Metamaterial Plate Vibration with WFE Method using Continuous Local Resonator Modeling

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Abstract: This research is an application of Wave Finite Element (WFE) method to model a metamaterial reinforced-plate elastic structure using Continuous Local Resonators (CLR) modeling. A CLR is a local resonator made with a solid cubical block (mass) connected to one end of four small beams and fixed in the others. This geometry provides a local resonator with many degrees-of-freedom that implies corresponding natural frequencies and mode shapes, which affects the plate metamaterial behavior. A reinforced-plate slice is modeled by Finite Element (FE) method, having periodic cells with a CLR in each one. The slices are used to find the wave numbers and corresponding modes by WFE to obtain dispersion diagrams and forced responses (FRFs). WFE results of FRFs are compared with conventional FE results. The results show that the bandgaps can be identified using both results (FRF and dispersion diagram). Also, the bandgaps occur for all wave modes tuned by the CLR, and the corresponding responses at these modes are attenuated.

Keywords: metamaterial, plate, bandgap, local resonator

INTRODUCTION

The effect of wave propagations in engineering periodic structures known as "propagation constant" by Mead (1970) are studied nowadays for many researchers that uses this concept together with the Bloch's theorem. This increase the number of possibilities to find new types of structures. Such as the elastic phononic crystals and metamaterials. Hussein et al. (2014) presented a historical approach to the evolution of research on these new type of materials. This review showed elastic configurations ranging from trusses and ribbed shells to phononic crystals and metamaterials.

New methods have been developed that use the same basic concepts of propagation constant together with approximated solution to find a way to solve complex engineering models that cannot be solved analytically neither numerically using traditional methods (Mencik and Ichchou, 2005; Duhamel et al., 2006; Mace et al., 2008; Waki et al., 2009). One of them is the WFE, which consists of modeling a small slice of elastic structure by conventional FE, then apply periodicity condition with Floquet-Bloch's theorem to obtain the transfer matrix eigenproblem. The method has been applied in various types of finite element model. The solution provides wavenumbers and corresponding wave-modes of a structure slice, from where dispersion diagram and frequency response functions (FRFs) are obtained.

Periodic structures with local resonators, also known as elastic metamaterial, contains an array of localized resonant structures. This kind of structure are responsible to create tuned bandpag in specific frequency bands (Wang, 2006). Miranda et al.(2018) investigated the wave propagating in an elastic metamaterial thin plate. They use the Plane Wave Expansion (PWE) method with the Kirchhoff-Love thin plate theory to calculate the bandgaps.

In this work, it is shown an elastic metamaterial reinforced-plate modeling by WFE and conventional FE methods. Continuous local resonators (CLR) are designed as a mass-spring system constructed using a solid cubical block (mass) connected by four very small beams (springs) to the plate stiffener-beams (Figure 1). In order to attenuate the plate excitation responses, the CLRs are designed to be tuning at the second plate natural frequency. Then, the CLR's first natural frequency is tuned approximately equal to that. The reinforced-plate metamaterial behavior are analyzed using the dispersion diagram and forced responses (FRFs). The forced response calculated by WFE is verified using conventional FE method. The results shows that bandgaps occur in more than one mode and the corresponding responses at these modes are attenuated.

METAMATERIAL ELASTIC PLATE MODELING BY WFE METHODS

Wave Finite Element

The WFE method consists to model a slice (Fig. 1) of the periodic structure by FE method and find the dynamic stiffness matrix using the equilibrium equation given as,

