Nonlinear Analysis of a 2DOF Rotordynamic System Using the Normal Form Method

D.J.G. Maldonado[1], **A. Karev** [2], **P. Hagedorn**[3], **T.G. Ritto**[4], **and R. Sampaio**[5]

[1] Universidade Federal do Rio de Janeiro, Av. Horacio Macedo, 2030, Cidade Universitária, Rio de Janeiro - RJ.

[2] Technische Universität Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany.

[3] Technische Universität Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany.

[4] Universidade Federal do Rio de Janeiro, Av. Horacio Macedo, 2030, Cidade Universitária, Rio de Janeiro - RJ.

[5] Pontifícia Universidade Católica Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro - RJ.

Abstract: This paper shows the computation of the fixed points of a nonlinear rotordynamic system using the normal forms method. The system has a flexible unrounded rotor, two bearings, and a nonlinear spring-damper at the middle. The system equations are taken to a normal form by applying a set of coordinates transformation. Then, the fixed points are computed and compared with numerical integration.

Keywords: normal forms, nonlinear dynamics, fixed points.

1 INTRODUCTION

In rotating machinery, some components present a nonlinear behavior, even for small perturbations. That is the case of hydrodynamic bearings or annular seals. Some authors like Tondl [8], Muszynska [5] and Gasch [1] studied the effect of nonlinearities on the response of the machine. Depending on the parameters value, different type of responses are found, such as forward whirl, backward whirl, chaotic responses, among others. One of the most important analysis in nonlinear systems are the fixed points, as their behavior around those points resembles a linear system. Nonlinear systems are analyzed by different techniques such as perturbation methods, averaging methods, multiple scales methods and the normal form method. The latter is used in this work due to its advantages over other methods.

A set of coordinate transformations can be used to simplify the nonlinear equations for a certain point, e.g. around a fundamental or combinational resonance. Those coordinate transformations are found by using the normal form method, which are explained and used by Nayfeh [7, 6], Karev [3], Murdock [4], Hochlenert [2]. The first step is to express the equations in an autonomous first-order form. Second, a polynomial expansion is applied to keep terms up to the third order. Third, a modal matrix is used to decouple the linear part of the system. Fourth, a near-identity transformation is applied to eliminate as many nonlinear terms as possible from the equations. Finally, the equations are simplified further by expressing them in polar coordinates.

2 SYSTEM EQUATIONS

The system shown in Figure 1 considers an unrounded shaft with k_ζ, k_η as its stiffness along its principal axis of inertia, a disk of mass m_d, two massless anisotropic bearings with stiffness k_x, k_y, and a nonlinear spring-damper system with cubic stiffness and damping $k_{nx}, k_{ny}, c_{nx}, c_{ny}$. The system equations are:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}(t)\mathbf{q}(t) + \mathbf{K}_N\mathbf{q}(t)^3 + \mathbf{C}\dot{\mathbf{q}}(t)^3 = \mathbf{0} \tag{1}$$

where,

$$\begin{aligned} &\mathbf{M} = \begin{bmatrix} m_d & 0 \\ 0 & m_d \end{bmatrix}, \mathbf{q} = \begin{bmatrix} u_x(t) \\ u_y(t) \end{bmatrix}, \mathbf{C}_N = \begin{bmatrix} c_{nx} & 0 \\ 0 & c_{ny} \end{bmatrix}, \mathbf{K}_N = \begin{bmatrix} k_{nx} & 0 \\ 0 & k_{ny} \end{bmatrix}, \mathbf{K}_b = 2\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}, \\ &\mathbf{K}(t) = \mathbf{K}_r(t) - \mathbf{K}_r(t)\left(\mathbf{K}_r(t) + \mathbf{K}_b\right)^{-1}\mathbf{K}_r(t), \;\; \mathbf{K}_r(t) = \begin{bmatrix} k_\zeta \cos^2\Omega t + k_\eta \sin^2\Omega t & (k_\zeta - k_\eta)\cos\Omega t)\sin\Omega t \\ (k_\zeta - k_\eta)\cos\Omega t \sin\Omega t & k_\zeta \sin^2\Omega t + k_\eta \cos^2\Omega t \end{bmatrix}. \end{aligned} \tag{2}$$

