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Discrete Mechanics and Optimal Control Applied to Mechanical Systems

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Abstract: Optimal control is extremely important in many research areas such as engineering, biology and economy. Normally, the interest is to guide some system in a way that some performance index is minimized. Some interesting engineering optimal control problems include the determination of the minimum-time path of a vehicle or the minimization of the control effort during a satellite maneuver. This work is concerned with an indirect approach for solving optimal control problems. The methodology consists in solving the two-point boundary value problem which comes from the necessary conditions for optimality. Generally, some discretization technique is used for solving the problem numerically such as collocation, shooting and multiple shooting. We propose the application of a discrete variational principle for obtaining the discrete-time necessary conditions for optimality. Hence, the discretization is carried-out in the beginning of the solution process. We illustrate numerically the method by optimizing a low thrust satellite orbit transfer.

Keywords: discrete mechanics, optimal control, indirect method

INTRODUCTION

Direct and indirect methods have been developed for the numerical resolution of optimal control problems. The indirect methods are based on the explicit expression of the necessary optimality conditions that are derived via the Pontryagin maximum principle. In contrast, direct methods transform the optimal control problem into a finite dimensional nonlinear optimization problem by means of a finite dimensional parametrization of the controls, or of both, states and controls. Junge *et al.* (2005) proposes the Discrete Mechanics and Optimal Control (DMOC) method, whereby the discrete equations of motion are obtained via discrete variational principle, presented by Marsden and West (2001). Then, the corresponding discrete-optimal control problem is solved by using, for example, a Sequential Quadratic Programming (SQP) algorithm, which is a direct approach. This approach has been widely used by many researches. For instance, Timmermann *et al.* (2011) utilizes the DMOC approach to design a feedforward control of a double pendulum mounted on a cart. Dubljevic *et al.* (2010) utilized the DMOC approach with Model Predictive Control to regulate the temperature in a reaction-diffusion system. Zu *et al.* (2017) proposed a trajectory design methodology for quadrotors which explores the benefits of both DMOC method and Dubins curve algorithms. Xu *et al.* (2017) developed a novel nonlinear Model Predictive Control formulation based on the DMOC approach. Differently from the aforementioned works, this paper is focused on the indirect approach to solve the optimal control problem for mechanical systems.

This paper is focused on the optimal control of dynamical systems whose behavior can be described by the Lagrange-d'Alembert's principle. It makes use of discrete calculus of variations to numerically solve the problem. Recently, many researchers have been useing this approach to derive the discrete equations of motion and then solve the optimal control problem via direct methods, where the problem is transformed into a finite dimensional nonlinear programming problem. This work proposes to apply the discrete variational principle to obtain the discrete equations of motion of the mechanical system under consideration and derive the discrete-time necessary conditions for optimality for the control problem. The corresponding two-point boundary value problem is solved via an indirect approach.

PROBLEM FORMULATION

This section discusses a computational approach for finding the optimal control input numerically while preserving the dynamic properties of the system, such as the total mechanical energy, in the case of conservative systems. This approach is based on the formulation of discrete mechanics discussed in Marsden and West (2001). Differently from traditional approaches, whereby the optimal control problem is discretized in the last step prior to numerical resolution, one can apply the discrete version of a variational principle to obtain a discrete dynamic model of a system, as well as the discrete necessary conditions for optimality. The discrete optimal control problem is then solved by using the indirect method.

Variational Mechanics

For a Lagrangian system with external forces $f(q(t),\dot{q}(t),u(t))$, where $u(t):[0,T]\rightarrow\mathscr{U}$ is a control parameter, the motion, represented by generalized coordinates $q(t)$ must satisfy the Hamilton's principle,

$$\delta \int_0^T L(q(t),\dot{q}(t))dt + \int_0^T f(q(t),\dot{q}(t),u(t)) \cdot \delta q(t)\, dt = 0 \tag{1}$$

for all variations δq satisfying $\delta q(0) = \delta q(T) = 0$.

