Shape optimization and forced response of phononic crystals using a spatial state space formulation

V.D. Lima, G.F.C.A. Assis, J.M.C. Dos Santos, J.F. Camino, J.R.F. Arruda

University of Campinas - UNICAMP, Rua Mendeleyev, 200, CEP-13083-860, Campinas, SP, Brazil.

vinidiaslima@gmail.com,{george.assis,zema,camino,arruda}@fem.unicamp.br

Abstract: Due to Bragg type interference, periodic structures exhibit frequency bands where waves cannot propagate, the so-called bandgaps. These bands can be investigated using dispersion diagrams (i.e., wavenumber versus frequency plots), which are usually obtained using the Plane Wave Expansion (PWE) method. The spatial periodicity can be of the geometrical type, material type, or both. In this work we investigate geometrically periodic acoustic ducts aiming at optimizing a low frequency band gap. The periodic cell shape is optimized using different parametric strategies. The possibility of optimizing the spatial Fourier coefficients of the PWE method is one of these strategies. Given that the optimized shape is arbitrary, a spatial state space formulation is used to obtain a spectral element representation of the periodic cell. This strategy consists of transforming the boundary value problem in an initial value problem and recasting the latter into an impedance formulation in the form of a Riccati equation. Imposing arbitrary boundary conditions, a set of linear equations allows the computation of a spectral dynamic stiffness matrix, which can be used to compute the forced response of a structure composed of a finite number of periodic cells. This spectral solution is validated using an axisymmetric acoustic Finite Element (FE) model of the whole structure. Experimental results are also shown to validate the proposed models and optimization strategy.

Keywords: phononic crystal, optimization, ducts, Riccati equation

STATE-SPACE FORMULATION

The wave equation describing the behavior of pressure in an acoustic duct with varying cross-section is (Blaubert and Xiang, 2008)

$$B\frac{\partial}{\partial x}\left[A(x)\frac{\partial p(x,t)}{\partial x}\right] = \rho A(x)\frac{\partial^2 p(x,t)}{\partial t^2} \tag{1}$$

where B is the adiabatic bulk modulus, ρ is the density, $A(x)$ is the cross-sectional area and $p(x,t)$ is the relative pressure. If the acoustic perturbations have small amplitude, the linear Euler's equation is

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{\rho}\frac{\partial p(x,t)}{\partial x} = 0 \tag{2}$$

where $u(x,t)$ is the particle velocity. Multiplying Eq. (2) by $A(x)$, one can write

$$v(x,t) = -\frac{A(x)}{\rho}\frac{\partial p(x,t)}{\partial x} \tag{3}$$

where $v(x,t) = A(x)\frac{\partial u(x,t)}{\partial t}$ is the volume acceleration.

A set of state-space equations is defined with Eq. (1) and Eq. (3) as

$$\frac{\partial \hat{v}(x)}{\partial x} = \frac{\omega^2 A(x)}{B}\hat{p}(x) \qquad \text{and} \qquad \frac{\partial \hat{p}(x)}{\partial x} = -\frac{\rho}{A(x)}\hat{v}(x) \tag{4}$$

which can be written in matrix form as:

$$\frac{\partial \hat{\mathbf{q}}}{\partial x} = \hat{\mathbf{H}}\hat{\mathbf{q}}, \qquad \hat{\mathbf{q}}(0) = \hat{\mathbf{q}}_0 \tag{5}$$

with the state $\hat{\mathbf{q}}(x) := \hat{\mathbf{q}}(x,\omega)$ and $\hat{\mathbf{H}}(x,\omega)$ expressed by

$$\hat{\mathbf{q}}(x) = \begin{bmatrix}\hat{p}(x)\\ \hat{v}(x)\end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{H}}(x,\omega) = \begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix} = \begin{bmatrix}0 & -\dfrac{\rho}{A(x)}\\ \frac{\omega^2 A(x)}{B} & 0\end{bmatrix} \tag{6}$$

The notation "ˆ" is used to denote a variable in the frequency domain.

