

Exact curvature computation of large displacements of beams

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Abstract: The use of highly flexible wings can lead to future long-range, fuel-efficient airplanes. This type of aircrafts is known as HALE (High-Altitude Long-Endurance), whose wings are known to undergo large flexural displacements, due to their relatively small stiffness. As it is important to evaluate these wings deformed shapes, we present a simple numerical solution by direct Runge-Kutta integration of the exact Euler-Bernoulli differential equation of the Elastica, implemented using ode45 function of MatLab. The model is a horizontal cantilever prismatic beam made of isotropic uniform linear elastic material. An excellent agreement with large-scale Finite Element models was obtained, with considerably lower computational cost, indicating a possible alternative numerical approach.

Keywords: HALE aircraft; Euler-Bernoulli elastica; Runge-Kutta; Finite Element Model.

INTRODUCTION

There is a desire for creating lighter and effective aircraft designs in order to reduce fuel consumption. One of the ways to achieve this goal is the use of light flexible materials that can undergo large deformations without exceeding their elastic limit. That has led to interest in analyzing that kind of structures subjected to static and dynamic loads (Cesnik et al, 2002). In aeronautical engineering, we mention the construction of high-aspect-ratio wings, whose geometry experiments large deformations during normal operating loads, described by geometrically nonlinear theories, whose solutions are critical challenges. According to Patil and Hodges (2000), these aircrafts are being considered for unmanned reconnaissance missions, environmental sensing, long-term surveillance and communications relay. Their slender wings present an aspect ratio of the order of 35. Due to their high flexibility, large geometric nonlinear deflections occur - up to 25% of wing semi-span, which makes necessary the application of exact curvature theory.

That is the purpose of our study, that is, introduce a simple numerical Matlab implementation of Runge-Kutta 4th order method, to integrate Euler's exact curvature differential equation to evaluate those large flexural displacements, as an *alternative* to fine mesh Finite Element models.

Background

The development of nonlinear beam theories started in the 18th century, by Jacob and Johann Bernoulli and Leonhard Euler. The first published work regarding deformation of flexible members was authored by Euler around 1744 in the appendix of his *De Curvis Elasticis* (Oldfather, Ellis and Brown; 1933). According to him, when a member is subjected to bending, it is not possible to neglect the slope of the deflection curve in the expression of the curvature - unless the deflections are small (Fertis, 2006).

It was also Euler who first introduced the law that states that the bending moment of a beam is proportional to the curvature produced by the action of load (Fertis, 1999), as shown in Equation (1). Thus, the Euler-Bernoulli law states that the bending moment function M is proportional to the curvature produced by the action of the load, as shown in Eq. (1), where R is the radius of curvature, θ is the slope at any point x , E is the modulus of elasticity and I is the cross-sectional moment of inertia. Equation (1) can be rewritten as Eq. (2), a second-order nonlinear differential equation showing that the deflection of a member is a nonlinear function of the bending moment. Its exact solution is quite difficult to obtain, as the principle of superposition can no longer be applied. For small deformations, it is usually assumed that $x = x_0$; hence, $L = L_0$. On the other hand, for large deformation (Figure 1), L is not L_0 (Patil and Hodges, 2000).

$$\frac{1}{R} = \frac{d\theta}{dx} = \frac{M}{EI} \quad (1)$$

$$\frac{1}{R} = \frac{y''}{[1 + (y')^2]^{\frac{3}{2}}} = -\frac{M(x)}{EI} \quad (2)$$

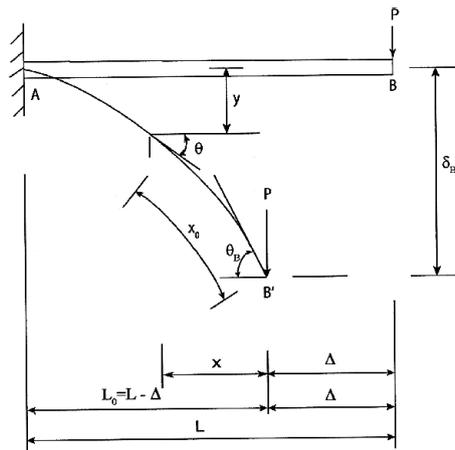


Figure 1 - Large deformation of cantilever beam with uniform cross section (Fertis, 2006)

