DINAME 2019 - Proceedings of the XVIII International Symposium on Dynamic Problems of Mechanics,
M.A. Savi, T.G. Ritto and W.M. Bessa (Editors), ABCM, Buzios, RJ, Brazil, March 10th to 15th, 2019

# A philosophically new approach to impact with and without vibrations

**Arindam Bhattacharjee**[1], **and Anindya Chatterjee**[1]

[1] Mechanical Engineering, Indian Institute of Technology Kanpur, 208016, India

*Abstract: In impacts between stiff solid bodies, even with simultaneous multiple impacts and/or vibrations, the net outcome obeys rigid body impulse momentum relations. However, the impulse remains indeterminate and must be separately predicted. We propose a new approach where various basic restrictions are directly retained using linear inequalities, and post-impact kinetic energy is minimized subject to new "outward" or rebound enhancing inequalities. The approach is philosophically new and algorithmically simple. It matches experimental data better than some popular approaches, and is in principle generalizable to other situations with new suitable outward inequalities.*

***Keywords: vibroimpact, impulse-momentum, kinetic energy, quadratic programming***

## INTRODUCTION

We consider impacts between stiff solid bodies, wherein a "rigid body" approximation is valid. Specifically, we assume that the overall contact interaction is brief though it may include multiple vibration-induced separations; motions during the impact interaction are small; contact and constraint forces are large compared with existing bounded forces like weight; velocity-squared acceleration terms can be neglected during the impact interaction; deformations during post-impact large motions are negligible; and the net effect of the contact impulses on those large "rigid body" or overall post-impact motions of the colliding objects is described accurately using rigid body impulse-momentum relations.

However, the impact impulses remain indeterminate, and must be modeled using separate constitutive models or impact laws. Of such models, some attempt to monitor deformations in the bodies; some model the interaction using evolution equations based on models of localized compliances between otherwise-rigid bodies (i.e., no vibrations in the bodies); and some predict impulses using simple rules with a few fitted parameters. In the simplest case with a single contact, one uses restitution in the normal direction, with some modeling of Coulomb friction: see Stewart and Trinkle (1996), Chatterjee and Ruina (1998), Rakshit and Chatterjee (2015) and references therein.

Simple models that predict only the net impulse are usually inaccurate. However, they find use in graphics, robotics, games, movies, and granular flows, where they offer an alternative to highly demanding computations. Here, for such net-impulse modeling, we suggest a new modeling philosophy that gives good results in many cases.

## GENERAL OBSERVATIONS, SIMULTANEOUS IMPACTS, AND OUTWARD INEQUALITIES

Fundamental restrictions on the contact interaction involve several inequalities that are *linear* or can be approximated as such; and non-increasing kinetic energy represents a *quadratic* inequality. As examples of linear inequalities, we usually assume the normal component of any contact impulse is nonnegative (no adhesion); and the normal component of the post-impact relative velocity at the contact point is nonnegative (the bodies do not interpenetrate). Further, assuming Coulomb friction at the contact, integrating over the impact yields $|P_t| \leq \mu P_n$, with $t$ and $n$ subscripts denoting tangential and normal components respectively. In planar cases, the above is effectively two linear inequalities, since $|a| \leq b$ is the same as $a \leq b$ and $-a \leq b$. In three dimensions, the friction inequality can be approximated using a set of $m$ linear ones, where larger $m$ improves the approximation (Stewart and Trinkle, 1996; Anitescu and Potra, 1997). Thus, fundamental restrictions of non-adhesive contact, non-interpenetration, and Coulomb friction are *all* expressible, at most with small approximation, as *linear inequality constraints* on the contact impulse vectors. We will explicitly incorporate such constraints, and never violate them. As opposed to the above linear inequalities, we note that post-impact kinetic energy is a *quadratic* function of the set of contact impulses. For kinetic energy, we will instead use minimization as described below.

Although the above restrictions seem simple, they have led to difficulties. Some impact laws can predict interpenetration (communication with authors of (Smith et al., 2012)), or increased kinetic energy (Glocker, 2013).

