Vortex-induced vibration analysis using invariant manifolds

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Abstract: In this paper, the non-linear normal modes are obtained for the problem of a rigid cylinder subjected to vortex-induced vibrations. The cylinder is constrained to oscillate in the cross-wise direction (i.e., the direction orthogonal to the free-stream velocity). The results are compared to the numerical simulations of the equations of motion. It is shown that the classical proposal of obtaining the modal manifolds using power expansions direct over the equations of motion leads to good results only when the structural damping ratio is high.

Keywords: invariant manifolds, non-linear normal modes, vortex-induced vibration

INTRODUCTION

Vortex-induced vibration (VIV) is, by it's own nature, a non-linear fluid-structure interaction problem. Then, it is expected that the use of techniques of applied dynamics on non-linear systems can give a deeper understanding of this phenomenon. Also, when a large number of degrees of freedom is present, the use of reduced-order models (ROMs) that can give qualitative results is desirable. Among other alternatives, ROMs can be obtained from the non-linear normal modes proposed by Shaw and Pierre (1993). In the VIV context, the application of this methodology considering the wake-oscillators in the modeling is novel. In the case of risers undergoing VIV, it can be mentioned the work of Keber and Wiercigroch (2008), in which the non-linear modes have been used to obtain a reduced order model for the structure, retaining terms that represents non-linear couplings between linear modes of the structure, with the wake oscillator being coupled after the model reduction.

NON-LINEAR NORMAL MODES

In linear dynamics, modal decomposition are usually employed aiming at obtaining the response. Such an approach leads to ROMs given by the corresponding modal equations. The response on physical coordinates can be easily obtained from the modal responses using the transformation given by the vibration modes. Advantages of modal decomposition scheme lie on both its capability of reducing the order of a model and in the fact that initial conditions matching one mode do not excite motions in any other modes.

In order to use those advantages and gains in non-linear problems, Shaw and Pierre (1993) made a contribution in defining and proposing a technique to determine normal modes for non-linear systems. For the construction of the non-linear modes, it is assumed that all the generalized coordinates and velocities can be written in terms of functional dependence with one single pair of displacement and velocity for singular modes. Those modes are them represented by manifolds that are constant in time in the phase space. For the application of the technique, the main ideas presented in Shaw and Pierre (1993) are exposed here. Consider the dynamical system given by the equations of motion in the form:

$$\dot{x}_i = y_i \qquad i = 1, ..., N \tag{1}$$

$$\dot{y}_i = f_i\left(\overrightarrow{x}; \overrightarrow{y}\right) \qquad i = 1, ..., N \tag{2}$$

Herein, overdots represents differentiation with respect to time. Let $\overrightarrow{x} = [x_1, ..., x_N]^T$ be the vector of generalized coordinates, $\overrightarrow{y} = [x_1, ..., x_N]^T$ the vector of generalized velocities and $f_i\left(\overrightarrow{x}; \overrightarrow{y}\right)$ the generalized forces normalized by the corresponding inertias. Now, it is assumed that exists at least one motion such that the generalized coordinates and velocities can be functionally related to one single pair of generalized coordinate with the associated generalized velocity. Assuming $(x_1, y_1) = (u, v)$, the sought relations are $x_i = X_i(u, v)$ and $y_i = Y_i(u, v)$ for $i = 1, ..., N$. Notice that $X_1 = u$ and $Y_1 = v$.

From this assumptions, it is clear that the manifolds will be two dimensional. This methodology can be applied for higher order manifolds by simply defining the functional relations in terms of more pairs instead of one, generating multimodes instead of singular modes. Using techniques of center manifold theory (see Nayfeh and Balachandran (1995)), an autonomous system of equations for the geometry of the manifolds can be obtained in the form of the following equations:

$$Y_i = \frac{\partial X_i}{\partial u} v + \frac{\partial X_i}{\partial v} f_1(u, X_2, ..., X_N; v, Y_2, ..., Y_N) \qquad i = 1, ..., N \tag{3}$$

$$f_i(u, X_1, ..., X_N; v, Y_1, ..., Y_N) = \frac{\partial Y_i}{\partial u} v + \frac{\partial Y_i}{\partial v} f_1(u, X_2, ..., X_N; v, Y_2, ..., Y_N) \qquad i = 1, ..., N \tag{4}$$