In (1), $L(q(t),\dot{q}(t) = K(\dot{q}) - U(q)$, K and U denoting the kinetic potential energies, respectively, is the Lagrangian function. In addition, the second term on the left-hand side of (1) is the virtual work done by f. Application of the rules of Variational Calculus leads to following Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = f(q,\dot{q},u). \tag{2}$$

Discrete Mechanics

The similar derivation can be performed in the framework of discrete Variational Mechanics. The time-discretization grid is defined by $\Delta t = \{t_k = kh \mid k = 0,\ldots,N\}, Nh = T$, where N is a positive integer and h is the step size. The continuous path $q : [0,T] \to Q$ is replaced by a discrete path $q_d : \{t_k\}_{k=0}^{N} \to Q$, where $q_k = q_d(kh) \to Q$ is an approximation to $q(kh)$ (Marsden and West, 2001). The continuous Lagrangian, $L(q,\dot{q})$, is replaced with a discrete Lagrangian, $L_d(q_k,q_{k+1},h)$ using the midpoint rule:

$$L_d(q_k,q_{k+1},h) = hL(\frac{q_k+q_{k+1}}{2}, \frac{q_{k+1}-q_k}{h}), \tag{3}$$

for the approximation of the action integral along the curve between q_k and q_{k+1}. Thus, one can write

$$\int_0^T L(q,\dot{q}) \approx \sum_{k=0}^{N-1} L_d(q_k,q_{k+1},h). \tag{4}$$

It should be noticed that it is possible to use more advanced quadrature rules to achieve integrators with higher orders of accuracy.

The variation of the discrete action with respect to q_k gives

$$\begin{aligned} &\delta \sum_{k=0}^{N-1} L_d(q_k,q_{k+1},h) \\ &= \sum_{k=0}^{N-1} \left[D_1 L_d(q_k,q_{k+1},h) \cdot \delta q_k + D_2 L_d(q_k,q_{k+1},h) \cdot \delta q_{k+1}\right] \\ &= \sum_{k=1}^{N-1} \left[D_1 L_d(q_k,q_{k+1},h) + D_2 L_d(q_{k-1},q_k,h)\right] \cdot \delta q_k \\ &+ D_1 L_d(q_0,q_1) \cdot \delta q_0 + D_2 Ld(q_{N-1},q_N) \cdot \delta q_N, \end{aligned} \tag{5}$$

Using a discrete integration by parts and the condition that $\delta q_0 = \delta q_N = 0$ yields

$$\delta \sum_{k=0}^{N-1} L_d(q_k,q_{k+1},h) = \sum_{k=1}^{N-1} \left[D_1 L_d(q_k,q_{k+1},h) \cdot + D_2 L_d(q_{k-1},q_k,h)\right] \cdot \delta q_k. \tag{6}$$

Note that D_1 and D_2 denote the derivative with respect to first and second arguments of L_d, respectively. The discrete Euler-Lagrange equations are obtained by enforcing the condition that the variations are required to vanish for all δq_k, resulting in the discrete-time Euler-Lagrange equation

$$D_2 L_d(q_{k-1},q_k,h) + D_1 L_d(q_k,q_{k+1},h) = 0. \tag{7}$$

The continuous virtual work term in Eq. (1) is approximated by

$$f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{hk}^{(k+1)h} f(q(t),\dot{q}(t),u(t)) \cdot \delta q(t)\, dt, \tag{8}$$

where f_k^-, f_k^+ are the left and right discrete forces, respectively, which are combined to represent the discrete force f_k.

One can interpret the left discrete force f_{k-1}^+ as the force resulting from the continuous control force acting during the time span $[t_{k-1},t_k]$ on the node q_k. The right discrete force f_k^- is the force acting on q_k resulting from the continuous control force during the time span $[t_k, t_{k+1}]$. Therefore, the discrete Lagrange-d'Alembert principle requires the discrete curve $\{q_k\}_{k=0}^{N}$ to satisfy

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}, h) + \sum_{k=0}^{N-1} [f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1}] = 0, \tag{9}$$

for all variations δq_k such that $\delta q_0 = \delta q_N = 0$. This leads to the discrete Euler-Lagrange equations

$$D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) + f_{k-1}^+ + f_k^- = 0. \tag{10}$$