Fertis (2006) claims that the difficulties in solving Eq.(2) for general loadings lead to popular use of the Finite Element Method, notable for its flexibility to account for complex domains and loading. Classical solutions for particular cases related to stability are to be found in Timoshenko (1961), for example. On the other hand, some other difficulties have arisen, such as the representation of rigid body motions of oriented bodies subjected to large deformation, which justifies why there is little analytical research on both material and geometric nonlinear behavior of flexible structures. Along the years, significant work has been done by Fertis (1999) and C.T. Lee (1991) on the nonlinear analysis of flexible bars, by using simplified nonlinear equivalent systems, for general nonlinear behavior of both prismatic and non-prismatic members.

Wang (1968; 1969) studied the deflection of both cantilever and simply support beams with constant cross-section under uniformly distributed load and concentrated loads, applying numerical methods (Runge-Kutta and Simpson's Rule) and obtaining good correlation with exact solutions. Figure 2 displays the comparison between linear and nonlinear theory for a uniformly distributed load. Prathap and Varadan (1976) studied the nonlinear deformation of a uniform cantilever beam of rectangular cross-section with concentrated load at the free end, considering nonlinear analysis for a Ramberg-Osgood material. Figure 3 show the comparison between linear material under linear theory and under nonlinear theory analysis.

Lo and Gupta (1978) also worked on the same problem as Prathap and Varadan (1976) but used the logarithmic strain definition for the regions where the material was stressed beyond its elastic limit (Fertis, 2006), obtaining the horizontal coordinate of beam tip within 1% error compared with the Euler-Bernoulli theory studied by Bishop and Drucker (1945). Changizi and Stiharu (2006) analysed the differences between linear and nonlinear analysis of a microcantilever beam's deflections for distributed load (Figure 4) and point load, asserting that nonlinearity has a very wide effect at the micro level and as effects grows exponentially in the micro level, it becomes hard to predict in which dimensions would be better to apply linear or nonlinear analysis. In this particular case, it is recommended to run nonlinear analysis in order to avoid errors larger than 10% (Changizi and Stiharu, 2006).

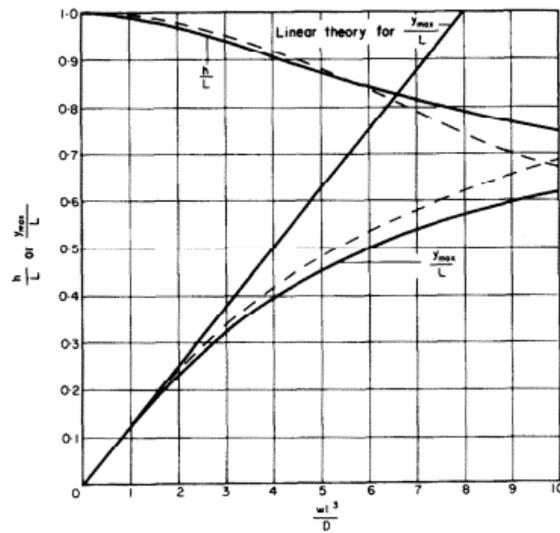


Figure 2 - Variation of $\frac{y_{max}}{L}$ and $\frac{h}{L}$ with $\frac{wL^3}{D}$, where D is the flexural rigidity, of cantilever beam under uniformly distributed load (Wang, 1969)

