An FFT-based Generalization of the ε-algorithm for the Integration of Oscillatory-Decaying Functions with Multiple Frequencies

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Abstract: This paper presents a method for numerical integration of improper integrals with oscillatory-decaying functions containing one or more than one frequency of oscillation. These integrals occur in the solution of a wide variety of dynamic problems; especially those in which some space transform is involved in the solution of the equations of motion. A classical integration method for the case of an integrand with a single oscillatory frequency is to extrapolate the integral from a sequence of partial sums over finite subintervals, in the framework of Eulers convergent series. This integration scheme, however, is unable to deal with the case in which the integrand has more than one oscillatory frequency. The integration scheme presented in this paper consists in using a fast Fourier transform to select appropriate endpoints to divide integrands with multiple oscillatory frequencies, together with a robust series extrapolation through the ε-algorithm. The method is used to integrate transcendental functions of multiple oscillatory frequencies, including a case in which a closed-form expression for the integrand is not known. The results are compared with classical adaptive quadrature implementations.

Keywords: Numerical integration, fast Fourier transform, ε-algorithm, multiple oscillatory frequencies

INTRODUCTION

Previous methods for the improper integration by the partition-extrapolation method for simple oscillatory-decaying integrands are reviewed in Lucas and Stone (1994), and Blakemore, Evans and Hyslop (1976). The core idea of these techniques was originally presented by Longman (1956), in which Eulers transformation of convergent series is used to obtain an extrapolation of the integral from the improper integral of the function between two of its consecutive roots. This approach was extended by Cavalcante, Vasconcelos and Labaki (2017) for the integration of functions with or without constant periods of oscillation with the help of a global homotopy continuation algorithm. Many variations of extrapolation techniques have been proposed after Longmans idea (Michalski, 1998), from which one can highlight Wynns ε-algorithm (Wynn, 1956; Chisholm, Genz and Rowland, 1972), W-transformation, and its modified version (mW-transformation), developed by Sidi (1982).

A comparison between Longmans method, the ε-algorithm and the mW-transformation is shown by Lucas and Stone (1994) which concluded that the latter is the most efficient extrapolator, which uses the average of successive zeros of the integrand as finite integration interval. A more recent study by Michalski and Mosig (2016) deals with multiple oscillatory frequencies, for the specific case of product of Bessel function, and resorts to the method of decomposition propose by Lucas (1995). However, this approach requires the generating expression of the integrand to be known. This information is unavailable in many practical problems, in which the integrands result from the solution of a linear system (Labaki, Mesquita and Rajapakse, 2014). Due to its irregular oscillatory behavior, one notorious difficulty in evaluating this type of integral is defining a partition that will yield appropriate series terms for posterior extrapolation (Lucas and Stone, 1994). In this work, we propose the use of fast Fourier transforms to break down the periods of oscillation of integrands with an arbitrary number of oscillatory frequencies. The highest frequency of oscillation is identified from the resulting frequency response signal, and used to define Sidis partition (Sidi, 1982). The integration of the integrand within the partitioned intervals is extrapolated according to the ε-algorithm. The article presents selected numerical results and comparisons with a classical adaptive quadrature.

STATEMENT OF THE PROBLEM

This work tackles the numerical integration of improper integrals I of oscillatory-decaying functions $f(x)$ with multiple oscillatory frequencies, with or without closed-form expressions:

$$I = \int_a^\infty f(x)dx \tag{1}$$

where a is the lower integration bound.

Two examples of integrands with such behavior are $f(x) = e^{-\lambda x}[cos(\omega_1 x) + cos(\omega_2 x)]$ and $f(x) = J_0(\delta_1 x)J_1(\delta_2 x)$, in which $J_m(x)$ are Bessel functions or the first kind and order m. However, in many practical applications, analytical expression for these integrands is not available. In Greens functions used in dynamic soil-foundation interaction analyses involving layered soils, for example, the integrand may arise from the solution of a linear system, which must be evaluated for specific values of the integration variable. This study proposes a numerical scheme to deal with both cases.

INTEGRATION METHOD

The integration method proposed in this work consists of two parts. In the first part, the integrand is divided into subintervals. A discussion of the appropriate subinterval endpoints is shown in this section. These endpoints must be selected so that the integral between consecutive endpoints satisfy some series extrapolation criteria, that is, the integral between consecutive subintervals must guarantee the convergence of the sum of subintervals. The second part consists in extrapolating the value of the integral from the integrals of the subintervals. This extrapolation is achieved through the ε-algorithm.

