses. However, the reported DFT responses resulted sensible with respect to displacement evaluation distance.

CONCLUSIONS

This work presented an extension of the Displacement Fitting Technique (DFT) to assess Stress Intensity Factors (SIF) in three-dimensional linear elastic fracture mechanics problems using the Boundary Element Method (BEM). The higher order terms of the asymptotic near crack front William´s solution were accounted into the correlation scheme, and resulted in more accurate SIF results compared with first order correlation techniques. However, the numerical SIF responses obtained with the DFT resulted very sensible with respect to the evaluation distance r , when the evaluation points are equally distributed in a circular path of radius r around the crack front, as evaluated in this study. Due to the lack of domain discretizations provided by the BEM, the number and location of the internal near crack front points for displacement evaluation can be arbitrarily chosen. To explore this advantage, such as explored in two-dimensional crack problems by Gonzalez et al. 2015 , is the objective of future studies.

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