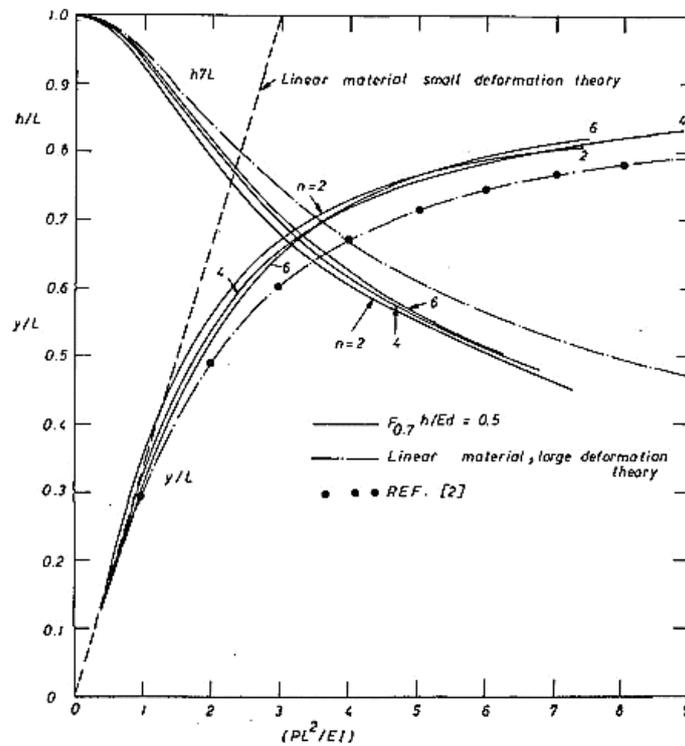


Figure 3 - Tip deflections for uniform cantilever beam under point load at free end (Prathap and Varadan, 1976)

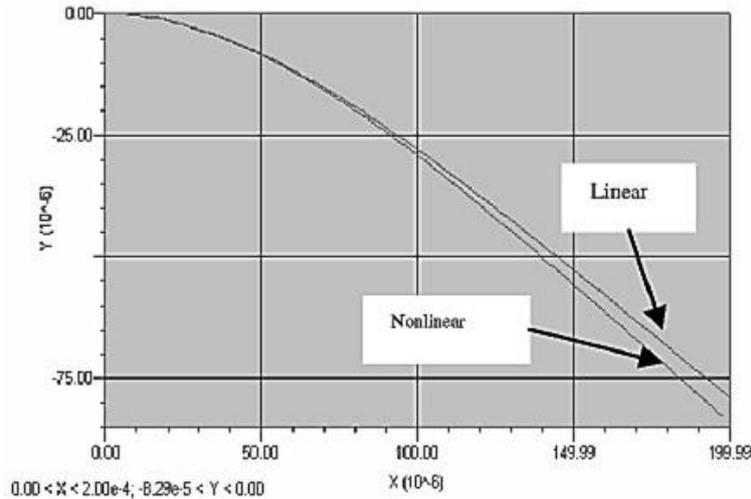


Figure 4 - Nonlinear/Linear deflection of the microcantilever under distributed force (Changizi and Stibaru, 2006)

As already said, closed form solutions of Eq. (2) for general loads is a very challenging issue. Due to this, we propose to write Eq. (2) as the set of two first order nonlinear ODE of Eqs. (4), using the substitution of variables of Eqs. (3),

$$x_1 = y \tag{3}$$

$$x_2 = y'$$

where

$$x_1' = x_2 \tag{4}$$

$$x_2' = \frac{M(x)}{EI} [1 + (x_2)^2]^{\frac{3}{2}}$$

Equations (4) are well suited to be numerically integrated using regular Runge-Kutta 4th and 5th order algorithms.

Computational Simulations

This work presents a geometrically nonlinear analysis of an isotropic uniform linear elastic material structure, a slender cantilever beam modeling of a high-aspect-ratio wing. The model adopted for this work is a horizontal prismatic cantilever beam, with a rectangular tube-shaped cross-section (Figure 5). For all examples, it will be considered $L = 1m$, beam made of aluminum ($E = 73,1GPa$), and $I = 4.92E - 10$. All finite element models are designed with beam elements, CBEAM type, which includes extension, bending (in two perpendicular planes), torsion and associated shear. This type of element supports also the effect of taper on transverse shear stiffness and of cross-sectional warping on torsional stiffness. For modeling issues, CBEAM allows the definition of cross-sectional properties at both ends, and the neutral axis does not need to coincide with the shear center. Moreover, CBEAM permits the application of either concentrated and/or distributed loads along the beam.

