Approximations using symbolic algebra coupled with Poincaré-Lindstedt method: some applications

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Abstract: Dynamical problems are governed by initial value problems (IVP), often nonlinear, that must be solved to understand the dynamical features of a problem. Numerical methods are very efficient and provide approximations with the precision required, if one is interested in solving a specific problem. Unfortunately, numerical approximations do not provide the insight necessary to understand how a solution depends on the parameters of the problem. Sometimes, perturbation methods help in the sense that they can provide an analytical approximation that shows how the parameters influence the solutions. However to solve a problem for large time intervals require high order approximations that are cumbersome to derive. This paper uses a symbolic method to derive approximation of an IVP using Poincaré- Lindstedt method. The resulting linear problems are combined to have several orders of approximations. The approximations are compared to understand their domain of validity. As a reference to estimate the quality of an approximation, a Runge-Kutta method is used for a specific value of the parameters, of course. To show the main features of the methodology, it is applied to a non-damped Duffing equation, the simplest nonlinear problem used in Mechanics. It is computed analytical approximations of displacement, velocity, and frequency, for any order of approximation and initial conditions desired by the user. To quantify how the order of approximation affect the results, the obtained analytical approximations are evaluated for different combinations of parameters values. The results show that the number of terms has a great influence in the accuracy of the approximation, specially when the term that controls the nonlinearities grows.

Keywords: Symbolic method, analytical approximations , Perturbation method.

INTRODUCTION

Dynamical systems are governed by initial value problems (IVP), which are, usually, nonlinear and frequently do not have a known solution (see Abarbanel and Sushchik (1993)). This article deals with two different approximation methods to nonlinear IVP solution, an analytical and a numerical method. Numerical methods are very efficient and provide approximations with the precision required, but it does not provide the needed insight to understand how a solution depends on the problem parameters. Alternatively, analytical approximations can often be found, particularly, if nonlinearity is relatively small, Wagg and Virgin (2012).

Perturbation methods are a powerful technique for calculating analytical approximations, see Rao (1995), Simmonds and Mann (1986), Wagg and Virgin (2012). The perturbation method chosen to be analyzed in this paper is the Poincaré-Lindstedt method, which gives an uniformly valid asymptotic expansion for periodic solutions of weakly nonlinear oscillations, He (2002). This method assumes that the nonlinear IVP solution is a power series of the perturbation parameter, which will be introduced in the nonlinear equation. To compute an analytical approximation, this series should be truncated according to the number of terms desired. As a reference to estimate the quality of an approximation, a Runge-Kutta method is used for a specific value of the parameters. An cumulative error was determined to quantify the accuracy of the approximations. Afterwards, we analyzed how the number of terms influences the analytical approximations domain of validity. This contribution presents the Poincaré-Lindstedt methodology by applying it to the non-damped Duffing equation, the simplest nonlinear problem used in Mechanics, more details in Kovacic and Brennan (2011).

ANALYTICAL APPROXIMATION TO DUFFING EQUATION

To explain the Poincaré-Lindstedt method, we will apply it to the non-damped Duffing equation,

$$\ddot{u}(t)+\omega_0^2\,u(t)+\varepsilon\,u^3(t)=0, \tag{1}$$

where u is the displacement, ε is the parameter that controls the nonlinearity and $\dot{\square}$ represents the derivation in relation to t. The initial conditions are $u(0)=A_0$ and $\dot{u}(0)=0$. Since this method is used to compute periodic approximations, we define a new non-dimensional parameter τ as

$$\tau=\omega\,t\,, \tag{2}$$

where the angular frequency, ω, is function of the initial amplitude, A_0, given by

$$\omega(A_0) = \omega_0 + \varepsilon\,\omega_1(A_0) + \varepsilon^2\,\omega_2(A_0) + \ldots + \varepsilon^N\,\omega_N(A_0)\,, \tag{3}$$

and each ω_i, $i = 1,2,3,..,N$ is also function of A_0 and them are chosen to avoid secular terms in the analytical approximation, i.e, terms that increase with t. Applying Eq. (2) into (1), we obtain