Interval Endpoints

Many choices of interval endpoints have been suggested for the integration of oscillatory integrands through series extrapolation techniques (Lucas and Stone, 1994; Longman, 1956; Michalski, 1998; Sidi, 1982; Lynes, 1985). One idea is based on the asymptotic expansion of Bessel functions for large arguments (Bromwich, 1965; Lucas, 1995),

$$J_n(\rho x) \cong \sqrt{\frac{2}{\pi\rho x}} cos\left(\rho x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{2}$$

which implies that the zeros of the Bessel functions are separated by approximately π. Hence, the evident choice of break points in this case are the equidistant points $x_n = b + nq$, $n \geq 0$, where $q = \pi/\rho$ is the asymptotic half-period of the Bessel functions, known as Sidi partition (Sidi, 1982), and b denotes the first breakpoint greater than a. Here it is suggested for simplicity that $b = a + q$ or even for specific cases that the value of b be adjusted to coincide with the first zero of the Bessel function after a. This choice of subinterval endpoints ensures that the partition yields a sequence of partial sums susceptible to extrapolation techniques.

We extended this concept for the case of functions with multiple oscillatory frequencies. From the spectral analysis, it can possible reveal a hidden periodicity due to one frequency, named dominant frequency, which carries the maximum energy among all founded frequencies (Telgársky, 2013). Our suggestion is that the argument ρ be the half of dominant frequency ($f_d/2$) of the integrand. The task of estimate this frequency consists in find the largest peak in the power spectral density of the signal then read the corresponding frequency. This step may look simple, and in fact, it is in cases that the noise can be neglected and there is a small number of frequencies relatively distant. Otherwise, some issues must be considered for determination of dominant frequency. A discussion about the algorithms and libraries can be found in Telgársky (2013).

Once the dominant frequency is identified, the interval endpoints are suitable to yield a sequence of partial sums adequated to extrapolation and are defined as

$$x_n = b + \frac{2\pi n}{f_d}, \quad n \geq 0. \tag{3}$$

The use of half the dominant frequency in Eq. 3 means that each integral involves a sequence of positive and negative integrand, resulting in a sequence which is no strictly alterning. However, the oscillation in the partial sums required in ε-algorithm will be smaller, and the convergence should be improved.

In this work, the identification of the frequencies of oscillation is obtained through classical fast Fourier transform (FFT). Figure 1a illustrates the FFT of an oscillatory-decaying transcendental function with two oscillation frequencies. Note that numerical FFT of discretized forms of the integrand incurs in additional numerical errors into the result of the integral. Numerical results in this work show, however, that these errors are negligible.

The reasoning used in this section for the definition of interval endpoints for transcendental functions also hold for the case of trigonometric functions, since the reasoning is based on the trigonometric expansion of such transcendental functions.

ε-algorithm

Classical Longmans extrapolation method of integration is inadequate for the present case of multiple frequencies. We use instead the ε-algorithm, an efficient series extrapolation algorithm introduced by Wynn (1956). Given a sequence of partial sums $\{S_n\}_{n=0}^{0}$, the transformation ε is defined as

$$\varepsilon_1(S_n) = 0, \quad \varepsilon_0(S_n) = S_n \tag{4}$$

$$\varepsilon_{s+1}(S_n) = \varepsilon_{s-1}(S_{n+1}) + [\varepsilon_s(S_{n+1}) - \varepsilon_s(S_n)]^{-1}, \quad n,s \geq 0 \tag{5}$$

Namely, $\varepsilon_{2s}(S_n)$ displays the transformed results $\varepsilon_s(S_n)$ for sequence $\{S_n\}$. The terms of the sequence in the present problem are the integrals within subintervals. The algorithm is robust and can deal with slowly convergent, and even some cases of divergent, series.

Algorithm Scheme

The proposed integration scheme can be summarized in three steps:

Step 1: Find the interval endpoints through FFT;

Step 2: Integrate the volumes between the endpoints intervals; and

Step 3: Extrapolate the value of integral through the ε-algorithm.

NUMERICAL RESULTS AND DISCUSSION

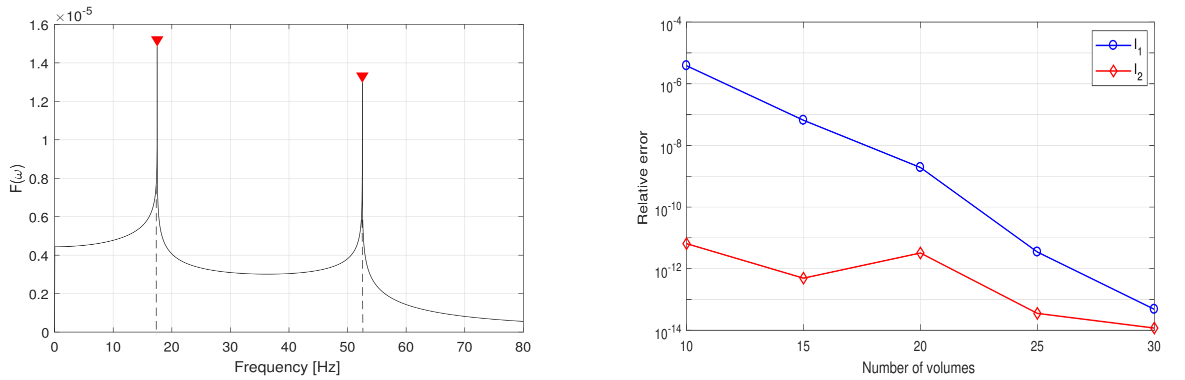
This section investigates the influence of parameters such as number of integration intervals (volumes), and the severity of decay and oscillation of the integrand, in the accuracy of integration of functions with multiple oscillatory frequencies, with or without closed-form expression for the integrands.