$\mathbf{M}$ is the mass matrix, $\mathbf{K}(t)$ is the stiffness matrix, $\mathbf{K}_r(t)$ is the rotor stiffness, $\mathbf{K}_b$ is the bearings stiffness matrix, $\mathbf{K}_N$ and $\mathbf{C}_N$ are the nonlinear stiffness and damping matrices, $\mathbf{q}(t)$ is the displacement vector in terms of the displacement of the center of disk u_x, u_y. The notation $\mathbf{q}^2$ and $\mathbf{q}^3$ correspond to a element-wise power operation.

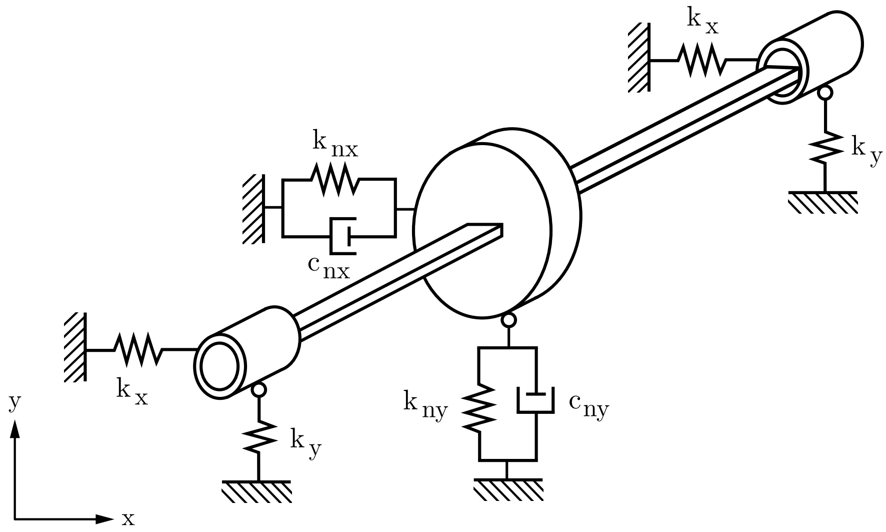

Figure 1: Rotordynamic system.

3 NORMAL FORM TRANSFORMATION

The objective of the normal form transformation is to determine the fixed points of Equation (1) by using a set of coordinate transformation in which the resulting equations are in a simpler form. The 5-step procedure is found below.

3.1 Augmented autonomous system

First, the equations are expressed in a first-order form by using the state vector

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^T = \begin{bmatrix} u_x & \dot{u}_x & u_y & \dot{u}_y & e^{i2\Omega t} & e^{-i2\Omega t} \end{bmatrix}^T, \tag{3}$$

where the complex exponentials were introduced to eliminate the time-dependent terms.

3.2 Polynomial expansion

A Taylor polynomial expansion of order 3 is applied and the equations take the form

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{f}_2(\mathbf{q}) + \mathbf{f}_3(\mathbf{q}), \tag{4}$$

where $\mathbf{A}$ is the Jacobian matrix, and $\mathbf{f}_2$, $\mathbf{f}_3$ are monomials of degree 2 and 3 respectively. Using (3) and (4) in (1), the following set of equations are obtained:

$$\begin{aligned}
\dot{q}_1 =& q_2, \\
\dot{q}_2 =& c_1 q_1 + c_2(q_5+q_6)q_1 + c_3(q_5-q_6)q_3 + c_4(q_5+q_6)^2 q_1 + c_5(q_6^2-q_5^2)q_3 + c_6 q_1^3 + c_7 q_2^3, \\
\dot{q}_3 =& q_4, \\
\dot{q}_4 =& c_8 q_3 + c_9(q_5+q_6)q_3 + c_{10}(q_5-q_6)q_1 + c_{11}(q_5+q_6)^2 q_3 + c_{12}(q_6^2-q_5^2)q_1 + c_{13} q_1^3 + c_{14} q_2^3, \\
\dot{q}_5 =& 2\Omega q_5 i, \\
\dot{q}_6 =& -2\Omega q_6 i,
\end{aligned} \tag{5}$$

where $c_1, \ldots, c_{14}$ are complex constants.