Equations 3 and 4 are, in general, as difficult to solve as the initial problem. However, it is possible to use series expansions to find an approximated solution near the equilibrium solution. Once the manifolds are determined, the modal Eqs. 5 and 6 are solved to obtain the modal dynamics for the problem.

$$\dot{u} = v \tag{5}$$

$$\dot{v} = f_1(u, X_2, ..., X_N; v, Y_2, ..., Y_N) \tag{6}$$

APPLICATION TO VORTEX-INDUCED VIBRATION

Consider now a rigid cylinder mounted on an elastic base and immersed in a fluid flow as sketched in Fig. 1. Both the spring and the dashpot are considered as linear elements of constant k and c respectively. The cylinder has diameter D, length L and dry mass m_c. The potential added mass is m_a and the mass of fluid displaced by the cylinder is $m_d = \rho \pi D^2 L/4$, with ρ being the fluid density. The natural frequency is still water is $\omega = \sqrt{k/(m_c + m_a)}$. The case in which the cylinder is excited by VIV is modeled using a wake-oscillator model, which makes use of a non-linear equation coupled to the structural oscillator aiming at representing the fluid dynamics. The model herein adopted is that proposed by Ogink and Metrikine (2010) and based on the previous study by Facchinetti, de Langre and Biolley (2004). Considering the parameters defined in Tab. 1, the dynamics of the hydro-elastic system is given by Eqs. 7 and 8.

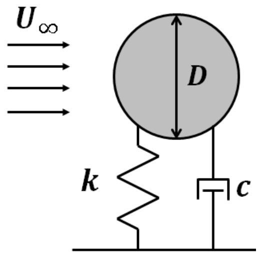

Figure 1: Problem sketch.

$$\ddot{y} + 2\zeta\dot{y} + y = \frac{U_r^2}{2\pi^3(m^* + C_a)} \left(\frac{q}{2} C_L^0 - C_D \frac{2\pi\dot{y}}{U_r} \right) \sqrt{1 + \left(\frac{2\pi\dot{y}}{U_r} \right)^2} \tag{7}$$

$$\ddot{q} + \varepsilon S_t U_r \left(q^2 - 1\right) \dot{q} + (S_t U_r)^2 q = A\ddot{y} \tag{8}$$

Hereafter, overdots are used for differentiation with respect to the non-dimensional time $\tau = \omega t$. Now, consider the following auxiliary definitions:

$$a = \frac{U_r^2}{2\pi^3(m^* + C_a)} \frac{C_L^0}{2} \tag{9}$$

$$b = \frac{U_r^2}{2\pi^3(m^* + C_a)} \frac{2\pi C_D}{U_r} \tag{10}$$

$$c = \varepsilon S_t U_r \tag{11}$$

$$d = (S_t U_r)^2 \tag{12}$$

Defining $y = u$ and $\dot{y} = v$, and writing the wake variable q and its derivative as $q = X(u, v)$ and $\dot{q} = Y(u, v)$, the manifolds can be determined solving Eqs. 13 and 14. The modal dynamics is given them by the solution of Eqs. 15 and 16. Those assumptions are valid considering that no internal resonance between y and q is involved. However, VIV is a resonant phenomenon, so, for this direct approach to work properly, there must be a slight shift in the frequency of the structure. This is achieved limiting the analysis to cases where the structural damping ratio is high (10% to 20%).

Table 1: Variables and parameters.