This linear system of equations of the first order, where the system parameters may vary along x, can be solved numerically for any type of variation of the parameters along x. However, the state cannot be known at one end, as the boundary condition is either in the volume acceleration (Neumann) or the pressure (Dirichlet), the other variable of the state being unknown. For a mixed boundary condition only the relation between the state variables is known. To overcome this issue, one can rewrite the problem in terms of the specific acoustical admittance:

$$u(x,t) = y(x,t)p(x,t) \tag{7}$$

Multiplying Eq. (7) by $A(x)$ and differentiating in time in the frequency domain results:

$$\hat{v}(x) = j\omega\hat{Y}(x)\hat{p}(x) \tag{8}$$

where $\hat{Y}(x)$ is the acoustic admittance. Substituting Eq. (8) into Eq. (5) yields to

$$\frac{\partial \hat{p}(x)}{\partial x} = H_{11}\hat{p}(x) + H_{12}\hat{v}(x) \qquad \text{and} \qquad j\omega\frac{\partial \hat{Y}(x)\hat{p}(x)}{\partial x} = H_{21}\hat{p}(x) + H_{22}\hat{v}(x) \tag{9}$$

which leads to:

$$j\omega\left(\frac{\partial \hat{Y}(x)}{\partial x}\hat{p}(x) + \hat{Y}(x)H_{11}\hat{p}(x) + \hat{Y}(x)H_{12}j\omega\hat{Y}(x)\hat{p}(x)\right) = H_{21}\hat{p}(x) + H_{22}j\omega\hat{Y}(x)\hat{p}(x) \tag{10}$$

Since $\hat{p}(x)$ cannot be identically zero, one finally obtains the following Riccati type equation:

$$\frac{\partial \hat{Y}(x)}{\partial x} + \hat{Y}(x)H_{11} - H_{22}\hat{Y}(x) + j\omega\hat{Y}(x)H_{12}\hat{Y}(x) = (j\omega)^{-1}H_{21} \tag{11}$$

For this equation an initial condition $Y(0) = 0$ can be used for a closed end, where $\hat{v}_0 = 0$ and p_0 is unknown, but not zero. The analogous formulation for an elastic rod is found at (Assis et al., 2018).

After evaluating the admittance at the opposite end, considering a unitary external input volume acceleration, the pressure is computed. With the state at $x = L$ known, and using it as a initial condition, the state at $x = 0$ is obtained by integration. The full state at both ends allows computing the transfer matrix, since it relates the state at the ends of a finite duct by

$$\hat{\mathbf{q}}(L) = \hat{\mathbf{T}}(\omega)\hat{\mathbf{q}}(0) \tag{12}$$

where $\hat{\mathbf{T}}$ is the $[2\times 2]$ transfer matrix. Applying the boundary conditions $\hat{v}(x = L) = 1$ and $\hat{v}(x = 0) = 0$ leads to

$$\begin{Bmatrix} \hat{p}_L \\ 1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \hat{p}_0 \\ 0 \end{Bmatrix} \tag{13}$$

The matrix elements of $\hat{\mathbf{T}}$ are

$$\begin{cases} T_{11} = \hat{p}_L/\hat{p}_0 \\ T_{21} = \hat{p}_0^{-1} \\ T_{22} = T_{11} \quad \text{(due to the cell symmetry)} \end{cases} \tag{14}$$

The relation between the off diagonal terms of matrix $\hat{\mathbf{T}}$ can be evaluated using the relation between the transfer matrix and the symmetric (due to reciprocity) matrix of coefficients $K_{ij}, i, j = 1, 2..$

$$\hat{\mathbf{T}}(\omega) = \begin{bmatrix} K_{12}^{-1}K_{11} & -K_{12}^{-1} \\ K_{21} - K_{22}K_{12}^{-1}K_{12}^{11} & K_{22}K_{12}^{-1} \end{bmatrix} \tag{15}$$

The symmetric matrix $\hat{\mathbf{K}}(\omega)$ is the reciprocal of the frequency dependent dynamic inertance matrix of a unit cell of a periodic duct system. Due to the symmetry of a unit cell, $K_{22} = K_{11}$, and because of the reciprocity of the duct system, $K_{12} = K_{21}$. With this properties, it can be shown that:

$$T_{12} = (T_{11}^2 - 1)T_{21}^{-1} \tag{16}$$

From the computed transfer matrix one can obtain the matrix $\hat{\mathbf{K}}(\omega)$ using Eq. (15).