Applying the Legendre transformation (Marsden and West, 2001) to the discrete equation (10), one obtains

$$p_k = -D_1 L_d(q_k, q_{k+1}) - f_k^-, \tag{11}$$

$$p_{k+1} = D_2 L_d(q_k, q_{k+1}) + f_k^+, . \tag{12}$$

Optimal Control

The optimal control problem is to find the control input sequence that minimizes a cost function given by

$$J_d = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k). \tag{13}$$

Several types of control problems can be considered, as mentioned before. For simplicity, this paper is focused on optimal control problems with a fixed terminal time and fixed terminal conditions. Hence, the discrete-time optimal control problem is stated as follows:

$$\min_{u_k} J_d = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k), \tag{14}$$

subject to

$$\begin{aligned} p_k &= -D_1 L_d(q_k, q_{k+1}) - f_k^-, \\ p_{k+1} &= D_2 L_d(q_k, q_{k+1}) + f_k^+, \end{aligned} \tag{15}$$

$$\begin{aligned} q(0) &= q_0, \\ p(0) &= p_0, \\ q(T) &= q_N, \\ p(T) &= p(N). \end{aligned} \tag{16}$$

The discrete cost function C_d, the discrete Lagrangian L_d, and the discrete forces are approximated using the midpoint rule, and constant control parameters are assumed on each time interval, according to:

$$C_d(u_k) = hC\left(\frac{q_k + q_{q+1}}{2}, \frac{q_{k+1} - q_k}{h}, u_k\right), \tag{17}$$

$$L_d(q_k, q_{k+1}, u_k) = hL\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}\right), \tag{18}$$

$$f_k^- = f_k^+ = \frac{h}{2} f\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}, u_k\right). \tag{19}$$

Discrete Mechanics and Optimal Control

The discrete-time necessary conditions for optimality must now be derived. Instead of discretizing the continuous-time necessary conditions to solve the optimal control problem, one can derive optimality conditions using a discrete-time version of the calculus of variations: discrete-time Hamilton's equations (11)–(12) are enforced by using Lagrange multipliers, and the variation of the corresponding augmented cost function is set to zero. The resulting necessary conditions are expressed as a two-point boundary value problem in a discrete-time setting. More explicitly, one can define the augmented cost functional as:

$$\mathscr{L}_d = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k) + \lambda_{k,0}^{\mathrm{T}}[p_k + D_1 L_d(q_k, q_{k+1}) + f_k^-] + \lambda_{k,1}^{\mathrm{T}}[p_{k+1} - D_2 L_d(q_k, q_{k+1}) - f_k^+]. \tag{20}$$

The variation of the augmented cost functional yields the discrete-time necessary conditions for optimality. By taking the first variation of $\mathscr{L}_d$ in (20), one obtains:

$$\begin{aligned}
\delta\mathscr{L}_d = \sum_{k=0}^{N-1} & D_{q_k}C_d(q_k,q_{k+1},u_k)\cdot\delta q_k + D_{q_{k+1}}C_d(q_k,q_{k+1},u_k)\cdot\delta q_{k+1} + D_{u_k}C_d(q_k,q_{k+1},u_k)\cdot\delta u_k \\
& + \lambda_{k,0}^{\mathrm{T}}\cdot\delta p_k + \lambda_{k,0}^{\mathrm{T}}D_{q_k}D_1L_d(q_k,q_{k+1})\cdot\delta q_k + \lambda_{k,0}^{\mathrm{T}}D_{q_{k+1}}D_1L_d(q_k,q_{k+1})\cdot\delta q_{k+1} \\
& + \lambda_{k,0}^{\mathrm{T}}D_{q_k}f_k^-\cdot\delta q_k + \lambda_{k,0}^{\mathrm{T}}D_{q_{k+1}}f_k^-\cdot\delta q_{k+1} + \lambda_{k,0}^{\mathrm{T}}D_{u_k}f_k^-\cdot\delta u_k \\
& + \lambda_{k,1}^{\mathrm{T}}\cdot\delta p_{k+1} - \lambda_{k,1}^{\mathrm{T}}D_{q_k}D_2L_d(q_k,q_{k+1})\cdot\delta q_k - \lambda_{k,1}^{\mathrm{T}}D_{q_{k+1}}D_2L_d(q_k,q_{k+1})\cdot\delta q_{k+1} \\
& - \lambda_{k,1}^{\mathrm{T}}D_{q_k}f_k^+\cdot\delta q_k - \lambda_{k,1}^{\mathrm{T}}D_{q_{k+1}}f_k^+\cdot\delta q_{k+1} - \lambda_{k,1}^{\mathrm{T}}D_{u_k}f_k^+\cdot\delta u_k,
\end{aligned} \tag{21}$$