Symbol	Definition	Description
y	y_r/D	Non-dimensional cylinder displacement
q	-	Non-dimensional wake variable
ζ	$c/\left(2\left(m_c+m_a\right)\omega\right)$	Damping ratio
U_r	$2\pi U_\infty/\left(\omega D\right)$	Reduced velocity
m^*	m_c/m_d	Non-dimensional mass
C_a	m_a/m_d	Potential added mass coefficient
C_L^0	-	Lift oscillation amplitude
C_D	-	Mean Drag coefficient
S_t	$\omega_{St}D/\left(2\pi U_\infty\right)$	Strouhal number
ε	-	Experimental correlation coefficient
A	-	Experimental correlation coefficient

$$Y-\frac{\partial X}{\partial u}v-\frac{\partial X}{\partial v}\left(-u-2\zeta v+(aX-bv)\sqrt{1+\left(\frac{2\pi v}{U_r}\right)^2}\right)=0 \tag{13}$$

$$\frac{\partial Y}{\partial u}v+\left(\frac{\partial Y}{\partial v}-A\right)\left(-u-2\zeta v+(aX-bv)\sqrt{1+\left(\frac{2\pi v}{U_r}\right)^2}\right)+c\left(X^2-1\right)Y+dX=0 \tag{14}$$

$$\dot{u}=v \tag{15}$$

$$\dot{v}=-u-2\zeta v+(aX-bv)\sqrt{1+\left(\frac{2\pi v}{U_r}\right)^2} \tag{16}$$

Based on Shaw and Pierre (1993), the following expansions are proposed:

$$X=a_1u+a_2v+a_3u^2+a_4uv+a_5v^2+a_6u^3+a_7u^2v+a_8uv^2+a_9v^3 \tag{17}$$

$$Y=b_1u+b_2v+b_3u^2+b_4uv+b_5v^2+b_6u^3+b_7u^2v+b_8uv^2+b_9v^3 \tag{18}$$

The substitutions of Eqs. 17 and 18 in Eqs. 13 and 14 are developed using Mathematica®. Then, the polynomial terms up to cubic order are collected. The determination on the coefficients a_i and b_i is then made from the algebraic equations obtained when it is required the vanishing of every collected polynomial coefficient, in order to satisfy Eqs. 13 and 14. This approach has been applied using a range of structural damping ratio from 0.001 to 0.3. However, as already mentioned, this direct approach works well only for values of the structural damping ratio above 0.1. The extension of the non-linear normal modes analysis for systems with small damping ratio needs to consider internal resonance and is intended for future works. Note that, as can be seen in Nayfeh, Chin and Nayfeh(1996), for systems with internal resonance is required that more variables are used when constructing the non-linear normal modes. That said, the extensions can only be made for systems with a higher number of degrees of freedom.

NUMERICAL EXAMPLE

The solution for the modal dynamics is presented and compared to that from numerical integration of the original equations of motion (Eqs. 7 and 8). The parameters are $\zeta=0.20$, $U_r=6$, $m^*=2.6$, $C_a=1$, $C_L^0=0.3842$, $C_D=1.1856$, $S_t=0.1932$, $\varepsilon=0.05$, and $A=4$. It is worth to emphasize that $U_r=6$ corresponds to a condition in which the cylinder experiences large oscillations amplitudes and that, excepted to the structural damping ratio ζ, the other parameters were extracted from Franzini and Bunzel (2018). Using these parameters and the methodology described in the previous section, the manifolds are given by Eqs. 19 and 20. In turn, Figs. 2 and 3 show the geometrical shape of the manifolds.