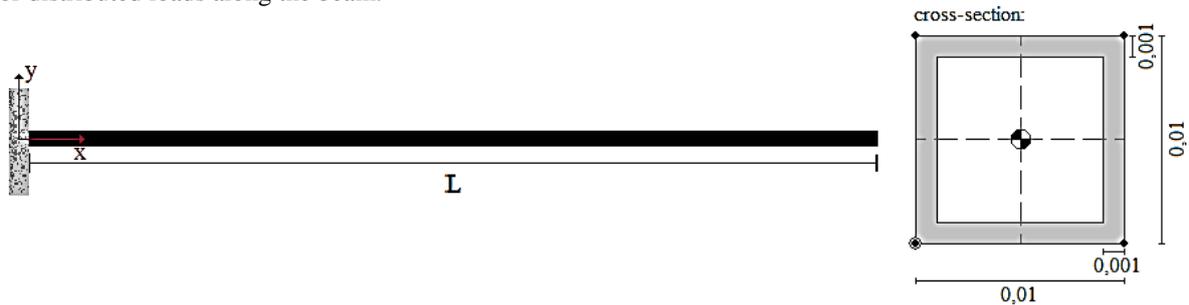


Figure 5 – Horizontal prismatic cantilever beam with rectangular-tube type cross-section.

First example: Parabolic distributed load

The first example considers the model of Figure 5, subjected to a distributed load q given by Eq. (5), which corresponds to a resultant force of 100N:

$$q(x) = 150 \left(\frac{x^2}{L^2} - 1 \right) \tag{5}$$

This load was implemented using ODE45 MatLab function with Eqs. (4) and in a finite element model (FEMAP), using 100 elements of type “beam” (CBEAM) with two nodes, accounting for large displacements with the arc-length

method, with Full Newton-Raphson (strategy of Modified Riks). The bending moment equation for this case is displayed in Eq. (6), while the results and comparison of both methods are displayed in Figure 7 and Table 1:

$$M(x) = Rx - M_R - 150 \left(\frac{x^2}{2} - \frac{x^4}{12L^2} \right) \tag{6}$$

where R and M_R are the support reaction force and moment, respectively, given by Eq. (7):

$$\begin{aligned} R &= 100L \\ M_R &= 37.5L^2 \end{aligned} \tag{7}$$

A convergence analysis was conducted to check model’s accuracy. The criterion for choosing the best number of elements was to reach 0.03% difference between the deflection found for previous analysis and for the current one (Figure 6). This criterion was met with a mesh of 100 elements, mesh density also applied for the next examples of this study. Through this convergence analysis, it is possible to assure that only one element is not possible to conduct the nonlinear analysis – reason why it was necessary to increase mesh density.

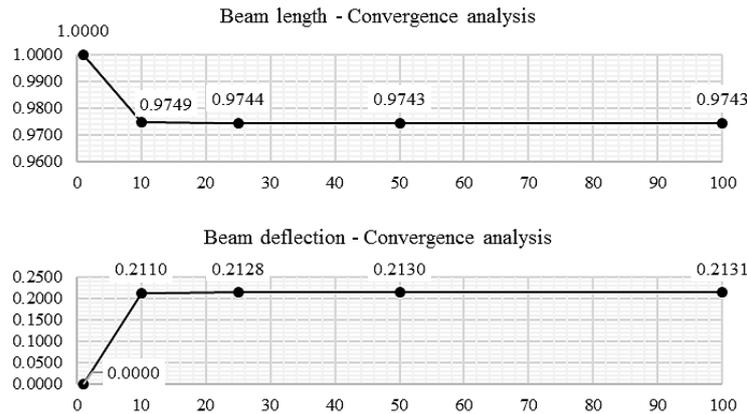


Figure 6 - Convergence analysis for finding most accurate mesh density.

Table 1 - Comparison with deflections in vertical direction and final position by applying MatLab ode45 and FEM with NASTRAN for parabolic distributed load.