$$\omega^2\,\hat{u}''(\tau) + \omega_0^2\,\hat{u}(\tau) + \varepsilon\,\hat{u}(\tau)^3 = 0\,, \tag{4}$$

where $\square'$ represents the derivation in relation to τ. The solution of equation (4) is considered to be a power series of ε, given by

$$\hat{u}(\tau) = \hat{u}_0(\tau) + \varepsilon\hat{u}_1(\tau) + \varepsilon^2\hat{u}_2(\tau) + \varepsilon^3\hat{u}_3(\tau) + \ldots + \varepsilon^N\hat{u}_N(\tau), \tag{5}$$

where each $\hat{u}_i$, $i = 0,1,2,3,..,N$ is an unknown function to be determined from the governing equations and the initial conditions. To calculate an approximation to the solution, Eqs. (3) and (5) should be truncated according to the order desired. The truncated equations should be replaced into Eq. (4). After, to form a linear family of IVP, the terms should be grouped according to the power of ε. To exemplify, we show the linear family of IVP formed considering an approximation of third order. This family is given by

$$\omega_0^2\hat{u}_0'' + \omega_0^2\hat{u}_0 = 0\,, \tag{6a}$$
$$\omega_0^2\hat{u}_1'' + \omega_0^2\hat{u}_1 = -\,\hat{u}_0^3 - 2\omega_0\omega_1\hat{u}_0''\,, \tag{6b}$$
$$\omega_0^2\hat{u}_2'' + \omega_0^2\hat{u}_2 = -\,\omega_1^2\hat{u}_0 - 2\omega_0\omega_1\hat{u}_1'' - 2\omega_0\omega_2\hat{u}_0'' - 3\hat{u}_1\hat{u}_0^2\,, \tag{6c}$$
$$\begin{aligned}\omega_0^2\hat{u}_3'' + \omega_0^2\hat{u}_3 = &-\,\omega_1^2\hat{u}_1'' - 2\omega_0\omega_1\hat{u}_2'' - 2\omega_0\omega_2\hat{u}_1'' - 2\omega_0\omega_3\hat{u}_0'' - 2\omega_1\omega_2\hat{u}_0'' \\ &- 3\hat{u}_0\hat{u}_1^2 - 3\hat{u}_0^2\hat{u}_2\,.\end{aligned} \tag{6d}$$

with initial conditions $\hat{u}_0(0) = A_0$, $\hat{u}_0'(0) = 0$ and $\hat{u}_i(0) = \hat{u}_i'(0) = 0$, $i = 1,2,3$. The Eqs. (6) should be solved hierarchically, since the high order equations are dependent of the lower order ones. So, Eq. (6a) is the first one to be solved as homogeneous equation, with solution $\hat{u}_0(\tau) = A_0\cos\tau$. Substituting it into Eq. (6b), we obtain

$$\omega_0^2\,\hat{u}_1'' + \omega_0^2\,\hat{u}_1 = \left(2A_0\,\omega_0\,\omega_1 - \frac{3A_0^3}{4}\right)\cos\tau - \frac{A_0^3}{4}\cos 3\tau\,. \tag{7}$$

Since we are interested in periodic approximations, we must eliminate secular terms, i. e, terms that grow without bound as $\tau\to\infty$. Observe that the right side of this equation contains a term $\cos\tau$, which results in $\tau\sin\tau$, a secular term. To avoid this, the coefficients from this term should be matching to zero in a way that the term is eliminated. Doing this, we find $\omega_1 = \frac{3A_0^2}{8\,\omega_0}$. After the resonant terms elimination, the Eq. (7) is solved, finding $\hat{u}_1$ as

$$\hat{u}_1(\tau) = \frac{A_0^3}{32\,\omega_0^2}(\cos 3\tau - \cos\tau), \tag{8}$$