where $D_{q_k}, D_{q_{k+1}}$ and D_{u_k} are the derivatives with respect to q_k, q_{k+1} and u_k.

Using the fact that the variations vanish at the end points, the summation can be reindexed as follows:

$$\begin{aligned}
\delta\mathscr{L}_d = \sum_{k=1}^{N-1} & D_{q_k}C_d(q_k,q_{k+1},u_k)\cdot\delta q_k + D_{q_k}C_d(q_{k-1},q_k,u_{k-1})\cdot\delta q_k + D_{u_k}C_d(q_k,q_{k+1},u_k)\cdot\delta u_k \\
& + \lambda_{k,0}^{\mathrm{T}}\cdot\delta p_k + \lambda_{k,0}^{\mathrm{T}}D_{q_k}D_1L_d(q_k,q_{k+1})\cdot\delta q_k + \lambda_{k-1,0}^{\mathrm{T}}D_{q_k}D_1L_d(q_{k-1},q_k)\cdot\delta q_k \\
& + \lambda_{k,0}^{\mathrm{T}}D_{q_k}f_k^-\cdot\delta q_k + \lambda_{k-1,0}^{\mathrm{T}}D_{q_k}f_{k-1}^-\cdot\delta q_k + \lambda_{k,0}^{\mathrm{T}}D_{u_k}f_k^-\cdot\delta u_k \\
& + \lambda_{k-1,1}^{\mathrm{T}}\cdot\delta p_k - \lambda_{k,1}^{\mathrm{T}}D_{q_k}D_2L_d(q_k,q_{k+1})\cdot\delta q_k - \lambda_{k-1,1}^{\mathrm{T}}D_{q_k}D_2L_d(q_{k-1},q_k)\cdot\delta q_k \\
& - \lambda_{k,1}^{\mathrm{T}}D_{q_k}f_k^+\cdot\delta q_k - \lambda_{k-1,1}^{\mathrm{T}}D_{q_k}f_{k-1}^+\cdot\delta q_k - \lambda_{k,1}^{\mathrm{T}}D_{u_k}f_k^+\cdot\delta u_k.
\end{aligned} \tag{22}$$

The variation $\mathscr{L}_d$ must be equal to zero for all possible variations along the optimal solution. Thus, the discrete-time necessary conditions for optimality for the discrete-time optimal control problem are given as follows:

- Stationarity Condition

$$D_{u_k}C_d(q_k,q_{k+1},u_k) + \lambda_{k,0}^{\mathrm{T}}D_{u_k}f_k^- - \lambda_{k,1}^{\mathrm{T}}D_{u_k}f_k^+ = 0. \tag{23}$$

- Costate equation

$$\begin{aligned}
\lambda_{k,0} + \lambda_{k-1,1} &= 0, \\
D_{q_k}C_d(q_k,q_{k+1},u_k) + D_{q_k}C_d(q_{k-1},q_k,u_{k-1}) & \\
+\lambda_{k,0}^{\mathrm{T}}D_{q_k}D_1L_d(q_k,q_{k+1}) + \lambda_{k-1,0}^{\mathrm{T}}D_{q_k}D_1L_d(q_{k-1},q_k) & \\
+\lambda_{k,0}^{\mathrm{T}}D_{q_k}f_k^- + \lambda_{k-1,0}^{\mathrm{T}}D_{q_k}f_{k-1}^- & \\
-\lambda_{k,1}^{\mathrm{T}}D_{q_k}D_2L_d(q_k,q_{k+1}) - \lambda_{k-1,1}^{\mathrm{T}}D_{q_k}D_2L_d(q_{k-1},q_k) & \\
-\lambda_{k,1}^{\mathrm{T}}D_{q_k}f_k^+ - \lambda_{k-1,1}^{\mathrm{T}}D_{q_k}f_{k-1}^+ &= 0.
\end{aligned} \tag{24}$$

