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Analytical approximation of the uncertainty of an identification problem with chaos polynomials

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Abstract: We aim at taking into account measurement uncertainties in an identification process with errors in the measurements. We understand that uncertainty is the cumulative distribution function (CDF). In the case of a continuous distribution the probability distribution function (PDF) characterizes the CDF and we want to find an analytical approximation of the PDF. Uncertainties are taken into account by modeling them as random variables and the distribution of the identified parameter is an unknown of an inverse problem, which is a result of an optimization process. More precisely, to focus on this branch of optimization, we are concerned by a four point bending static test applied to a beam, and we model the identified elastic modulus with polynomial chaos expansions, using Hermite, and Chebyshev of 2nd kind, polynomials to see the influence of the polynomial used in the identification. Two different hypotheses are made about the uncertainties in the loading, leading to the investigation of two applications. This spectral theory for the quantification of uncertainties of this kind of identification problems is verified practically on both of these applications with successful results for an adequate order of truncation of the expansion basis.

Keywords: identification, inverse problem, quantification of uncertainty, polynomial chaos expansion

INTRODUCTION

Considering uncertainties becomes a major issue in current engineering design and in mechanical inverse problems. In this work, we consider the identification of the elastic modulus of a homogeneous beam which is subjected to a four points static bending test. But, if there is uncertainties in measurements, the parameter identification which results from the solution of an optimization problem shows that it has a statistical distribution that is not known *a priori*. Thus, we aim at taking into account some uncertainties throughout the identification process, in order to model the desired elastic modulus.

More precisely, the uncertainties are taken into account by modeling each of them as a random variable. The input random vector is then transformed into a random variable given as the output of the identification procedure, based on the minimization of a residual involving the input and the analytical model. Then, the compound model is represented by a chosen polynomial chaos expansion. Two simple examples are used to show the methodology, each considering different hypotheses about the uncertainty in the loading.

This study is organized as follows. The deterministic four point bending test is described and a model of it is presented in the first Section. In this section, the linear Euler-Bernoulli model for the static response of a beam in bending is introduced, taking into account the boundary conditions of the four point bending test. The deterministic optimization problem for the identification of the elastic modulus is presented, when the measurements, the test, and the analytical model, are free from errors, but this is not shown due to lack of space. Then, the problem is recast by considering there are uncertainties in the measured displacements as well as in the total load measurements. Uncertainties in the positions of the loads was also done but are not shown in this paper because they give no new results. To represent the results of this identification problem, the random elastic modulus, representations by polynomial chaos expansion are used. We then focus on the numerical strategies that we have used to determine the coefficients of the polynomial expansion, and the results are compared. Unfortunately, only parts of the results are shown.

As a control of the results we have used Monte Carlo simulations, but no discussion is presented. This methodology was used since it is standard and it is good enough for our proposes.

TOY PROBLEM: F OUR POINT BENDING TEST DESCRIPTION AND ANALYTICAL MODEL

The problem we want to model is a three dimensional body, a slab, say, but here we discuss a toy problem to simplify the description. Let a uniform homogeneous specimen be subjected to a static four-point bending test. It is a classical experiment in mechanics to determine the elastic modulus e of the specimen. This test consists in placing the beam specimen, of length l, supported by its ends, in contact with two pin, or roller supports, where two concentrated loads, q_1 and q_2, as sketched below, will act (see Figure 1). The distances between the left side support and the actuator pins are l_1

and l_2, respectively. This geometry is often used in material testing due to its convenience, as it avoids the need to clamp the specimen to the testing apparatus.

Modeling the beam as an one-dimensional object, and using the Euler-Bernoulli theory, we get the relationship between the beam's deflection and the applied load as:

$$e\, i_{yy} \frac{\partial^4 w}{\partial x^4} = q_1 \delta(x - l_1) + q_2 \delta(x - l_2) \qquad \text{for} \quad 0 < x < l$$

where $w(x)$ is the deflection of the beam in the transverse direction at some position x, $\delta(x)$ is the Dirac function and i_{yy} is the second moment of area of the beam's cross-section. It is calculated with respect to the axis which passes through the centroid of the cross-section being perpendicular to the applied loading. For the four-point bending test, boundaries conditions are:

$$\begin{cases} w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 & \text{for} \quad x = 0 \\ w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 & \text{for} \quad x = l \end{cases}$$

The application of two point loads at different locations leads to $w(x)$ being a sum of piecewise functions:

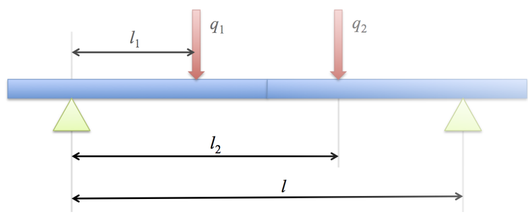

Figure 1 – Sketch of the four point bending test

$$w(x; e, q_1, q_2) = \frac{(l - l_1) q_1}{6 e\, i_{yy} l} \left(\frac{l}{l - l_1} \langle x - l_1 \rangle^3 - x^3 + l_1 (2l - l_1) x \right) + \frac{(l - l_2) q_2}{6 e\, i_{yy} l} \left(\frac{l}{l - l_2} \langle x - l_2 \rangle^3 - x^3 + l_2 (2l - l_2) x \right)$$

where $\langle \bullet \rangle$ are singularity brackets such that $\langle x - b \rangle^n = \frac{(x-b)^n}{n+1} H(x - b)$ for integer n, $n \geq 0$ and $H(\bullet)$ is the Heaviside step function: $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$.