DUCT SHAPE OPTIMIZATION

Preliminary results obtained using the PWE (Sigalas and Economou,1994) and optimizing only for the band gap width are available in the M.Sc. thesis of the first author (Lima, 2018). Two optimization problems are investigated for a duct system with rectangular profile: maximizing the lengths of sections with different diameters and maximizing the expansion chamber. The initial model has tubes with 150 *mm* length and 37.5 *mm* of diameter, and expansion chamber with 165 *mm* length and 145 *mm* of diameter. Figure 1 illustrates this type of system.

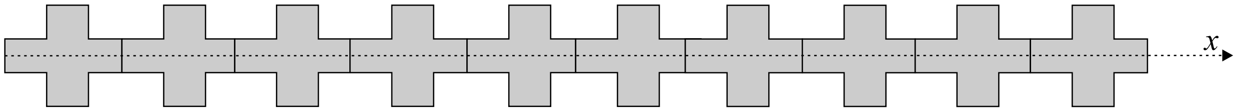

Figure 1 – Duct system with rectangular profile.

In the first problem, the design variables are the lengths of the expansion chamber and the tubes with fixed diameters. The objective function for this problem is

$$\begin{array}{cl} \max\limits_{L_a, L_b} & \Delta\omega(L_a, L_b) = \omega_{n+1} - \omega_n \\ \text{subjected to} & 50\ mm \leq L_a \leq 250\ mm \\ & 200\ mm \leq L_b \leq 400\ mm \end{array} \tag{17}$$

Figure 2 shows the optimized first bandgap. The bandgap obtained has a size of 637.34 Hz and the final geometry is $L_a = 127\ mm$ and $L_b = 20\ mm$. There is a gain of 213.89 Hz when compared with the initial first bandgap of 423.45 Hz. This is a clear improvement of the first bandgap. Note that there is a small shift, from approximately 120 Hz to 170 Hz, where a band of frequencies become a propagative band after the optimization. However, a much larger frequency band become attenuated after the optimization procedure.

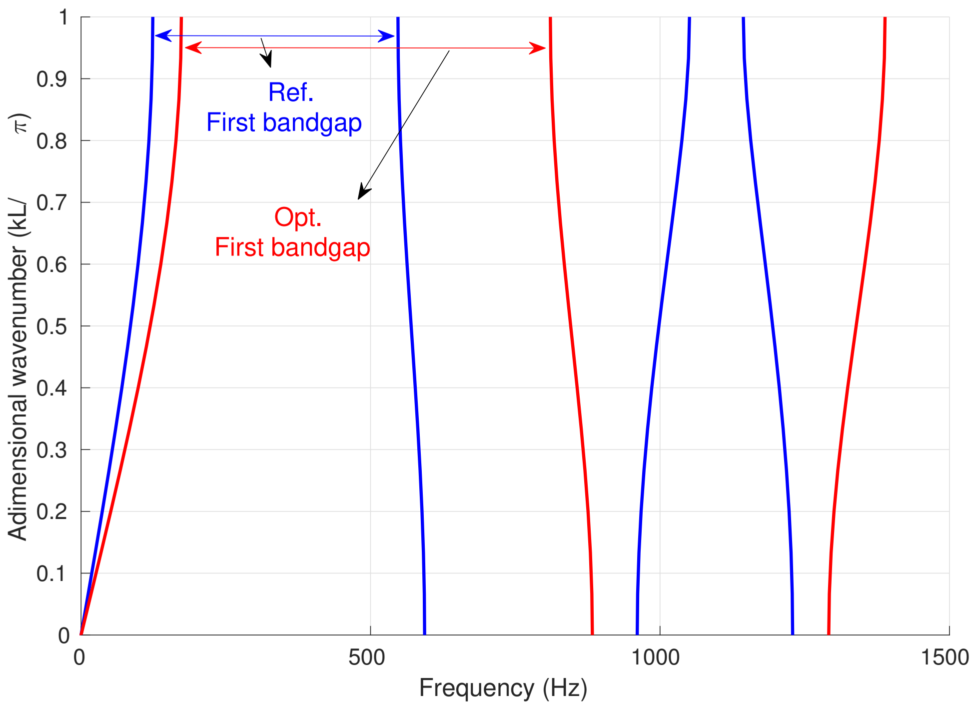

Figure 2 – Reference dispersion compared with maximized first bandgap for the lengths optimization.