$$X=-9.35764u+46.1269u^3+25.7682v-178.748u^2v+24.8814uv^2-236.082v^3 \tag{19}$$

$$Y=-33.2378u+267.384u^3-4.25491v-103.887u^2v+581.673uv^2-295.368v^3 \tag{20}$$

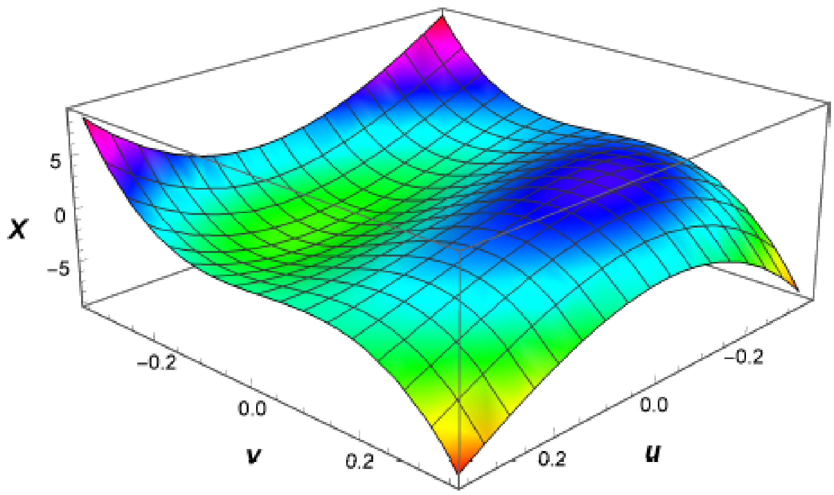

Figure 2: X Manifold for $\zeta = 0.20$.

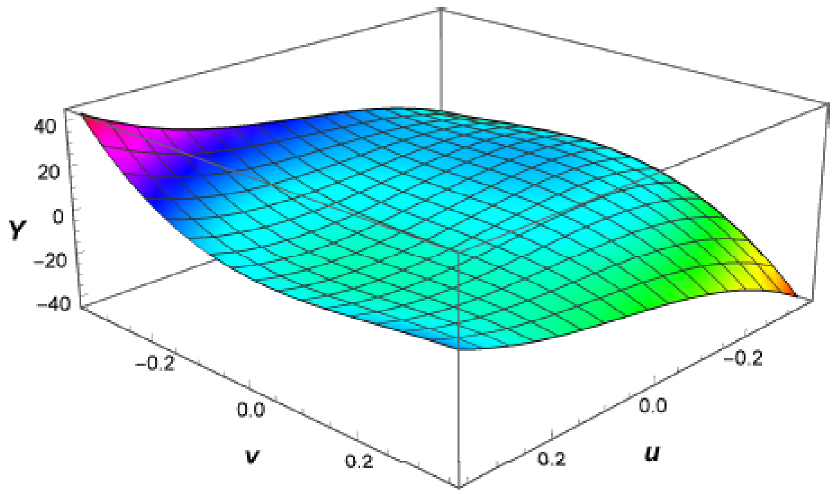

Figure 3: Y Manifold for $\zeta = 0.20$.

Figs. 4 and 5 show the time histories obtained from both the modal solution using the invariant manifolds and the numerical integration of Eqs. 7 and 8. Notice that those figures display the responses for $900 < \tau < 1000$ and that the oscillation frequency is well recovered. In turn, there is a relative difference of order of 10% in the amplitude prediction.

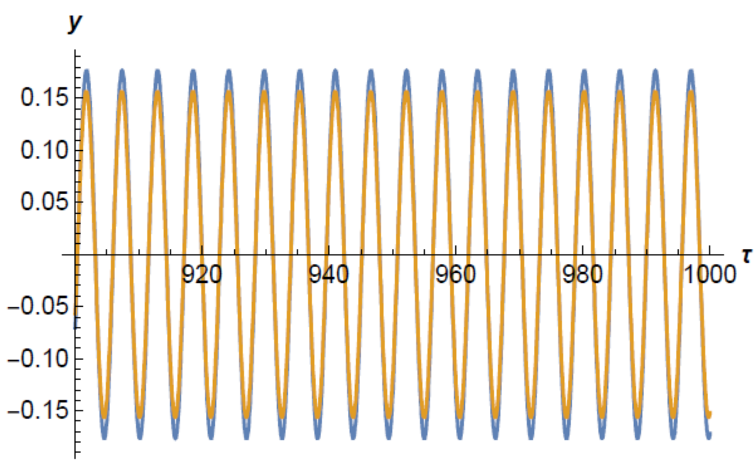

Figure 4: Displacement time history for $\zeta = 0.20$.