- Boundary Conditions

$$\begin{aligned}
q(0) &= q_0, \\
p(0) &= p_0, \\
q(N) &= q_N, \\
p(N) &= p_N.
\end{aligned} \tag{25}$$

The necessary conditions for optimality are represented by a discrete-time two point boundary value problem. One must find the control input sequence u_k, state (q_k, p_k), and costates $(\lambda_{k,0},\lambda_{k,1})$ that satisfy the stationary condition (23), the equations of motion (11)–(12), and the costate equations (24), under the given boundary conditions. Furthermore, by using the standard Legendre transformation, one can obtain $p_k = D_2L(q_k,\dot{q}_k)$. Then, $p_0 = D_2L(q_0,\dot{q}_0)$ and $p_N = D_2L(q_N,\dot{q}_N)$, where $(q_0,\dot{q}_0)$ and $(q_N,\dot{q}_N)$ are fixed.

Numerical Example - a Satellite low thrust orbit transfer

One considers as example the optimal control problem associated to a satellite low thrust orbital transfer, formulated using polar coordinates $q=(r,\psi)$. For this task, one considers to transfer the satellite from one circular orbit to one in the same plane with a larger radius, while the number of revolutions d around the Earth is fixed. In addition, one considers a given initial radius and angular position and velocity $(r(0),\psi(0),\dot{r}(0),\dot{\psi}(0))$ to a desired terminal condition $(r(T),\psi(T),\dot{r}(T),\dot{\psi}(T))$ during a fixed time Nh, while minimizing the control effort. The system dynamics is obtained via discrete variational principle, by using the discrete forced Legendre transformation. The discrete optimal control problem is formulated as follows.

$$\min_{u_k} J_d = \frac{h}{2}\sum_{k=0}^{N-1} u_k^2, \tag{26}$$

subject to

$$\begin{aligned}
m\dot{r}_k + m\left(\frac{r_k - r_{k+1}}{h}\right) + m\left(\frac{r_k + r_{k+1}}{4}\right)\frac{(\psi_k-\psi_{k+1})^2}{h} - \frac{2hM\gamma m}{(r_k+r_{k+1})^2} &= 0,\\
m\dot{\psi}_k r_k^2 + \frac{hu_k(r_k+r_{k+1})}{4} + \frac{(r_k+r_{k+1})(\psi_k-\psi_{k+1})}{2h} &= 0,\\
m\dot{r}_{k+1} + m\left(\frac{r_k - r_{k+1}}{h}\right) - m\left(\frac{r_k + r_{k+1}}{4}\right)\frac{(\psi_k-\psi_{k+1})^2}{h} + \frac{2hM\gamma m}{(r_k+r_{k+1})^2} &= 0,\\
m\dot{\psi}_{k+1} r_{k+1}^2 - \frac{hu_k(r_k+r_{k+1})}{4} + \frac{(r_k+r_{k+1})(\psi_k-\psi_{k+1})}{2h} &= 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&(r(0),\psi(0)) = (r_0,\psi_0),\\
&(\dot{r}(0),\dot{\psi}(0)) = (\dot{r}_0,\dot{\psi}_0),\\
&(r(T),\psi(T)) = (r_N,\psi_N),\\
&(\dot{r}(T),\dot{\psi}(T)) = (\dot{r}_N,\dot{\psi}_N).
\end{aligned} \tag{28}$$

One can obtain the discrete-time necessary conditions for optimality by using equations (23)–(24). They are given as follows.