and subsequently replaced in Eq. (6c), which becomes

$$\begin{aligned}\omega_0\,\hat{u}_2'' + \omega_0\,\hat{u}_2 = &\left(\frac{\cos(3\tau)}{4} + \frac{3\cos(\tau)}{4}\right)\left(\frac{3A_0^5}{8\,\omega_0^2} + \frac{9A_0^3\,\omega_1}{4\,\omega_0}\right) \\ &+ \cos(\tau)\left(A_0\,\omega_1^2 + 2A_0\,\omega_0\,\omega_2 - \frac{7A_0^3\,\omega_1}{4\,\omega_0}\right) \\ &- \left(\frac{3A_0^5}{8\,\omega_0^2}\right)\left(\frac{5\cos(3\tau)}{16} + \frac{\cos(5\tau)}{16} + \frac{5\cos(\tau)}{8}\right).\end{aligned} \tag{9}$$

The same steps used to find $\hat{u}_1$ should be done to solve Eq. (9). Substituting all the terms $\hat{u}_i$ and ω_i calculated into Eqs. (5) and (3), respectively, we obtain an analytical approximation to $\hat{u}$ as function of τ. If desired, the approximation can be written as function of t.

Observing the Eq. (9), it is possible to verify that this equation has more terms that Eq. (7), consequently, to solve it is more irksome. This happens because we are solving an equation that corresponds to a higher order of ε. Table 1 shows the relationship between approximation order, quantity of equations in the family of IVP, number of terms in the right side of each equation and the total of terms. Looking at Tab. 1, we observe that the greater the approximation order, greater the quantity of equations to solve and total of terms. Since approximations with high order are cumbersome to solve by hand, a routine in MATLAB software using symbolic algebra was developed. This routine allows the computation of approximations to solution of u according to the order desired by the user.

Table 1: Relationship between approximation order, quantity of equations in the family of IVP, number of terms in the right side of each equation and the total of terms.

Approximation order	Quantity of equations in the family IVP	Quantity of terms in each equation	Total
1	2	0 2	2
2	3	0 2 4	6
3	4	0 2 4 7	13
4	5	0 2 4 7 11	24
5	6	0 2 4 7 11 15	39

DISCUSSION

The symbolic algebra routine implemented in MATLAB software allows the computation of approximations to the displacement, velocity and frequency, ω, with desired order and initial conditions. Another routine, also implemented in MATLAB, was developed to calculate numerical approximations to u through Runge-Kutta method, using the command ODE45. The obtained analytical approximations are evaluated for different combinations of parameters values and compared with the numerical approximation to understand their domain of validity. In Fig. 1, we can observe the analytical and numerical approximations curves for specific value during a interval of time. We select the first and the fifth order analytical approximation to compare with the numerical one, and it is possible to conclude that according to the time evolution the curves distance. For the first interval of T, both analytical approximation are coincident of the numerical one, while during the second interval, the first order and the numerical one begins to separate and the same happens with the fifth order in the third interval.