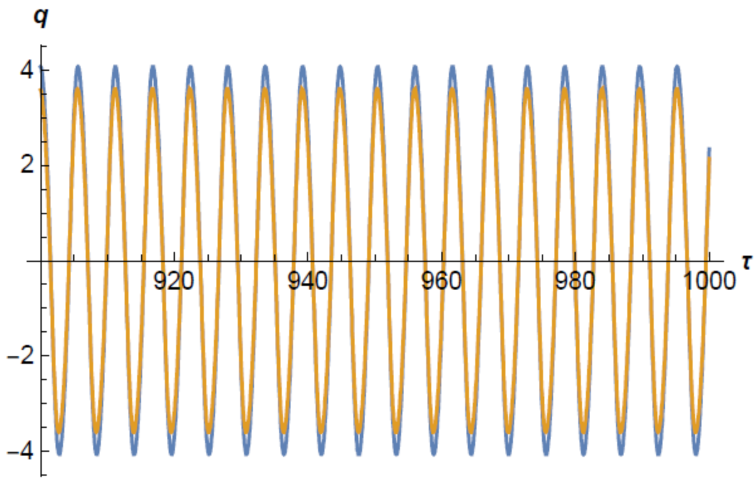

Figure 5: Wake variable time history for $\zeta = 0.20$.

To exemplify how the representation gets better when the system is shifted from the 1:1 resonance, the damping ratio is now considered as $\zeta = 0.30$. In this case, the manifolds are given by Eqs. 21 and 22. In turn, Figs. 6 and 7 show the geometrical shape of the manifolds.

$$X = -9.85519u + 65.9289u^3 + 31.5041v - 288.399u^2v + 2.95846uv^2 - 387.12v^3 \tag{21}$$

$$Y = -41.122u + 440.785u^3 - 4.31955v - 142.066u^2v + 937.767uv^2 - 565.505v^3 \tag{22}$$

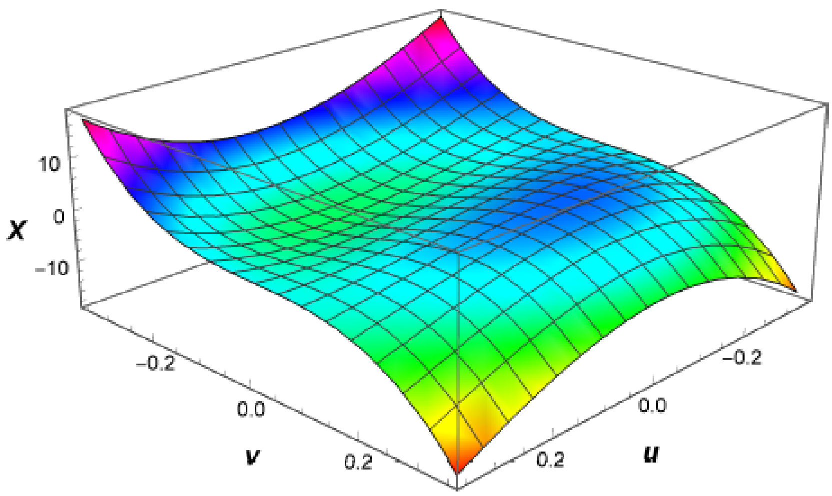

Figure 6: X Manifold for $\zeta = 0.30$.

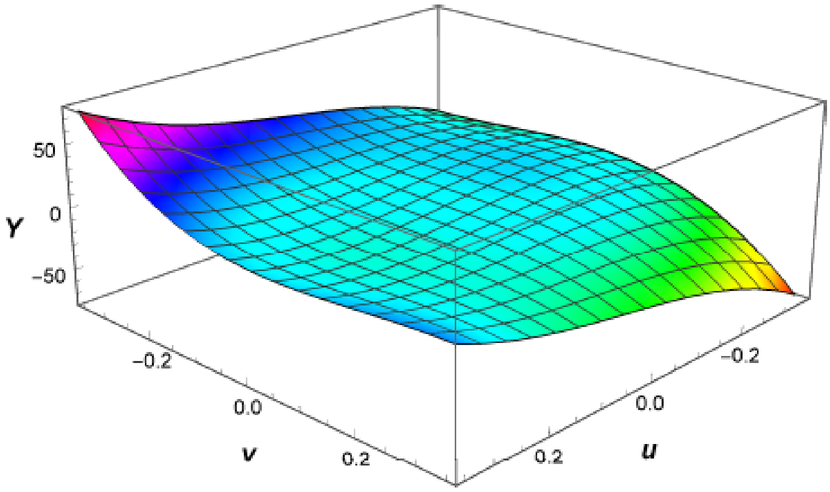

Figure 7: Y Manifold for $\zeta = 0.30$.