	Y[m]	% difference	X [m]	% difference
MatLab ode45	0.2173		0.9650	
NASTRAN	0.2131	1.9318%	0.9743	-0.9582%

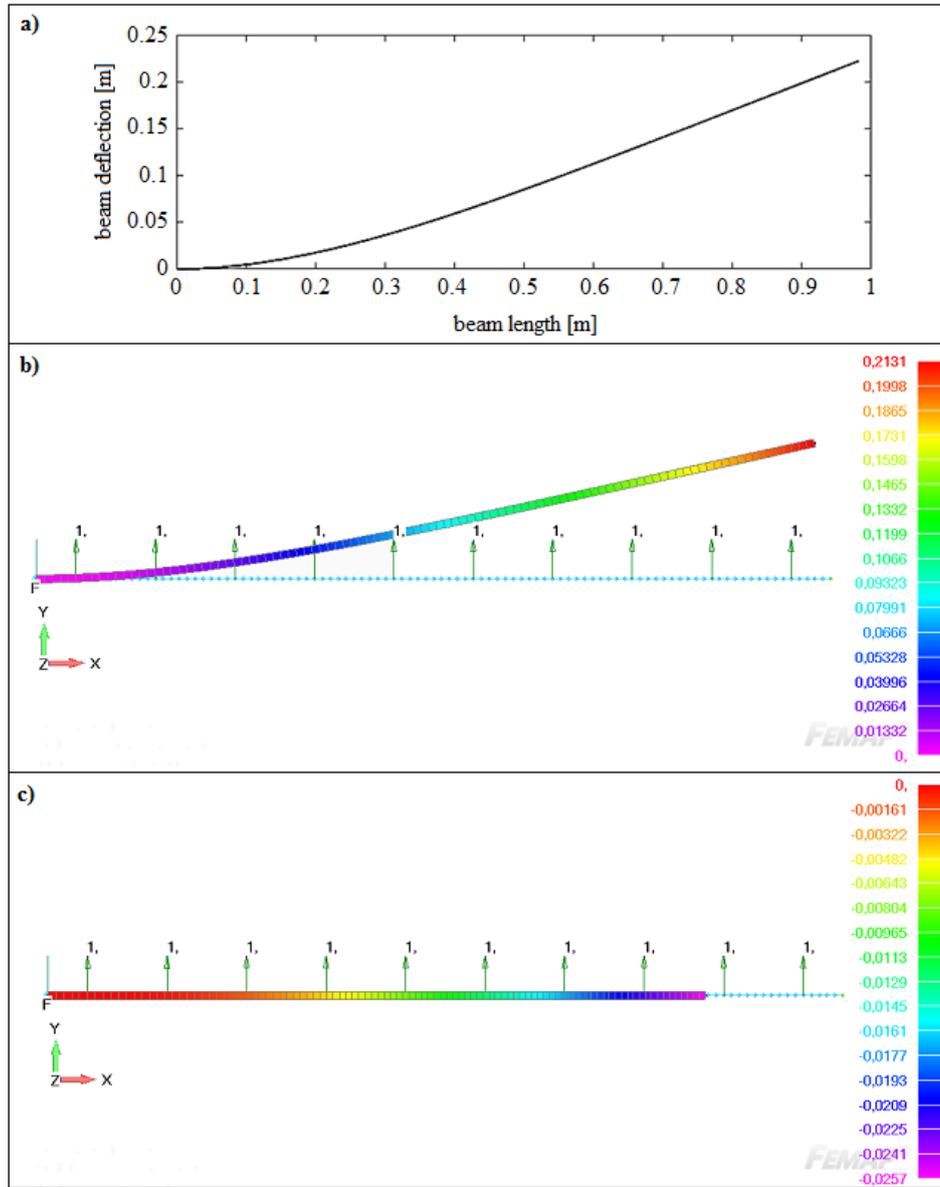


Figure 7 - Deformed beam under parabolic distributed load given by a) MatLab ode45, b) NASTRAN Y direction and c) NASTRAN X direction.