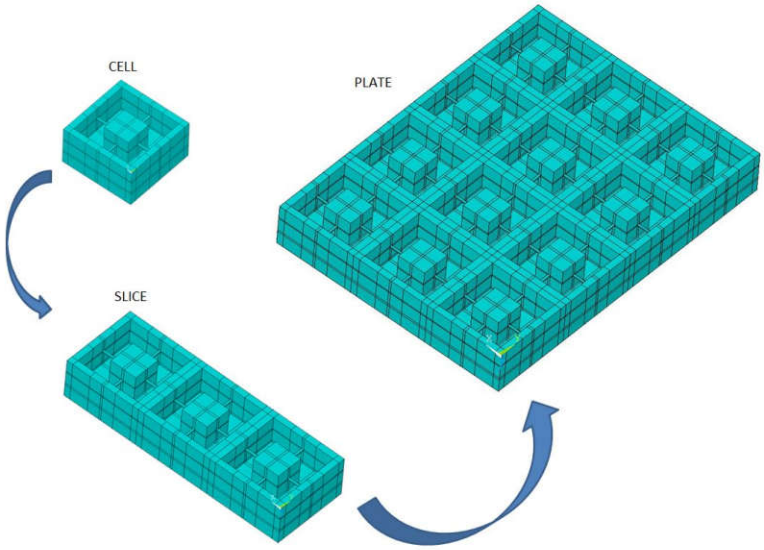


Figure 1: Plate model.

$$(\mathbf{K}-\omega^2\mathbf{M})\mathbf{u}=\mathbf{F} \quad \text{or} \quad \mathbf{D}\mathbf{u}=\mathbf{F}, \tag{1}$$

where, $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, $\mathbf{u}$ is the displacement vector, $\mathbf{F}$ is the external force vector, ω is the circular frequency. The dynamic stiffness matrix, $\mathbf{D}$, can be partitioned as:

$$\begin{bmatrix} \mathbf{D}_{ii} & \mathbf{D}_{il} & \mathbf{D}_{ir} \\ \mathbf{D}_{li} & \mathbf{D}_{ll} & \mathbf{D}_{lr} \\ \mathbf{D}_{ri} & \mathbf{D}_{rl} & \mathbf{D}_{rr} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{u}_l \\ \mathbf{u}_r \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_i \\ \mathbf{F}_l \\ \mathbf{F}_r \end{Bmatrix}, \tag{2}$$

where l, i and r represents the left, internal and right degrees-of-freedom (DOF), respectively. Considering $\mathbf{F}_i = 0$, the internal displacement from the Eq. (2) can be obtained as:

$$\mathbf{u}_i = \mathbf{D}_{ii}^{-1}(\mathbf{D}_{il}u_l + \mathbf{D}_{ir}u_r). \tag{3}$$

Substituting Eq. (3) into Eq. (2) the condensed dynamic stiffness matrix is obtained as,

$$\begin{bmatrix} \mathbf{D}_{ll} & \mathbf{D}_{lr} \\ \mathbf{D}_{rl} & \mathbf{D}_{rr} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_l \\ \mathbf{F}_r \end{Bmatrix}, \tag{4}$$

where, $\mathbf{D}_{ll} = \mathbf{D}_{ll} - \mathbf{D}_{li}\mathbf{D}_{ii}^{-1}\mathbf{D}_{il}$, $\mathbf{D}_{rl} = \mathbf{D}_{rl} - \mathbf{D}_{ri}\mathbf{D}_{ii}^{-1}\mathbf{D}_{il}$, $\mathbf{D}_{lr} = \mathbf{D}_{lr} - \mathbf{D}_{li}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir}$, $\mathbf{D}_{rr} = \mathbf{D}_{rr} - \mathbf{D}_{ri}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir}$.

Equation (4) can be transformed in a transfer matrix formulation by creating a state vector $\mathbf{q}_j = \{\mathbf{u}_j \quad \mathbf{F}_j\}^T$ with $j = l, r$, and applying to the Eq. (4) to obtain,

$$\underbrace{\begin{Bmatrix} \mathbf{u}_r \\ -\mathbf{F}_r \end{Bmatrix}}_{\mathbf{q}_r} = \underbrace{\begin{bmatrix} -\mathbf{D}_{lr}^{-1}\mathbf{D}_{ll} & -\mathbf{D}_{lr}^{-1} \\ \mathbf{D}_{rl} - \mathbf{D}_{rr}\mathbf{D}_{lr}^{-1}\mathbf{D}_{ll} & -\mathbf{D}_{rr}\mathbf{D}_{lr}^{-1} \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{Bmatrix} \mathbf{u}_l \\ \mathbf{F}_l \end{Bmatrix}}_{\mathbf{q}_l}. \tag{5}$$