3.3 Modal transformation

By using the eigenvectors matrix $\mathbf{q} = \mathbf{R}\mathbf{x}$ of the linear part, Equation (5) is written as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{\Lambda}\mathbf{x} + \mathbf{F}_2\mathbf{x}^2 + \mathbf{F}_3\mathbf{x}^3 \tag{6}$$

where $\mathbf{\Lambda} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathrm{diag}(i\omega_1, -i\omega_1, i\omega_2, -i\omega_2, 2i\Omega, -2i\Omega)$, $\mathbf{F}_2 = \mathbf{R}^{-1}\mathbf{f}_2$, $\mathbf{F}_3 = \mathbf{R}^{-1}\mathbf{f}_3$. ω_1 and ω_2 are the natural frequencies and Ω the rotor speed.

3.4 Normal forms transformation

The objective of the normal form transformation is to take Equation 6 into the form

$$\dot{\mathbf{y}} = \mathbf{h}(\mathbf{y}) = \mathbf{\Lambda}\mathbf{y} + \mathbf{H}_2\mathbf{y}^2 + \mathbf{H}_3\mathbf{y}^3 \tag{7}$$

by using the near-identity transformation

$$\mathbf{x} = \mathbf{g}(\mathbf{y}) = \mathbf{y} + \mathbf{G}_2\mathbf{y}^2 + \mathbf{G}_3\mathbf{y}^3. \tag{8}$$

By substituting Equations (7) and (8) into (6) and equating the coefficients, matrices $\mathbf{G}_2$ and $\mathbf{G}_3$ can be chose such that the most amount of terms from $\mathbf{H}_2$ and $\mathbf{H}_3$ are eliminated. Thus, all nonlinear $y_1^{m_{k1}}, \cdots y_6^{m_{kl}}$ terms can be eliminated except those that fulfill the resonance condition

$$\lambda_j = \sum_{l=1}^{6} m_{kl}\lambda_l \ , \ \text{with} \ \sum_{l=1}^{6} m_{kl} = i \ , \ \text{for} \ \ i = 1,2,3. \tag{9}$$

where n and k are index related to the j-th row and the k-th column of matrices $\mathbf{H}$ and $\mathbf{G}$.

3.5 Normal Form in Polar Coordinates

The final transformation

$$y_1 = r_1 e^{i(\varphi_1+\Omega t)},\ y_2 = r_1 e^{-i(\varphi_1+\Omega t)},\ y_3 = r_2 e^{i\varphi_2},\ y_4 = r_2 e^{-i\varphi_2}, \tag{10}$$

takes the complex 6-dim system of Equation (7) to a real 4-dim set of Equations. Inserting (10) into (7), and considering an analysis around $\Omega \approx \omega_2$, yields the equation in normal form

$$\begin{aligned} \dot{r}_1 &= d_1 r_1^3 + d_2 \sin(2\varphi_1) r_1, \\ \dot{\varphi}_1 &= d_3 r_1^2 + d_4 \cos(2\varphi_1) + d_5 - \Omega, \\ \dot{r}_2 &= d_6 r_2^3, \\ \dot{\varphi}_2 &= d_7 r_2^2 + d_8. \end{aligned} \tag{11}$$

where $d_1 \ldots, d_8$ are real constants.