Second example: cosine distributed load

This example also considers the model of Figure 5, but the distributed load q is now given by Eq. (8), which in turn also corresponds to a resultant force of 100N:

$$q(x) = 157.0796 \cos\left(\frac{\pi x}{2L}\right) \quad (8)$$

This load was implemented using ODE45 MatLab function with Eqs. (4) and in a finite element model (FEMAP), using 100 elements of type “beam” (CBEAM) with two nodes, accounting for large displacements with the arc-length method, with Full Newton-Raphson (strategy of Modified Riks). Convergence was checked as previously described. The bending moment equation for this case is displayed in Eq. (6), while the results and comparison of both methods are displayed in 7 and Table 2:

$$M(x) = Rx - M_R - 157.0796 \left(\frac{2xL}{\pi} \sin(\theta) - \frac{2L}{\pi^2} \{ \pi x \sin(\theta) + 2L[\cos(\theta) - 1] \} \right) \quad (9)$$

where R and M_R are the support reaction force and moment, respectively, given by Eq. (10):

$$R = \frac{314.1592L}{\pi} \quad (10)$$

$$M_R = 314.1592(\pi - 2) \left(\frac{L}{\pi}\right)^2$$

$$\theta = \frac{\pi X}{2L}$$

Table 2 - Comparison with deflections in vertical direction and final position by applying MatLab ode45 and FEM with NASTRAN for cosine distributed load.

	Y[m]	% difference	X [m]	% difference
MatLab ode45	0.2062		0.9655	
NASTRAN	0.2034	1.3591%	0.9766	-1.1525%

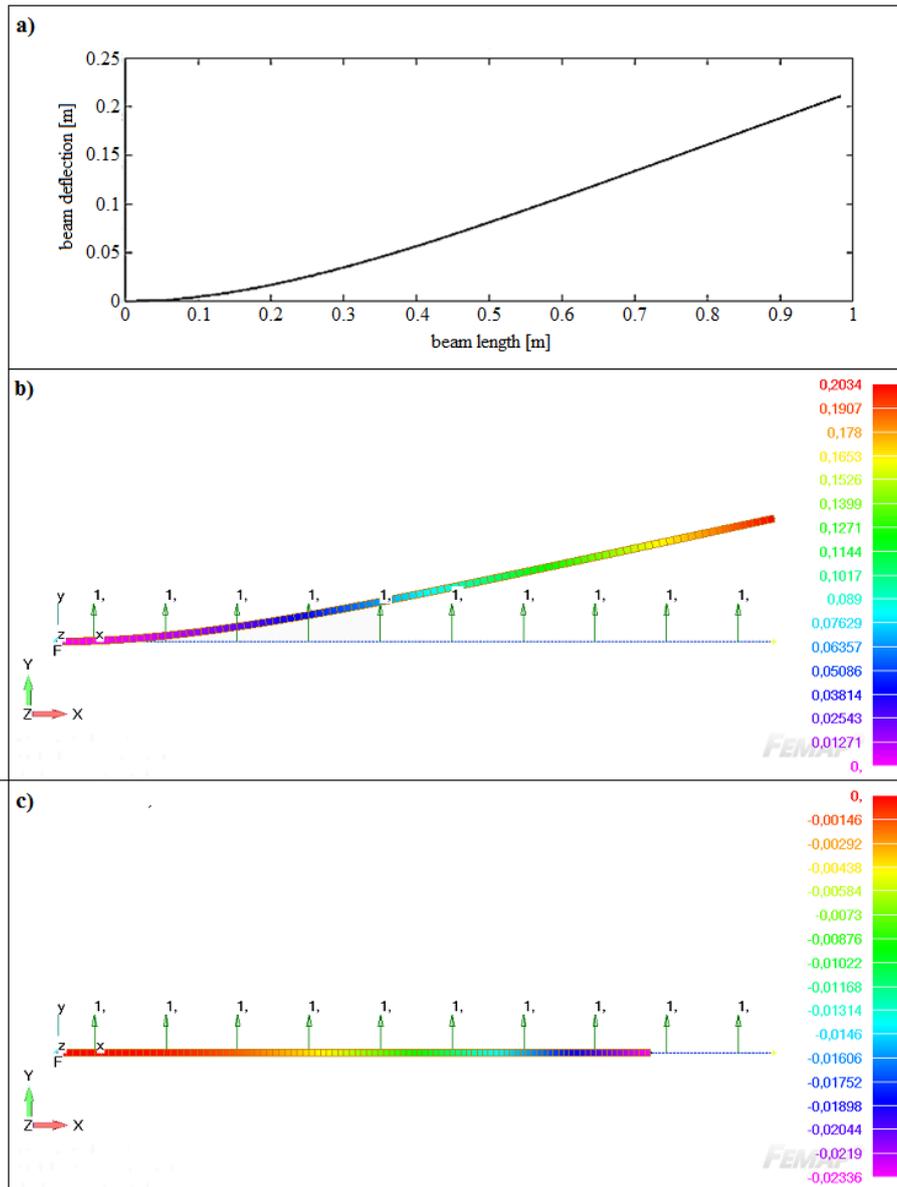


Figure 8 - Deformed beam under cosine distributed load given by a) MatLab ode45, b) NASTRAN Y direction and c) NASTRAN X direction.