Applying the Floquet-Bloch's theorem and considering consecutive unit-cells, displacement continuity condition, and the force balance (Mead, 1973) the Eq. (5) becomes an eigenproblem given by,

$$\mathbf{T}\mathbf{q}_l = e^{\mu}\mathbf{q}_l, \tag{6}$$

where e^{μ} are the eigenvalues, $\mu = -ikL$ is the attenuation constant given as function of k the wavenumber, L the cell length and i the imaginary unit. To avoid ill-conditioning likely to occur in matrix $\mathbf{D}_{lr}^{-1}$, Eq. (6) can be rewritten in a representation by displacement vector alone by (Zhong and Williams, 1995):

$$e^{\mu}\underbrace{\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}_{ll} & -\mathbf{D}_{lr} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{Bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{Bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{D}_{rl} & \mathbf{D}_{rr} \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{Bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{Bmatrix}}_{\mathbf{w}} \tag{7}$$

where $\mathbf{w}$ is the displacement vector associated to the unit-cell. It can also be shown that $e^{-\mu}$ and $\mathbf{Lw} = \Phi$ are eigenvalues and eigenvectors of Eq. (7), respectively. The wave that traveling to the right and left directions are represented by $|e^{\mu_j}| \leq 1$ and $|e^{-\mu_j}| \geq 1$ eigenvalues respectively where $j = 1, 2, \ldots, n$.

Then, the state vectors $\mathbf{q}^{(m)}$ for a finite structure can be expressed as (Silva, 2014):

$$\mathbf{q}^{(m)} = \sum_j \Phi_j \mathbf{Q}_j^{(m+1)} = \sum_j \Phi_j e^{-ik_j d} \mathbf{Q}_j^{m}, \quad \text{with} \quad m = 1, 2, 3, \ldots, N, N+1. \tag{8}$$

where N is the slice number, k_j is the wavenumber. $\mathbf{Q}_j^{(m+1)}$ and $\mathbf{Q}_j^{(m)}$ are the wave vector amplitudes at the interface $m+1$ and m, respectively.

Modal organization

The wave mode are computed for a several discrete frequencies ω_m. The problem is to identify between all the defined modes in a frequency ω_{m+1} which correspond the mode i defined in the previous frequency ω_m. Then, for a mode j in the angular frequency $\omega + \Delta\omega$ to a $\Delta\omega$ small enough has been:

$$\left| \frac{\Phi_i(\omega)^H \Phi_i(\omega+\Delta\omega)}{||\Phi_i(\omega)|| ||\Phi_i(\omega+\Delta\omega)||} \right| = max \left\{ \left| \frac{\Phi_i(\omega)^H \Phi_j(\omega+\Delta\omega)}{||\Phi_i(\omega)|| ||\Phi_j(\omega+\Delta\omega)||} \right| \right\}. \tag{9}$$

This is the Modal Assurance Criterion - MAC. It is used to estimate a correlation between the wave shapes, where j is the mode of frequency $\omega + \Delta\omega$ with higher correlation with the mode i of the frequency ω.