4 NUMERICAL RESULTS

The normal forms is validated by comparing the numerical integration of Equation (5) and the results of Equation (11) for $\Omega \approx \omega_2 \approx 31.62$, as shown in Figure 2. The difference between both curves is relative small, showing a good agreement.

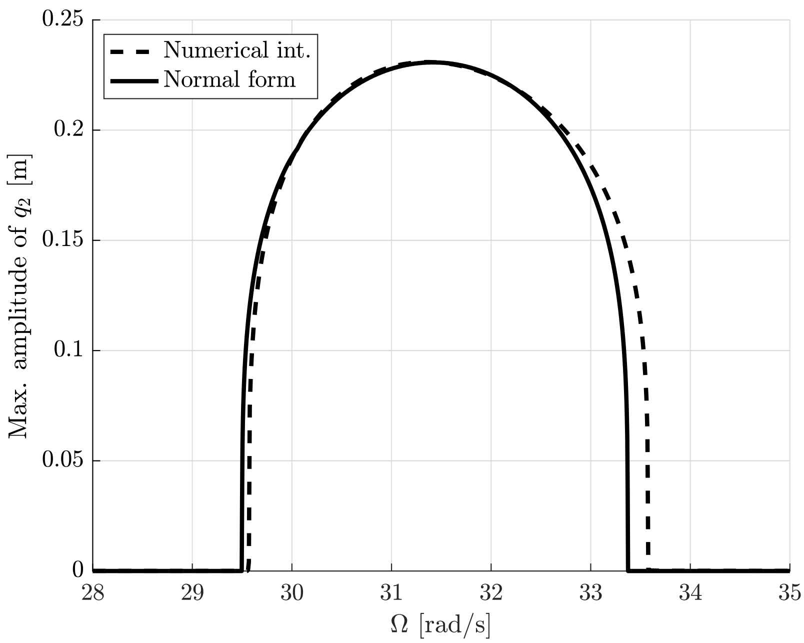

Figure 2: Fixed point variation with the excitation frequency for $k_x = 5 \cdot 10^2$N/m, $k_y = 1.75 \cdot 10^3$N/m, $k_\eta = 2 \cdot 10^3$N/m, $k_\zeta = 10^3$N/m, $m = 1kg$, $c_{nx} = c_{ny} = 50$Ns/m.

4.1 Conclusions

In this paper it was able to simplify the nonlinear equations of a rotordynamic system. A set of 5 coordinate transformations was used to eliminate as many nonlinear terms as possible. The most important transformation is the normal form. It is based on a near-identity transformation that can eliminate all nonlinear terms except those that fulfill a resonance condition. The final equations are written in polar coordinates and represent the behavior of the system near $\Omega \approx \omega_2$. The method was compared with the original equations and they showed a good agreement.

REFERENCES

[1] R. Gasch and H. Pfützner. *Rotordynamik: Eine Einführung*. Springer-Verlag, March 2013.

[2] D Hochlenert. *Normalformen und Einzugsbereiche nichtlinearer dynamischer Systeme: Beispiele und technische Anwendungen (Normal forms and domains of attraction of nonlinear dynamical systems: examples and technical applications)*. Habilitation Thesis, Technische Universität Berlin, 2012.

[3] Artem Karev, Daniel Hochlenert, and Peter Hagedorn. Asynchronous parametric excitation, total instability and its occurrence in engineering structures. *Journal of Sound and Vibration*, August 2018.

[4] James Murdock. *Normal Forms and Unfoldings for Local Dynamical Systems*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[5] Agnieszka Muszynska. *Rotordynamics*. CRC Press, May 2005.

[6] Ali H. Nayfeh. *The Method of Normal Forms*. John Wiley & Sons, August 2011.

[7] Ali H. Nayfeh and Dean T. Mook. *Nonlinear Oscillations*. John Wiley & Sons, September 2008.

[8] Ales Tondl. *Some Problems of Rotor Dynamics*. Chapman and Hall, March 1966.

RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.