Third example: equivalent lumped load

Still considering the model of Figure 5, it is adopted now a concentrated load of 100N, implemented using ODE45 MatLab function with Eq. (4) and in a finite element model (FEMAP), using 100 elements of type “beam” (CBEAM) with two nodes, accounting for large displacements by the arc-length method, with Full Newton-Raphson (strategy of

Modified Riks). Convergence was checked as previously described. The results are displayed in Figure 9 and Table 3. The reactions and bending moment equation are not displayed, as they are quite obvious in this case.

Table 3 - Comparison with deflections in vertical direction and final position by applying MatLab ode45 and FEM with NASTRAN for concentrated equivalent load.

	Y[m]	% difference	X [m]	% difference
MatLab ode45	0.1685		0.9718	
NASTRAN	0.1678	0.4002%	0.9847	-1.3272%

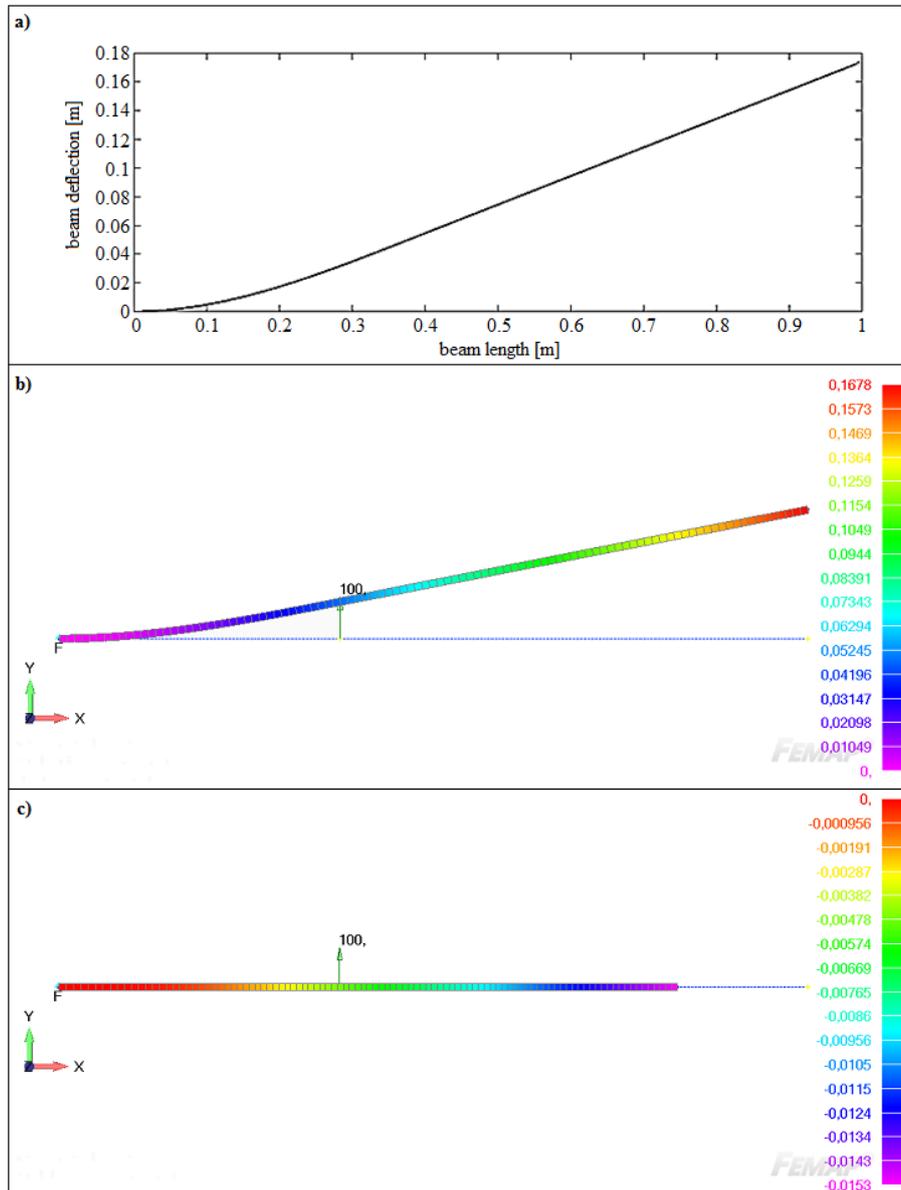


Figure 9 - Deformed beam under concentrated load given by a) MatLab ode45, b) NASTRAN Y direction and c) NASTRAN X direction.

CONCLUSIONS

The results obtained with both methods present good correlation and good representation of the present geometric nonlinearity, leading to displacements different of those of linear theories. In addition, this work indicates that in practice it is possible to apply either parabolic distributed load or cosine distributed load, as both loadings lead to very similar values of deflection and displacements. Hence, the MatLab algorithm seems to be an alternative more efficient in terms of computational cost than the finite element solution provided by NASTRAN, whose convergence was checked.

ACKNOWLEDGMENTS

The authors acknowledge support by CAPES, CNPq and FAPESP, all Brazilian research funding agencies.

REFERENCES

- Bishop, K. E.; Drucker, D. C. Large Deflections of Cantilever Beams. **Quarterly of Applied Mechanics**, v. III, n. 3, p. 272-275, April 1945.