Our adoption of quadratic programming in this work is prompted by three things: (i) our linear inequality constraints suggest it, (ii) quadratic programming offers easily available routines, and (iii) we will find simple model parameterizations which match experiments well. There remains one final modeling element that requires attention, following a preliminary discussion of restitution in impact. This element, an "outward inequality", is our main new contribution. See Fig. 1 (left). Consider a restitution model of the form $V_{f,n} = -eV_{i,n}$, where $n$, $f$ and $i$ denote "normal", "final" and "initial" respectively. Here, the dropping ball causes new separation to occur at the right end of the rod, and no finite $e$ can model

**Figure 1 – Left: usual kinematic (Newtonian) restitution is problematic here. Right: In Newton's cradle, the four yellow balls must move together if complementarity holds.**

the impact. Modelers can therefore use an inequality of the form

$$V_{f,n} \geq -eV_{i,n}, \quad \text{or} \quad V_{f,n} + eV_{i,n} \geq 0. \tag{1}$$

Subsequently, a common approach uses *linear complementarity* conditions as follows:

$$V_{f,n} + eV_{i,n} \geq 0, \quad P_n \geq 0, \quad \text{and} \quad \left(V_{f,n} + eV_{i,n}\right) P_n = 0.$$

The above seems sensible. However, consideration of Newton's cradle, Fig. 1 (right), shows that if such complementarity conditions hold then for any finite $e$ we *cannot* have separation at the rightmost contact, contradicting common experience. The above motivates our main idea, namely energy minimization subject to suitable "outward" inequalities.

The outward inequality idea, as an alternative to linear complementarity, can be motivated using a single vertically dropped ball. We adopt Eq. (1), with $V_{i,n} < 0$ to ensure an impact and $0 \leq e \leq 1$ as usual. Minimizing the post-impact kinetic energy gives $V_{f,n} = -eV_{i,n}$, a satisfactory outcome. Here, Eq. (1) serves as an outward inequality. Without it, there would be zero rebound.

Now consider a more general case. For simultaneous multiple impacts between several bodies, Rakshit and Chatterjee (2015) used impulse, velocity and friction inequalities as above, and minimized post-impact kinetic energy subject to

$$\sum_{k=1}^{n_c} V_{f,n}^{(k)} m_k^{\alpha} \geq -\sum_{k=1}^{n_c} e_k V_{i,n}^{(k)} m_k^{\alpha}, \text{ suggesting } \alpha = \frac{1}{2}. \tag{2}$$

Equation (2) is a single outward inequality for all contact locations; $k$ refers to impact location; $e_k$ allows different notional restitution for each contact; and $m_k$ refers to a "local" normal inertia (see the paper for details). For reasons of space, Fig. 2 shows just a few outcomes predicted by that model.

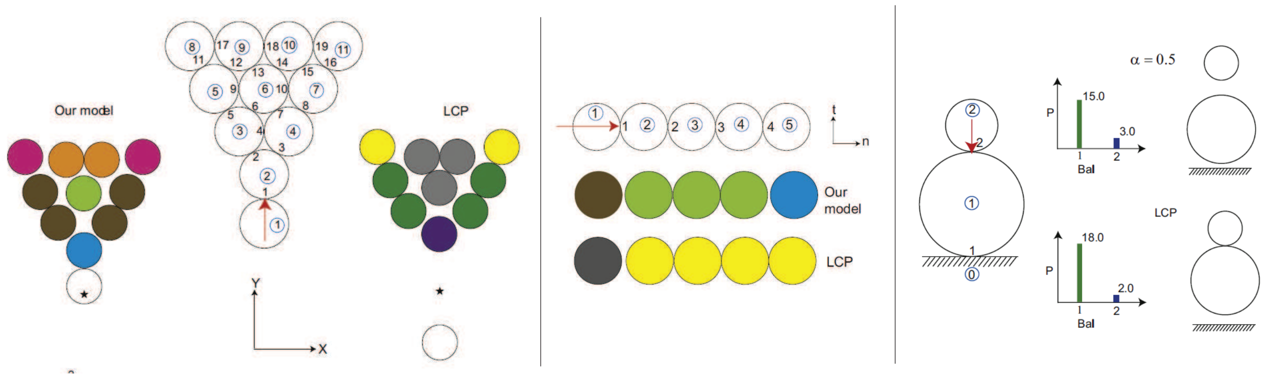


**Figure 2 – Rakshit and Chatterjee's approach compared with linear complementarity: (i) pool break, (ii) Newton's cradle, with some separation predicted rather than none. (iii) two balls dropped together.**