For the second problem, the design variables are the diameters of the expansion chamber and the tubes. The lengths are fixed. The objective function for this problem is

$$\begin{array}{cl} \max\limits_{D_a, D_b} & \Delta\omega(D_a, D_b) = \omega_{n+1} - \omega_n \\ \text{subjected to} & 100\ mm \leq D_a \leq 200\ mm \\ & 40\ mm \leq D_b \leq 100\ mm \end{array} \tag{18}$$

Figure 3 shows the results of the optimization for the first bandgap. The optimum band gaps is 522.38 Hz with a gain of 98.93 Hz. The optimized diameters are $D_a = 200\ mm$ and $D_b = 40\ mm$, the superior limit for D_a and inferior limit to D_b. This occurs because as much the difference between D_a and D_b increase, the impedance also increase. The superpositions of plane waves in the tubes and chambers system are intensified, improving the attenuation and the size of the frequency band.

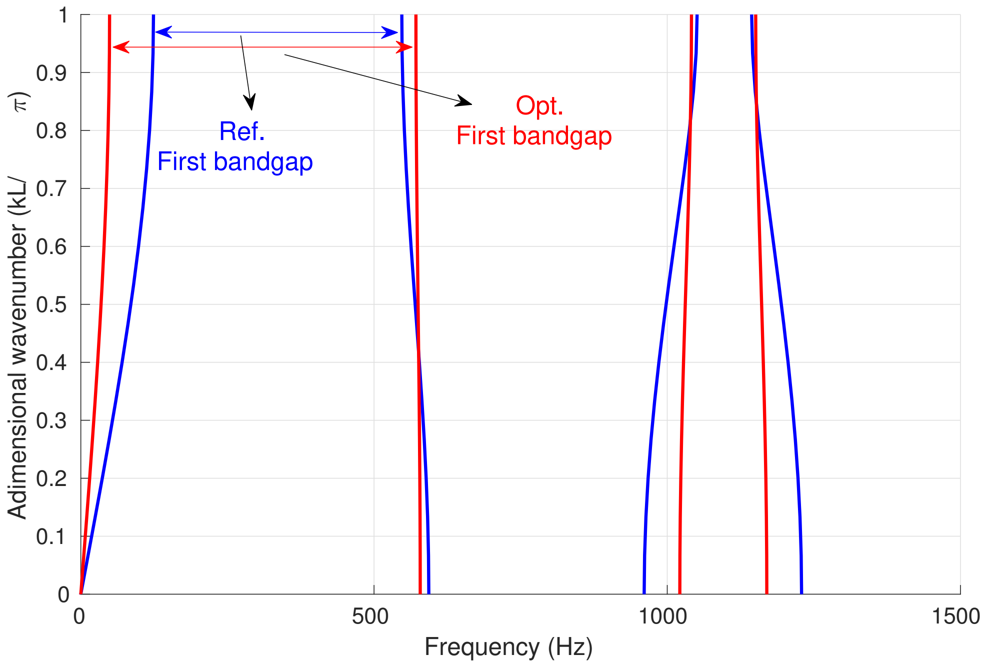

Figure 3 – Reference dispersion compared with maximized first bandgap for the diameters optimization.

CONCLUSIONS

The first band gap is optimized for the lowest and broadest possible solution given geometrical size constraints. The shape parameter optimization led to expected results, i.e., the highest possible impedance contrast between low and high impedance sections of the duct cell. The Fourier coefficient optimization is still under investigation and new results will be presented in the full version of the paper.

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