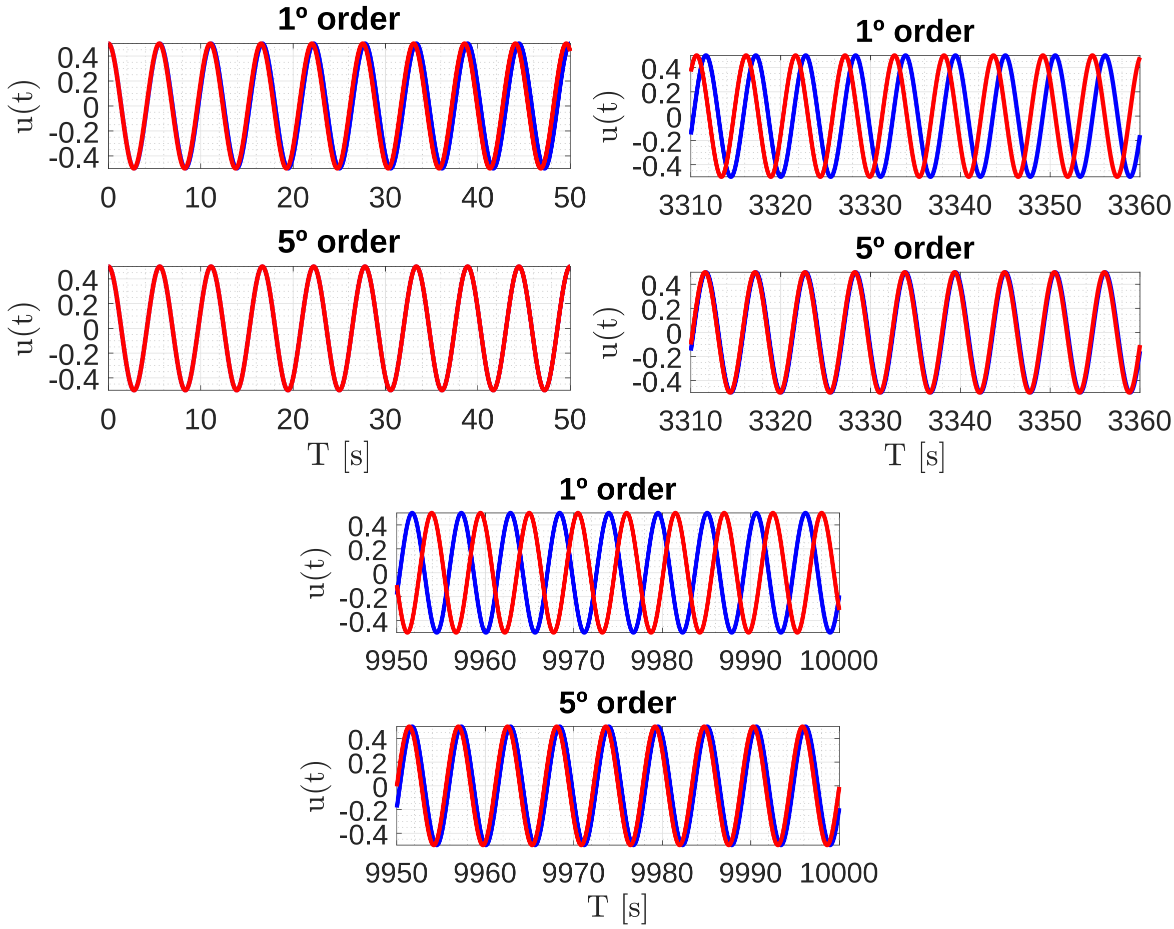


Figure 1: Numerical approximation, (red) and analytical approximation, (blue). Considering $\varepsilon = 1.5$, $\omega_0 = 1$, $A_0 = 0.5$, $0 < T < 10000$.

To improve the study about the domain of validity of the approximations, we created a measure of the noncumulative error, between the analytical and numerical approximations. The error is given by the module of the maximum difference between the curves during a time interval $[0, T]$ for fixed parameters values, that is

$$\max_{0<t<T} |u^{a_i}(t) - u^n(t)|, \tag{10}$$

where u^{a_i} is the analytical approximation of order i and u^n is the numerical approximation. Since system trajectory is periodic, their maximum amplitude has the same value of the initial displacement, A_0. Using this metric, there is a maximum error which happens when, simultaneously, one approximation are in the maximum displacement and the other are in the minimum, as exemplified by Fig. 2, therefore the maximum error value is $2A_0$.