Figs. 8 and 9 show the time histories obtained from both the modal solution using the invariant manifolds and the numerical integration of Eqs. 7 and 8.

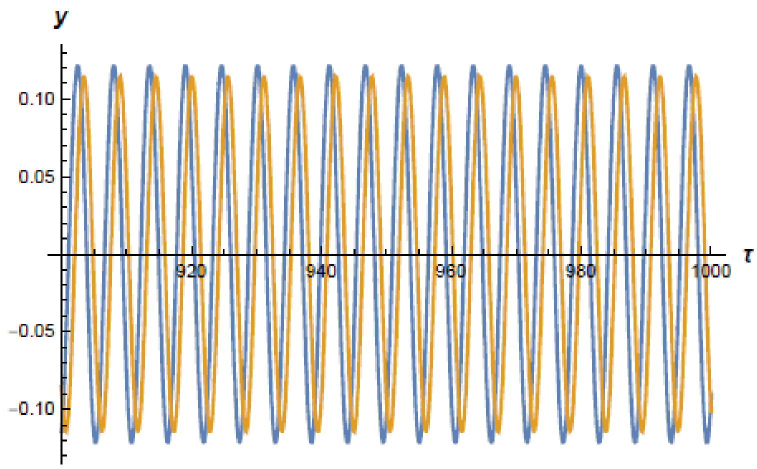

Figure 8: Displacement time history for $\zeta = 0.30$.

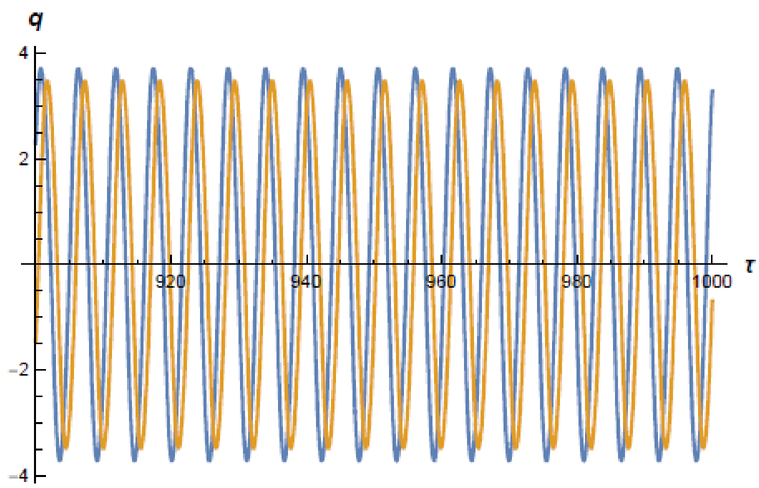

Figure 9: Wake variable time history for $\zeta = 0.30$.

CONCLUDING REMARKS

The use of the non-linear normal modes approach, as defined by Shaw and Pierre (1993) can give a useful tool for the creation of reduced-order models for systems without internal resonance. This paper dealt with the vortex-induced vibrations (VIV) phenomenon. By this moment, the results showed an agreement between the analysis of the modal equations and the direct numeric integration of the equations of motion if the structural damping ratio is large. The large structural damping ratio is responsible for ensuring that the procedure proposed works for the VIV problem, which is resonant. Alternative methods of determining the manifolds are in study to extend the methodology for small values of the structural damping ratio. As shown for example in Nayfeh, Chin and Nayfeh(1996), the use of systems with more degrees of freedom is necessary for a proper consideration of internal resonances.

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