- Cesnik, C. E. S. Et Al. X-HALE: A Very Flexible Unmanned Aerial Vehicle For Nonlinear Aeroelastic Tests. **AIAA Journal**, Michigan, V. 50, N. 12, p. 2820-2834, December 2012.
- Changizi, M. A.; Stiharu, I. Nonlinear versus Linear Deflection Analysis of Microcantilever. **IEEE ISIE**, Montréal, p. 3322-3327, July 2006.
- Fertis, D. G. **Nonlinear Mechanics**. 2. Ed. Boca Raton: Crc, V. 1, 1999.
- Fertis, D. G. **Nonlinear Structural Engineering: With Unique Theories and Methods to Solve Effectively Complex Nonlinear Problems**. 1. ed. Berlin: Springer-Verlag Berlin Heidelberg, v. 1, 2006.
- Fertis, D. G.; Lee, C. T. Inelastic Analysis of Prismatic and Nonprismatic Members with Axial Restraints. **Mechanics of Structures and Machines: An International Journal**, Ohio, v. 19, n. 3, p. 357-383, July 1991.
- Lo, C. C.; Gupta, S. D. Bending of a nonlinear rectangular beam in large deflection. **Journal of Applied Mechanics - ASME**, v. 45, p. 213-215, March 1978.
- Oldfather, W. A.; Ellis, C. A.; Brown, D. M. **Leonhard Euler's Elastic Curves**. 1. ed. Chicago: The University of Chicago Press, v. 20, 1933.
- Patil, M. J.; Hodges, D. H. On The Importance Of Aerodynamic And Structural Geometrical Nonlinearities In Aeroelastic Behaviour Of High-Aspect-Ratio Wings. **41st AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, And Materials Conference**, Atlanta, April 2000. 11.
- Prathap, G.; Varadan, T. K. The inelastic large deformation of beams. **Journal of Applied Mechanics**, v. 43, n. 4, p. 689-690, February 1976.
- Timoshenko, S.P.; Gere, J.M.. **Theory of Elastic Stability**, 2nd Ed. McGraw-Hill. New York, 1961.
- Wang, T. M. Non-linear Bending of Beams with Concentrated Loads. **Journal of the Franklin Institute**, v. 285, n. 5, p. 386-390, May 1968.
- Wang, T. M. Non-linear Bending of Beams with Uniformly Distributed Loads. **International Journal of Non-linear Mechanics**, v. 4, n. 4, p. 389-395, December 1969.

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