The four point bending test is classically designed to have a symmetry, when one chooses $q_1 = q_2$ and $l_2 = l - l_1$.

Identification problem in presence of uncertainties

We are now interested in the identification results obtained when there are uncertainties in the displacement measurements and in the loading. Since the loading is driven to respect the requested load in the experiments, it is assumed that the error comes from a difference between the true loading and the load measurement. Then, denoting random variables by capital letters, the applied loads Q_1 and Q_2 are now modeled as random variables, such that:

$$Q_1 = \frac{1}{2}(\bar{q} + \xi_q) \qquad \text{and} \qquad Q_2 = \frac{1}{2}(\bar{q} + \xi_q)$$

where ξ_q is a random variable which follows an uniform law:

$$\xi_q \sim \mathcal{U}(-\delta_q, +\delta_q) \qquad \text{for} \qquad \delta_q = c_q \bar{q}$$

Moreover, the measured displacements vector becomes a random variable vector $\mathbf{W}_{\text{meas}}$ such that:

$$\mathbf{W}_{\text{meas}} = \mathbf{w}(\mathbf{x}; e_{\text{true}}, Q_1, Q_2) + \xi_w$$

where ξ_w is a random vector which follows a multi-normal law:

$$\xi_w \sim \mathcal{N}_5(0, \sigma_w) \qquad \text{for} \qquad \sigma_w = c_w \max(|\mathbf{w}_{\text{meas}}|),$$

assuming the noise in measurements have no correlation. Hence, the coefficients of variation c_q and c_w enable to drive the level of noise in measurements.

Then, assuming the known information is identical to the one of the deterministic situation, we recast the identification problem as:

$$E^* = \arg_{e \in \mathbb{R}^+} \min \left(\sum_{k=1}^{5} \left(\{\mathbf{W}_{\text{meas}}\}_k - w\left(x_k; e, \tfrac{1}{2}\bar{q}, \tfrac{1}{2}\bar{q}\right) \right)^2 \right)$$

where the optimal solution E^* becomes now a random variable having an unknown distribution. In other words, we have defined here the stochastic model that leads to the random variable output E^* from an input random variable vector $\mathbf{Z} = \{Q_1; \mathbf{W}_{\text{meas}}\}$, in the form $E^* = \mathscr{H}(\mathbf{Z})$, $\mathscr{H}$ being the mathematical function $\mathscr{H} : \mathbb{R}^6 \mapsto \mathbb{R}$ that model the inverse problem. Its output depends on the coefficients of variation c_q and c_w. At the limit, when all these coefficients of variation tend to be null, the previous deterministic situation would be recovered, to produce $E^* = e_{\text{true}}$ for the distribution of the optimal solution. Finally, one has to note that the centered statistical moments defined by:

$$\mu_i = \mathrm{E}\left[(E^* - \mathrm{E}[E^*])^i\right] \qquad \text{for } i \geq 2$$

are also valuable quantities to characterize the RV E^* and they will be used in the following sections.

MODELING OF THE RANDOM VARIABLE OPTIMAL SOLUTION

Polynomial chaos expansion

Let the stochastic quantities considered to be in the probability space (Ω, A, Γ) with space of events Ω, σ -algebra A, and probability measure Γ. Then, to model the random variable (RV) optimal solution E^*, we choose to represent it using a polynomial expansion having the form :

$$E_p^* = \sum_{k=0}^{p} a_k \psi_k(y), \qquad p \in \mathbb{N} \tag{1}$$

where p denotes the order of expansion, a_k are the expansion coefficients to be determined and $\psi_k(y)$ are the polynomials forming the orthogonal basis $\{\psi_0, \ldots, \psi_p\}$ with respect to the measure Γ. Hence, for Y a suitable random variable having f_{Y} as the probability density function (PDF) defined onto the support $\mathscr{S}$, the following useful property holds:

$$\int_{y \in \mathscr{S}} \psi_l(y)\, \psi_k(y)\, f_{\mathrm{Y}}(y)\, \mathrm{d}y = \mathrm{E}[\psi_l(y)\, \psi_k(y)] = \delta_{kl} \tag{2}$$

where $\mathrm{E}[\bullet]$ denotes the expected operator and δ_{kl} the Kronecker symbol.