In order to illustrate the influence of the number of integration volumes (the number of terms in the extrapolated series; the number of integration subintervals) in the result of the integral, consider the improper integrals given in Eqs. 7 and 8, the close-form solution of which are known.

$$I_1 = \int_0^\infty J_1(3x/2)J_0(x)dx = 2/3 \tag{6}$$

$$I_2 = \int_0^\infty x^{-4}J_0(x)J_5(2x)dx = 27/4096 \tag{7}$$

Figure 1b shows the relative error of the present method compared with the analytical solution, for different numbers of volumes. A considerably low number of volumes around 30 is enough to obtain relative errors close to double-precision machine epsilon in both cases.



(a) (b)

Figure 1: (a) illustration of FFT of an integrand with two oscillatory frequencies and (b) accuracy of present method in solving Eq. 7 and 8.

In order to illustrate the cases in which a closed-form expression of the integrand is unavailable, consider the problem in which the integrand results from the solution of a linear system,

$$[A]\{s\} = \{b\} \tag{8}$$

where

$$[A] = \begin{bmatrix} 1 & 1 \\ e^{\lambda x}J_0(\omega_2 x)^{-1} & 0 \end{bmatrix} \quad \text{and} \quad \{b\} = \{J_1(\omega_1 x) \quad J_0(\omega_2 x)\}^T \tag{9}$$

Clearly, the solution $\{s\}$ will involve multiplication of Bessel functions of different orders and oscillation-decay characteristics, and in this particular case $\{s\}$ is trivial to obtain. Nevertheless, this problem can represent generalized situations in which integrals such as

$$I_3(\lambda, \omega_1, \omega_2) = \int_0^\infty \{s\}dx \tag{10}$$

need to be approximated. Figure 2 shows the results obtained with the present scheme in the solution of Eq. 11. Different parameters λ, ω_1, and ω_2 are considered. The integration is compared with a classical adaptive quadrature integration ("Reference") and good agreement is observed.

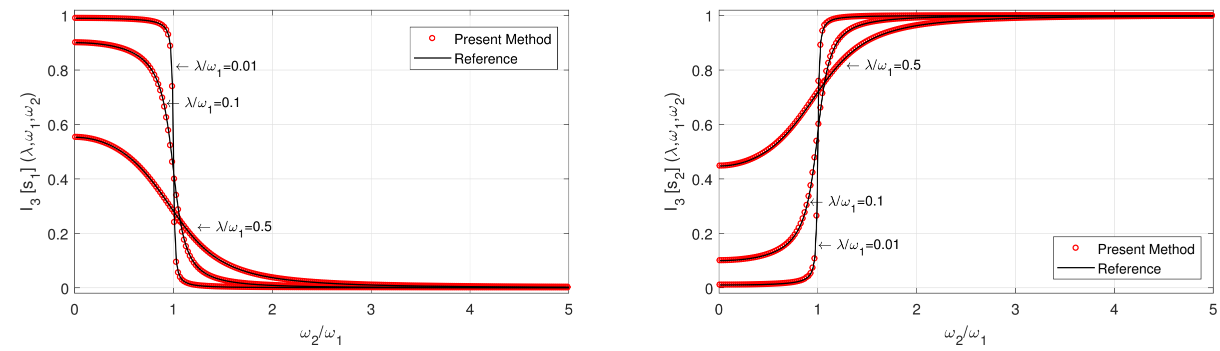


Figure 2: Integration of Eq. 11 for different parameters.

CONCLUSION

This work proposed a numerical integration scheme for oscillatory-decaying integrands containing more than one oscillatory frequency. The integration scheme uses series extrapolation with the ε-algorithm, wherein the terms of the series are integrated between interval endpoints determined with aid of fast Fourier transforms. The integration scheme has shown to be robust and capable of solving the title problem for different parameters, including the case in which closed-form expressions of the integrand are unavailable.

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