Meta material and Continuous local resonator

Basically in this case, the metamaterial consists of a structure composed by divided in identical 12 identical and symmetric unit-cells arranged as an array of 3 x 6 cells (Fig. 1). Each unit-cell consists of a plate surrounded by stiffener-beams with a Continuous local resonator (CLR) inside. The CLR is a local resonator made with a solid cubical block (mass) connected by four small beams (springs) to the stiffener-beams. The cell was modeled with the first natural frequency tuned to the second natural frequency of the reinforced-plate, in order to create a bangap on this frequency. A numerical modal analyses of the CLR was performed with commercial software ANSYS® considering the solid cubical block with the four small beams free and the stiffeners-beams and plate fixed. The results shows that the six first natural frequencies in a frequency band DC up to 2000 Hz are 778, 1209, 1209, 1454, 1454, 1846 with the corresponding mode shapes flexural-z, torsional-x, torsional-y, longitudinal-x, longitudinal-y, torsional-z (Fig. 2).

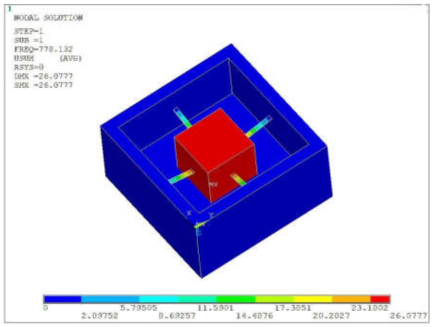
NUMERICAL RESULTS

In this section the force response by WFE method is verified by conventional FE method and dispersion curves of a metamaterial reinforced-plate slice are calculated. The slice (Fig. 3b) was modeled by commercial software ANSYS® using 2610 element type solid (SOLID185). Geometric parameters and material property are shown in Table 1. The plate was exited free-free in the direction z in position $y = 0$ with a load $F = 1\ N$ per node.

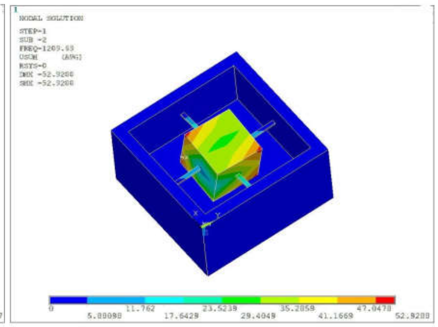
Table 1: Simulated metamaterial plate geometric parameters and material properties.

Geometry/Property	Value
Cell length$[m]$	$37x10^{-3}$
Plate thickness $[m]$	$1x10^{-3}$
Stiffener beam cross section base $[m]$	$3x10^{-3}$
Stiffener beam cross section height $[m]$	$13.5x10^{-3}$
Resonator beam cross section base $[m]$	$1x10^{-3}$
Resonator beam cross section height $[m]$	$0.5x10^{-3}$
Resonator beam length $[m]$	$9x10^{-3}$
Resonator mass side (cube) $[m]$	$7x10^{-3}$
Young's modulus (E) $[Pa]$	$0.72x10^{9}$
Mass density (ρ) $[kg/m^3]$	700
Structural damping (η)	0.02

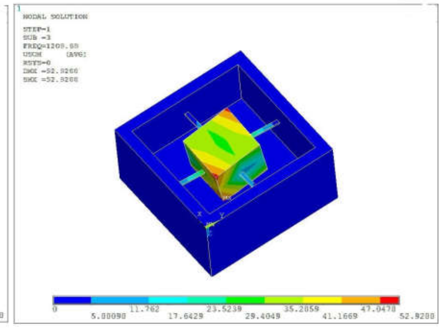
To verify the results, the WFE and conventional FE methods are compared for a plate with six slices and three cell in each slide (Fig. 1). Figure 4 shows the plate receptance calculated by both methods, and a good agreement between them can be observed. Figure 5 shows a comparison between two plates with six slices of three cells, but one with resonators (Fig. 3a) and the other without resonators (Fig. 3b). As expected, the result for the plate with resonators