- Stationarity condition

$$8u_k + (r_k+r_{k+1})(\lambda^{\psi}_{k,0} - \lambda^{\psi}_{k,1}) = 0 \tag{29}$$

- Costate equation

$$\begin{aligned}
&h\lambda^r_{k,1}((m(-\psi_k^2+2\psi_k\psi_{k+1}-\psi_{k+1}^2+4))/(4h^2) - (4M\gamma m)/(r_k+r_{k+1})^3)\\
&-h\lambda^r_{k-1,0}((m(-\psi_k^2+2\psi_k\psi_{k-1}-\psi_{k-1}^2+4))/(4h^2) - (4M\gamma m)/(r_k+r_{k-1})^3)\\
&+h\lambda^r_{k,0}((m(\psi_k^2-2\psi_k\psi_{k+1}+\psi_{k+1}^2+4))/(4h^2) + (4M\gamma m)/(r_k+r_{k+1})^3)\\
&-h\lambda^r_{k-1,1}((m(\psi_k^2-2\psi_k\psi_{k-1}+\psi_{k-1}^2+4))/(4h^2) + (4M\gamma m)/(r_k+r_{k-1})^3)\\
&+(h\lambda^{\psi}_{k,0}u_k)/4 - (h\lambda^{\psi}_{k,1}u_k)/4 + (h\lambda^{\psi}_{k-1,0}u_{k-1})/4-\\
&(h\lambda^{\psi}_{k-1,1}u_{k-1})/4 + (\lambda^{\psi}_{k,0}m(r_k/2+r_{k+1}/2)(2\psi_k-2\psi_{k+1}))/(2h)\\
&+(\lambda^{\psi}_{k,1}m(r_k/2+r_{k+1}/2)(2\psi_k-2\psi_{k+1}))/(2h)\\
&-(\lambda^{\psi}_{k-1,0}m(r_k/2+r_{k-1}/2)(2\psi_k-2\psi_{k-1}))/(2h)-\\
&(\lambda^{\psi}_{k-1,1}m(r_k/2+r_{k-1}/2)(2\psi_k-2\psi_{k-1}))/(2h) = 0\\
&(\lambda^{\psi}_{k,0}m(r_k+r_{k+1})^2)/(4h) - (\lambda^{\psi}_{k-1,1}m(r_k+r_{k-1})^2)/(4h)\\
&-(\lambda^{\psi}_{k-1,0}m(r_k+r_{k-1})^2)/(4h) + (\lambda^{\psi}_{k-1,0}m(r_k+r_{k+1})^2)/(4h)\\
&+(\lambda^r_{k,0}m(r_k/2+r_{k+1}/2)(2\psi_k-2\psi_{k+1}))/(2h)\\
&-(\lambda^r_{k,1}m(r_k/2+r_{k+1}/2)(2\psi_k-2\psi_{k+1}))/(2h)\\
&+(\lambda^r_{k-1,0}m(r_k/2+r_{k-1}/2)(2\psi_k-2\psi_{k-1}))/(2h)\\
&-(\lambda^r_{k-1,1}m(r_k/2+r_{k-1}/2)(2\psi_k-2\psi_{k-1}))/(2h) = 0\\
&\lambda^r_{k,0} + \lambda^r_{k-1,1} = 0\\
&\lambda^{\psi}_{k,0} + \lambda^{\psi}_{k-1,1} = 0.
\end{aligned} \tag{30}$$