## APPLICATION TO VIBRATION-DOMINATED BALL-BEAM IMPACT

We now turn to vibration-dominated impact. See Fig. 3. A ball (of mass $M$) strikes a lightly damped Euler-Bernoulli beam (length $L$, mass per unit length $\overline{m}$) which vibrates upon impact. We neglect vibrations in the ball. The rebound velocity ($v_{\mathrm{f}}$) of the ball defines the restitution. The beam's displacement is written using its *normal* modes as $y(x,t) \approx \sum_i^N a_i(t)\phi_i(x)$, $i = 1..N$, where $a_i(t)$ are coordinates corresponding to normal modes $\phi_i(x)$. Including light modal beam damping, $a_i(t)$ and ball position ($z$) satisfy

$$\ddot{a}_i + 2\zeta\omega_i\dot{a}_i + \omega_i^2 a_i = -\phi_i(b)F, \quad i = 1\ldots N \quad \text{(beam)}, \qquad \text{and} \qquad M\ddot{z} = F \quad \text{(ball)} \tag{3}$$

respectively; here dots denote time derivatives, $\omega_i$ are natural frequencies, and $F$ is Hertzian. Initial conditions: $y(x,0) = 0$, $\dot{y}(x,0) = 0$, $z(0) = 0$ and $\dot{z}(0) = -1$ (impact velocity). Upon numerical simulation, the dynamics is seen to be rich, with multiple subimpacts. Modal damping regularizes the impact outcome significantly. Final simulations are with $N = 40$.

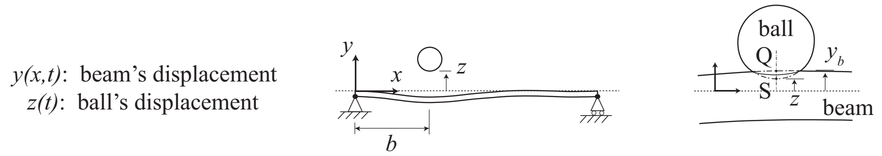


**Figure 3 – Left: Impact of a Hertzian body on an EB beam. Right: Ball-beam contact. Solid lines denote the actual configuration. S and Q are the notional contact points on the undeformed ball and beam respectively.**

## Impact model

We define $Z = M\sum_{i=1}^{N}\phi_i(b)^2$, which is dimensionless because the mode shapes are mass-normalized. For small and large $Z$, we observed numerically that restitution is relatively high. For intermediate $Z$, restitution is low. We accordingly empirically define three dimensionless parameters $\mu_0$, $\alpha_0$ and $\alpha_1$ and define the dependent quantities

$$\bar{e} = \frac{\mu_0^2 Z^2 + 1}{\mu_0^2 Z^2 + \mu_0 Z + 1}\, e, \qquad \beta_0 = \frac{\alpha_0 Z}{N + \alpha_0 Z}, \qquad \beta_1 = \frac{\alpha_1 M \phi_1(b)^2}{1 + \alpha_1 M \phi_1(b)^2}. \tag{4}$$

Here $\bar{e}$ is a *vibration-incorporating* restitution for a given notional restitution value $0 \le e \le 1$ (we use $e = 1$); $\beta_0$ incorporates an average value over all active modes; $\beta_1$ incorporates the pseudostatic or low frequency response. The three fitted parameters $\mu_0$, $\alpha_0$, and $\alpha_1$ are all assumed positive; $\bar{e}$, $\beta_0$ and $\beta_1$ are then all between 0 and 1.

For our impact model, the known quantities are (i) $v_{\mathrm{i}} < 0$, (ii) $e = 1$, (iii) $N$, and (iv) $\phi_i(b) \ge 0$; and the fitted parameters are $\mu_0$, $\alpha_0$ and $\alpha_1$. All $\dot{a}_i = 0$ at the start of the interaction. The unknowns are $v_{\mathrm{f}}$ and the post-impact $\dot{a}_i$. Of these, we are here interested in $v_{\mathrm{f}}$, which defines the restitution level.