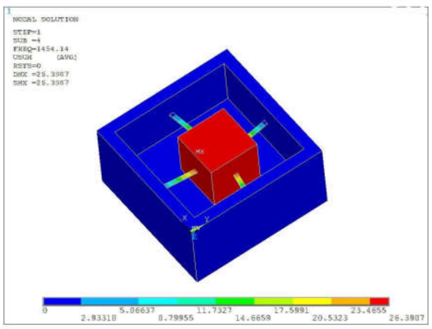
(a) Mode 1: 778 Hz



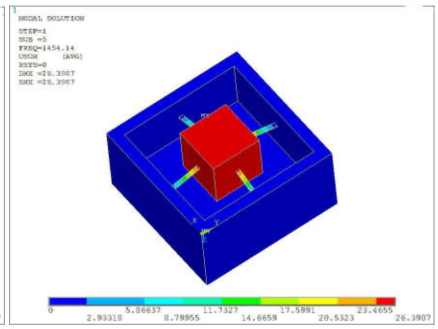
(b) Mode 2: 1209 Hz



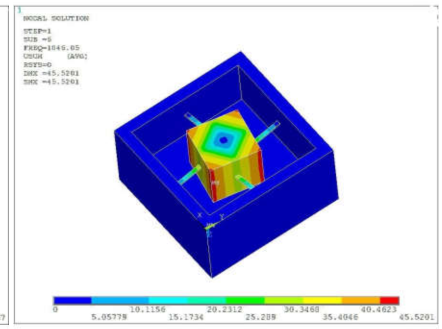
(c) Mode 3: 1209 Hz



(d) Mode 4: 1454 Hz



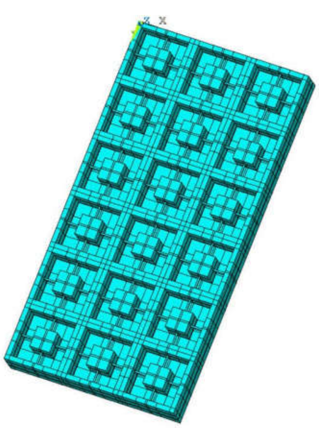
(e) Mode 5: 1454 Hz



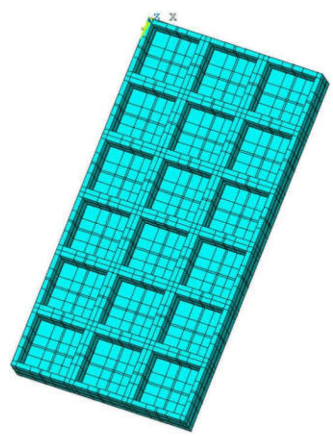
(f) Mode 6: 1846 Hz

Figure 2: Modal analysis of a cell: the fist six modes

split the second natural frequency in the plate without resonators in two others, and creates a bandgap with a significant amplitude attenuation. However, this caused a bandgap in the interest band making a shift in the natural frequency of the structure. Moreover, the transmittance of plate with resonators caused a bandgap in the same frequencies of the natural frequencies of the cell as can seen more precisely in the comparison of figure (2a) to figure (5). Where the difference was 2 Hz.



(a)



(b)

Figure 3: Plate with six slices:(a) with resonators; (b) without resonators

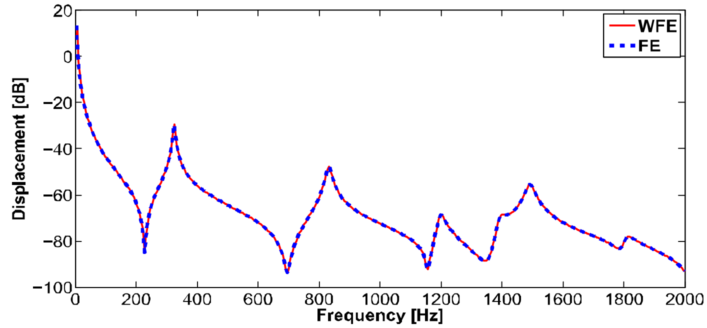


Figure 4: Comparison of WFE (red solid line) and FE (blue dashed line) by displacement: results from plate without resonators, Fig. (3b), measured in coordinates $(0,0,0)$ of the plate.