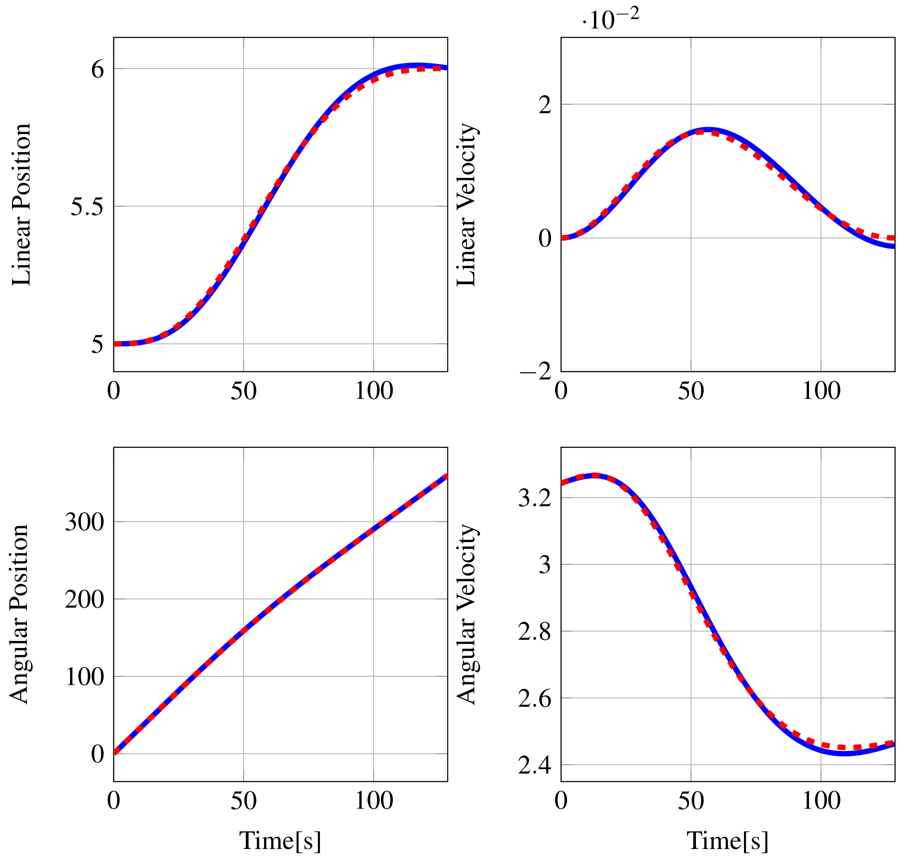

Figure 1: Satellite

The discrete-time necessary conditions for optimality are solved using the shooting method.

Simulation Results

The mass of the satellite is chosen as $m = 100$ kg and is moving in the gravitational field of the Earth which has mass equal $M = 6 \cdot 10^{24}$ kg. Furthermore, the gravitational constant is denoted by $\gamma = 6.673 \cdot 10^{-26}$. The boundary conditions are chosen as $(r(0), \psi(0)) = (5, 0)$; $(\dot{r}(0), \dot{\psi}(0)) = (0, \sqrt{\gamma M / r_0^3})$ and $(r(T), \psi(T)) = (6, 0)$; $(\dot{r}(T), \dot{\psi}(T)) = (0, \sqrt{\gamma M / r_1^3})$. The path must be completed in $T = d(T_0 + T_1)/2$ s, where $T_0 = 2\pi\sqrt{r_0^3/(\gamma M)}$, $T_1 = 2\pi\sqrt{r_1^3/(\gamma M)}$ and $d = 1$ revolution. The step size is $h = 0.01$ and the number of integration steps is $N = 12848$. Figure 1 presents the optimal path obtained for the satellite. The dashed red lines represent the solution obtained via *bvp5c*, which is an algorithm from MATLAB for solving boundary value problems. The blue lines are the solution obtained from discrete mechanics. Figure 2 presents a polar view of the satellite optimal trajectory. One can see that the mission is completed in one circular orbit, as specified. One can see a little mismatch between the two solutions, which is more evident in the graphic related to the control efforts in Figure 2. This can be due to the sensitivity of the solution with respect to the initial guesses of the non-specfied Lagrange multipliers. In this work, the initial guesses for the Lagrange multipliers are ajusted manually.

CONCLUSIONS

This paper concerns to the formulation of the discrete optimal control problem by using the discrete variational principle. The discrete mechanics approach provides an alternative way for solving optimal control problems. One advantage of this method is that the problem is discretized in the beginning of the solution process, which provides accurate mathematical models. Based on this approach, we developed the discrete-time necessary conditions for optimality, which are posed in a two-point boundary value problem, and is numerically solved via an indirect method using shooting method. The method was evaluated via computational simulations applied to an aerospace engineering problem. For future works, the indirect algorithm which was implemented to solve the discrete-time boundary value problem will be improved to providing more accurate solutions using the multiple shooting algorithm, for instance.