The post-impact kinetic energy of the system is $KE_{\mathrm{final}} = \frac{1}{2}\left(M v_{\mathrm{f}}^2 + \sum_{i=1}^{N}\dot{a}_i^2\right)$. As discussed earlier, we minimize the post-impact kinetic energy of the system subject to various inequality constraints. These are:

$$|\dot{a}_i(t)| \le \phi_i(b) M (v_{\mathrm{f}} - v_{\mathrm{i}}),\ i = 1\ldots N, \qquad v_{\mathrm{f}} - \sum_{i=1}^{N}\dot{a}_i\phi_i(b) \ge -\bar{e}v_{\mathrm{i}}, \qquad v_{\mathrm{f}} - \beta_0\sum_{i=1}^{N}|\dot{a}_{\mathrm{i}}|\phi_i(b) \ge -\beta_1\bar{e}\,v_{\mathrm{i}}. \tag{5}$$

The inequality on the left constrains the modal response, and is obtained when we treat the net impulse as non-instantaneous. The middle and the right contain our two outward inequalities. The middle inequality is not common, but aligned with usual restitution ideas. The crucial right hand-side inequality approximately incorporates multiple subimpacts due to higher mode oscillations, and relatively slow lower-mode dynamics. It can be shown that the above inequalities ensure non-interpenetration and net positive impulse.

An optimization problem must have a nonzero feasible set. To this end, note that $\dot{a}_i = 0$ for all $i$ and $v_f = -\bar{e}v_{\mathrm{i}}$ is both a feasible point and predicts a reduction in net kinetic energy. It follows that the minimized kinetic energy will be no larger, and so all constraints will be met. The above minimization problem can be restated as an exactly equivalent quadratic program (details omitted), and therefore solved efficiently using built-in routines in Matlab.

## Model parameter fitting results

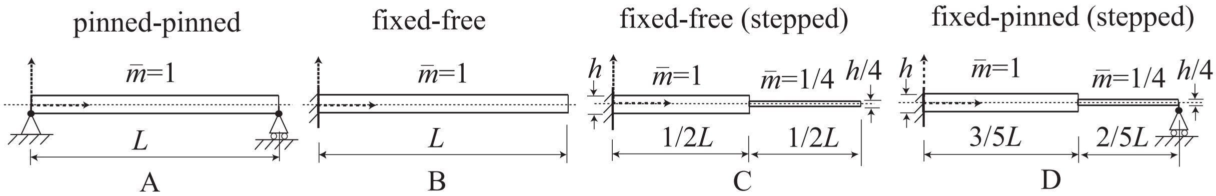


**Figure 4 – Four beams are considered for model validation.**

We validate the impact model for four beams (see Fig. 4). For each beam, we numerically find $v_{\mathrm{f}}$ for varying $M$ and $b$. We fit our model to this numerically generated data: see results of fit in Fig. 5. The average behavior and trends are captured reasonably well. For example, the overall variation of restitution with $M$, the sudden changes observed for stepped beams, and the rising trends near free ends, are all captured reasonably well on average. Smaller-scale variations with impact location are not captured. We emphasize that for many masses and many impact locations on a given beam, we are fitting a complicated function of two variables; and even a linear function of two variables has three parameters, which is as many as we have allowed ourselves. Thus, the performance of our model can be considered very good.