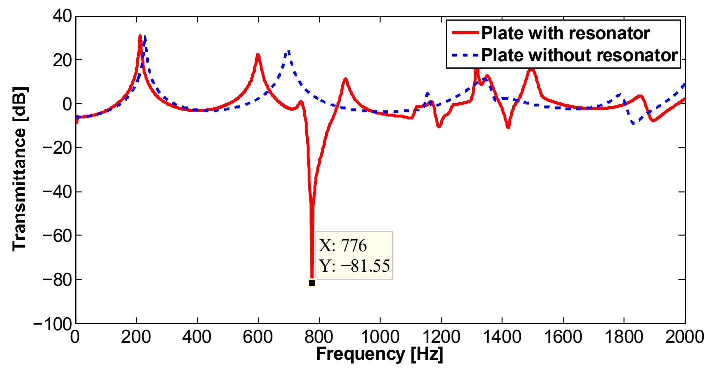


Figure 5: Comparison of WFE for plates with resonators (blue sold line) and without (red dashed line) resonators.

The dispersion curves were computed to a slice without and with resonators, Fig. (6) and (7), respectively. This results show the six first plate modes. It is possible to see that the slice with resonator caused a bandgaps around of each natural frequencies of cell (computed by modal analyze (Fig. 2)). It can seen that occur in several wave modes. Probability, this was caused because of the influence of resonators that created a coupling bandgaps, where many modes have gap in the similar band of frequency.

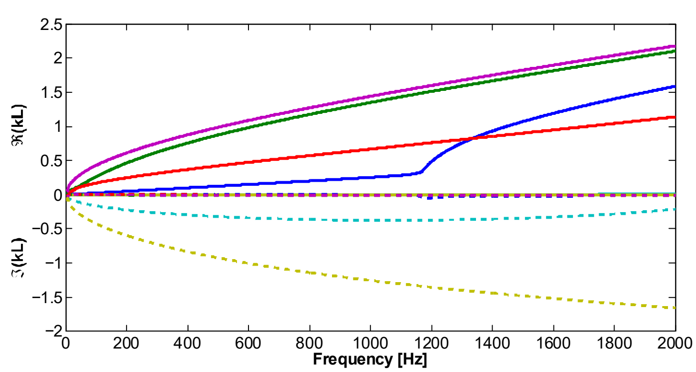


Figure 6: Dispersion curves of the six first wave modes of slice: without resonators.

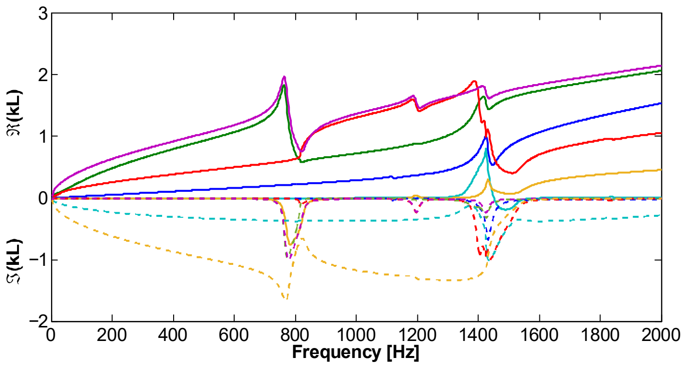


Figure 7: Dispersion curves of the six first wave modes of slice: with resonators.

CONCLUSION

Elastic metamaterial plates are investigated. A WFE method was proposed as engineering tool to calculate these kinds of structures. The vibration analysis was computed as well. Further, the results of proposed method (WFE) and FE (by ANSYS®) agree well. The comparison of plate with and without resonators shown that we can tune resonators to have bandgaps in undesirable resonance frequencies. Furthermore, we shown that bandgaps occur in the same frequency of the cell modes. Finally, the dispersion curves of a plate slice were shown and many modes had the similar frequency-band gap. Future works will involve comparisons of numerical and experimental results.

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