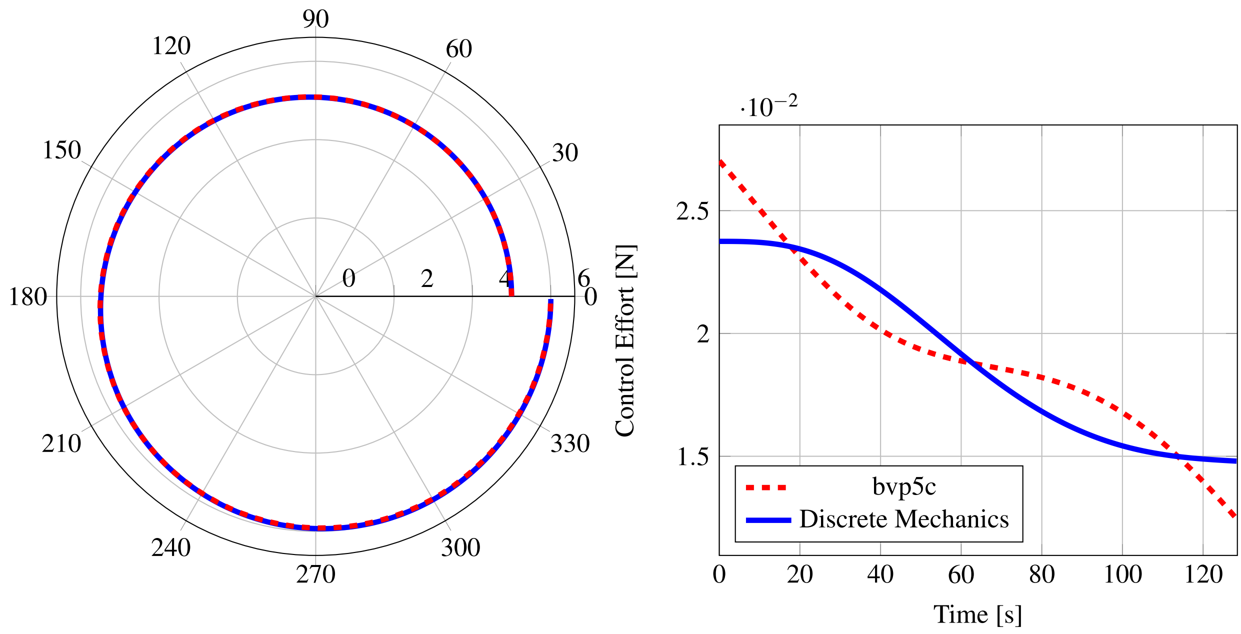

Figure 2: Satellite low thrust orbital transfer.

REFERENCES

Dubljevic, S., Kobilarov, M. and Ng, J., 2010. "Discrete mechanics optimal control (DMOC) and model predictive control (MPC) synthesis for reaction-diffusion process system with moving actuator". In *Proceedings of the 2010 American Control Conference*. IEEE. doi:10.1109/acc.2010.5531140.

Junge, O., Marsden, J.E. and Ober-Blobaum, S., 2005. "Discrete mechanics and optimal control". *IFAC Proceedings Volumes*, Vol. 38, No. 1, pp. 538–543. doi:10.3182/20050703-6-cz-1902.00745.

Marsden, J.E. and West, M., 2001. "Discrete mechanics and variational integrators". *Acta Numerica 2001*, Vol. 10, pp. 357–514. doi:10.1017/s096249290100006x.

Timmermann, J., Khatab, S., Ober-Blobaum, S. and Trachtler, A., 2011. "Discrete mechanics and optimal control and its application to a double pendulum on a cart". *IFAC Proceedings Volumes*, Vol. 44, No. 1, pp. 10199–10206. doi:10.3182/20110828-6-it-1002.01985.

Xu, K., Timmermann, J. and Trachtler, A., 2017. "Nonlinear model predictive control with discrete mechanics and optimal control". In *2017 IEEE International Conference on Advanced Intelligent Mechatronics (AIM)*. IEEE. doi:10.1109/aim.2017.8014272.

Zu, Y., Zhang, W. and Shan, J., 2017. "A trajectory design method for quadrotor based on DMOC method and dubins path algorithm". In *2017 36th Chinese Control Conference (CCC)*. IEEE. doi:10.23919/chicc.2017.8027741.

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