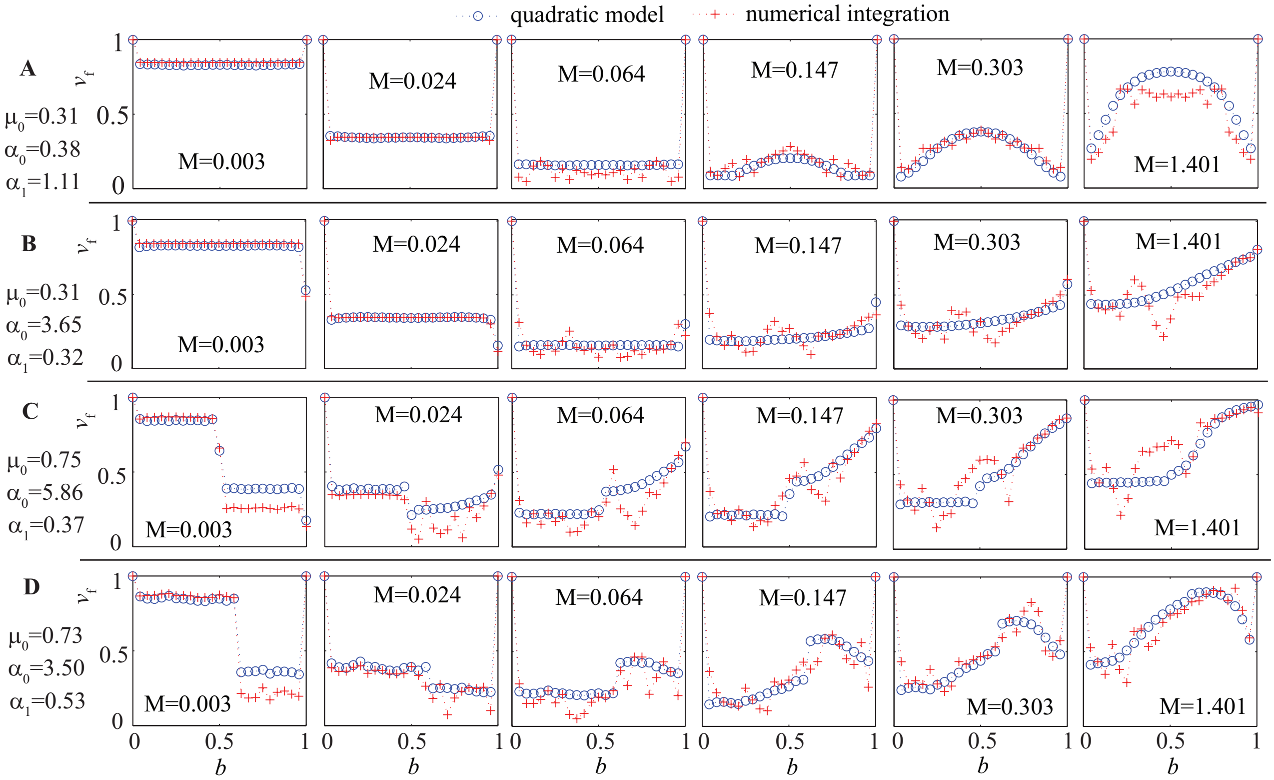


**Figure 5 – Results of fit for the four beams considered.**

## CONCLUSIONS

A new algebraic quadratic program based, and philosophically new, approach is proposed here for modeling impacts between two or more stiff objects, with and without vibrations, where contact occurs through single or multiple points. In general, we minimize the post-impact kinetic energy of the system subject to several linear inequalities. Other than the usual non-interpenetration, non-negative impulse and frictional (approximated) linear constraints, our formulation includes an additional but crucial "outward" inequality constraint. This non-unique, unknown and to-be-developed "outward" inequality depends on the material, geometry and the specific problem. The quadratic program based approach gives better results for two broad problems: (i) simultaneous impacts between rigid bodies (multiple contact points), and (ii) impacts on a flexible body (single contact point) wherein vibration effects like mutiple subimpacts are significant.


Anitescu, M. and Potra, F.A., 1997, "Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems", Nonlinear Dynamics, Vol.14, No. 3, pp. 231-247.

Chatterjee, A. and Ruina, A., 1998, "A new algebraic rigid-body collision law based on impulse space considerations", Journal of Applied Mechanics, Vol.65, No. 4, pp. 939-951.

Glocker, C., 2014, "Energetic consistency conditions for standard impacts", Multibody System Dynamics, Vol.32, No. 4, pp. 445-509.

Rakshit, S. and Chatterjee, A., 2015, "Scalar generalization of Newtonian restitution for simultaneous impact". International Journal of Mechanical Sciences, Vol.103, pp. 141-157.

Smith, B., Kaufman, D.M., Vouga, E., Tamstorf, R. and Grinspun, E., 2012, "Reflections on simultaneous impact", ACM Transactions on Graphics (TOG), Vol.31, No. 4, pp. 106.

Stewart, D.E. and Trinkle, J.C., 1996, "An implicit timestepping scheme for rigid body dynamics with inelastic collisions and coulomb friction", International Journal for Numerical Methods in Engineering, Vol.39, No. 15, pp. 2673-2691.

Todorov, E., 2011, "A convex, smooth and invertible contact model for trajectory optimization", In Robotics and Automation (ICRA), IEEE International Conference on, pp. 1071-1076.


## RESPONSIBILITY NOTICE