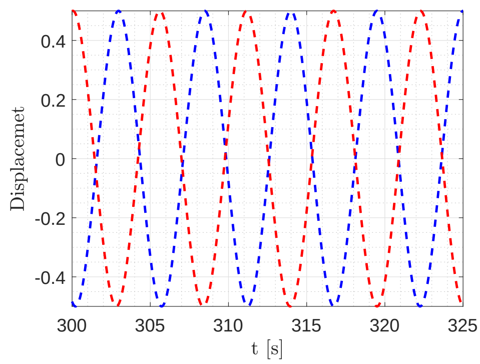
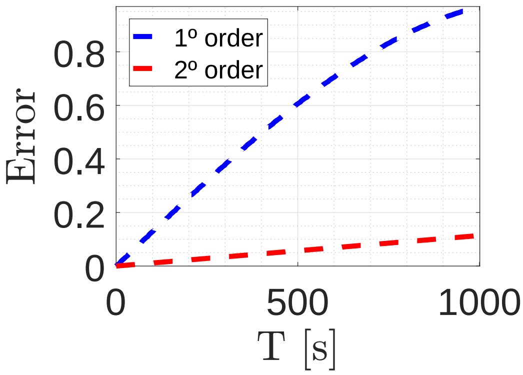
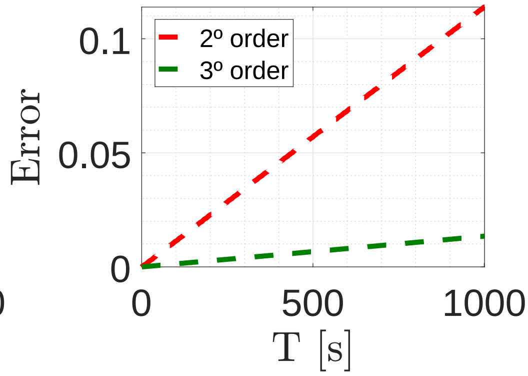


Figure 2: Example: error maximum in the displacement

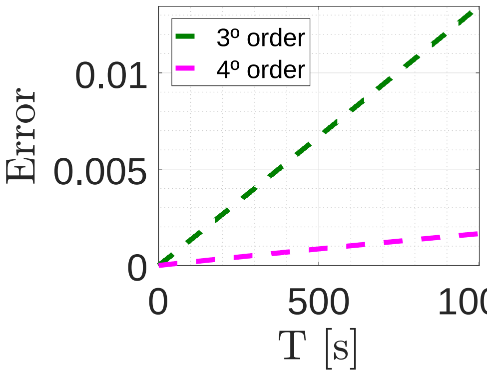
Figure 3a shows the approximation errors of first and second order as function of T. Observe that the first order approximation achieved the maximum error when $T \approx 1000$, while the second order approximation presents error around 0.1 when $T \approx 1000$. Figure 3 shows, as aforementioned, that increase the approximation order improves the domain of validity.



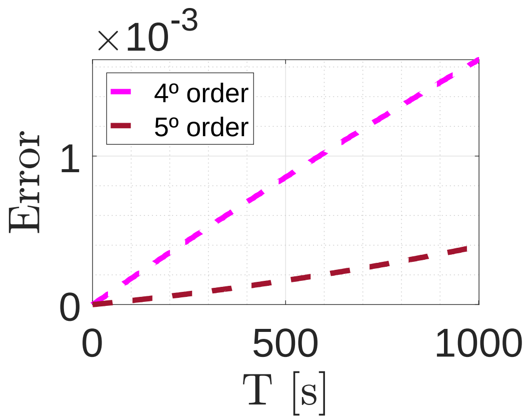
(a) Error between 1 and 2 order.



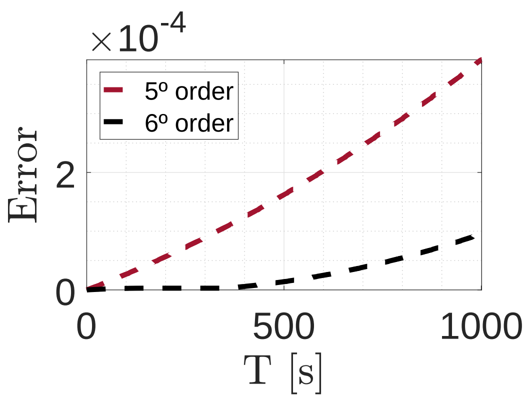
(b) Error between 2 and 3 order.



(c) Error between 3 and 4 order.



(d) Error between 4 and 5 order.



(e) Error between 5 and 6 order.

Figure 3: Noncumulative error between the analytical and numerical approaches, considering $\varepsilon = 0.75$, $\omega_0 = 1$ and $A_0 = 0.5$.

The next analyzes will be about the influence of ε, the parameter that controls the nonlinearity, and the initial condition of displacement, A_0, in the domain of validity of the analytical approximations.

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