Moreover, for a proper Monte-Carlo (MC) simulation based strategy, it is interesting to express the link between RVs. Then, denoting F_{Y} the cumulative distribution function (CDF) of the RV Y and F_{E^*} the CDF of E^*, we introduce the isoprobabilist transformation $E^* \longmapsto Y$ such that $F_{E^*}(e) = F_Y(y)$, leading to:

$$e = F_{E^*}^{-1}(F_{\mathrm{Y}}(y)) \qquad \text{or} \qquad y = F_Y^{-1}(F_{E^*}(e)) \tag{3}$$

Numerical strategies

Monte-Carlo simulations enable us to have access to "measurements" vector data $\hat{\mathbf{E}}^*$ which is a sampling of E^* enabling us to construct an empirical CDF $\widehat{F}_{E^*}(e)$. Next, we can use numerical strategies to obtain the expansion coefficients, when limiting the order of expansion, p, to a small, practical value. For this, we consider the variational method, the collocation or least squares method, and the moment method in their straightforward form since the interest here is not in efficiency but in applicability.

The variational method exploits the orthogonality of the polynomial basis. From relations 1 and 2 we have:

$$\int_{y \in \mathscr{S}} \psi_k(y)\, E^* f_{\mathrm{Y}}(y)\, \mathrm{d}y = a_k \int_{y \in \mathscr{S}} \psi_k(y)\, \psi_k(y)\, f_{\mathrm{Y}}(y)\, \mathrm{d}y$$

and, using the relation 3:

$$a_k = \frac{\mathrm{E}[\psi_k(y(e))\, E^*]}{\mathrm{E}\left[\psi_k^2(y(e))\right]}. \tag{4}$$

The collocation method consists in solving directly in the least squares sense the relation 1 for the over-determined set of collected data. Hence:

$$a_k = \arg\min(\|R\|) \qquad \text{where} \qquad R = E^* - \sum_{k=0}^{p} a_k \psi_k(y(e)). \tag{5}$$

But, for the data considered here, it is obvious that both the variational method and the collocation method will produce the same results, since relations 4 and 5 leads to the same expressions when using the vector data sampling $\hat{\mathbf{E}}^*$.

At last, the moment method consists in minimizing the gap between the center moments of E^* and the ones given by the expansion:

$$a_k = \arg\min \left(\sum_{i=2}^{p+1} \left(\mathrm{E}\left[(E^*)^i \right] - \mathrm{E}\left[\left(\sum_{k=0}^{p} a_k \psi_k (y(e)) \right)^i \right] \right)^2 \right)$$

However, for a better numerical conditioning (this is a crucial point, the components of the vector must have equal weights in the optimisation otherwise there is problems), we express this relation as:

$$\begin{cases} a_0 = \mathrm{E}[E^*] \\ a_k = \arg\min \left(\sum_{i=2}^{p+1} \left(1 - \frac{1}{\mu_i} \mathrm{E}\left[\left(\sum_{k=1}^{p} a_k \psi_k (y(e)) \right)^i \right] \right)^2 \right) & \text{for } 1 \le k \le p. \end{cases}$$

Since this leads to a nonlinear set of equations, a tolerance value ε has to be defined in a relative stopping criterion.

Table 1 – Statistical moments and major quantiles of E^* for the two applications

App.	Mean $\mathrm{E}[E^*]$	Variance μ_2	Skewness $\mu_3 \mu_2^{-\frac{3}{2}}$	Kurtosis $\mu_4 \mu_2^{-2}$	Quantile 5%	Quantile 95%
1	200	1.57	0.02	2.13	198	202
2	201	1.19	-0.34	4.5	199	202

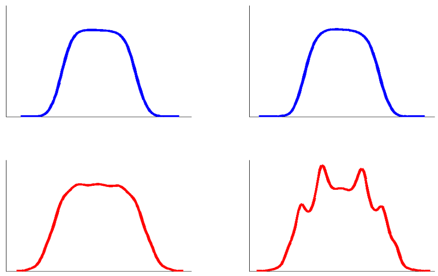

Figure 2 – Results for the first application: left, empirical PDF of E^* ; right, empirical PDF issued from the polynomial expansion at the order 15; blue curves are for the Hermite polynomial basis while red curves are for the Chebyshev of 2nd kind polynomial basis; left curves are from the variational method while right curves are from the moments method

CONCLUSIONS

The paper adresses two main issues. 1) Uncertainty quantification problem: how the distributions of different kind of noise (on both the input and output) affect the distribution of the minimizer, the elasticity modulus, in a static bending test with several discrete measurements of the deformation of a beam?; 2) A modelization problem: how to model the uncertainty using a polynomial chaos expansion in order to avoid using only the two firsts moments to plot an envelope graph, as one sees frequently in the literature.

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