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One-Dimensional Phononic Crystal Dynamic Analysis by Spectral Transfer Matrix Method

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Abstract: The attenuation and control of vibration and noise observed in periodic systems increases the number of researches on this class of materials, commonly referred as phononic crystals. The evaluation of the wave propagation problem in phononic crystals is significantly simplified due to periodicity. It enabling faster and more accurate numerical analysis. Elastic phononic crystals are calculated using the dynamic stiffness matrix formulation, which is conveniently transformed in a transfer matrix to apply Floquet-Bloch theorem. This procedure can generates an ill-conditioned transfer matrix. In the present work a new procedure to evaluate one-dimensional phononic crystals is proposed, which uses directly a spectral transfer matrix (STM). The simulated results presents good agreement with the conventional numerical methods and allows to obtain dispersion diagrams and force responses of periodic structures.

Keywords: Phononic Crystal, Band Gap, Spectral Transfer Matrix, Wave Propagation

INTRODUCTION

Most of the structures present in several areas of engineering can be evaluated as periodic systems. Such systems consist of several identical elements uniformly coupled along the structure, denominated cells. The consideration of periodicity in the domain provides significant simplification in the modal analysis and the dynamic response of the structure allowing the extraction of the main natural modes and frequencies (Mead, 1973).

Recently, several researchers have investigated the behavior of wave propagation in structures with periodic pattern. These periodicity can be made with a change of properties, as reported by Xiang and Shi (2009), or cross-sectional area change, as studied by Nobrega *et al* (2016). Structures with such configurations are commonly referred to as phononic crystals and have the property of "wave filtering"in certain frequency bands, known as band gap (Xiao *et al*, 2012). This behavior comes from destructive interference of waves due to geometric and/or material discontinuities. Allowing its application in problems of noise and vibration control (Ruzzene *et al*, 2014).

The determination and optimization of band gaps of a phononic crystal can be performed through numerical simulations. Among the available approaches, the Finite Element Method (FEM) is distinguished by the wide generality in the evaluation of irregular domains. However, it presents high computational cost in medium and high frequency bands (von Flotow, 1987). In order to overcome these limitation, the Spectral Element Method (SEM) (Doyle, 1997; Lee, 2000) was proposed. It is an accurate method since its formulation is the exact analytical solution of the problem. Then, one SEM element is equivalent to infinity FEM elements. However, its evaluation in phononic crystals requires a new element each time that a discontinuity appears in the structure. In addition, it requires the analytical solution of wave propagation limiting the exact analysis to simpler structures (rod, beam).

In general, elastic phononic crystals are calculated from a dynamic stiffness matrix (DSM) formulation, which is conveniently transformed in a transfer matrix (TM) approach to apply the Floquet-Bloch theorem. In this procedure the DSM is partitioned and inverted generating an ill-conditioned TM (Nobrega *et al*, 2016). In the present work a new procedure to evaluate one-dimensional phononic crystals is proposed, which uses directly the Spectral Transfer Matrix (STM) (Pestel and Leckie, 1963; Lee, 2000). The approach is applied for an elastic rod phononic crystal and simulated results of dispersion diagram and force response are obtained. The results are compared with that obtained by FEM and presents very good agreement.

SPECTRAL TRANSFER MATRIX METHOD

The STM is an alternative approach to derive the spectral element without need the exact wave solutions of the frequency-domain governing equations (Lee, 2000). The method is based on the representation of the governing equations in a set of first oder ordinary differential equations as show in Eq. (1):

$$\frac{d\mathbf{y}}{dx} = \mathbf{G}\,\mathbf{y}, \tag{1}$$

where $\mathbf{y}$ is the state-vector and is written as $\mathbf{y} = \{\mathbf{u} \;\; \mathbf{F}\}^T$ with $\mathbf{u}$ representing the displacement and $\mathbf{F}$ indicating the internal forces.

The solution of the Eq. (1) is given by:

$$\mathbf{y}(x) = e^{\mathbf{G}x}\mathbf{y}(0). \tag{2}$$

Applying for the 1-D structural member of length L it has:

$$\mathbf{y}(L) = e^{\mathbf{G}L}\mathbf{y}(0) = \mathbf{T}\mathbf{y}(0), \tag{3}$$

where $\mathbf{T} = e^{\mathbf{G}L}$ is the transfer matrix.

STM for rods

The simple rod homogeneous differential equation in the frequency domain and the internal force can be written as:

$$\begin{cases} \dfrac{d}{dx}\left[EA\,\dfrac{du(x)}{dx}\right] + \rho A\omega^2 u(x) = 0, \\ \\ F(x) = EA\dfrac{d\,u(x)}{dx}, \end{cases} \tag{4}$$

where E, A, ρ and ω are, respectively, Young's modulus, cross-sectional area, mass density and circular frequency. The internal force F is applied in the axial orientation.

The Eq.(4) can be rewritten as:

$$\begin{cases} \dfrac{du}{dx} = \dfrac{F}{EA} \\ \\ \dfrac{dF}{dx} = -EAk^2\,u \end{cases} \Rightarrow \frac{d}{dx}\begin{Bmatrix} u \\ F \end{Bmatrix} = \begin{bmatrix} 0 & 1/EA \\ -EAk^2 & 0 \end{bmatrix}\begin{Bmatrix} u \\ F \end{Bmatrix}. \tag{5}$$

Thus, the matrix $\mathbf{G}$ for rod is given by:

$$\mathbf{G} = \begin{bmatrix} 0 & 1/EA \\ -EAk^2 & 0 \end{bmatrix}, \tag{6}$$

where $k = \omega\sqrt{\rho/E}$ is the wave number.

STM for beams

The homogeneous dynamic equation for Euler-Bernoulli beam and the internal forces are given as:

$$\begin{cases} EI\dfrac{d^4w(x)}{dx^4} - \omega^2\rho A\,w(x) = 0, \\ \\ M(x) = EI\dfrac{d^2w(x)}{dx^2}, \\ V(x) = -EI\dfrac{d^3w(x)}{dx^3}. \end{cases} \tag{7}$$

Then, the set of differential equations can be rewritten as:

$$\begin{cases} \dfrac{dw}{dx} = \theta \\ \\ \dfrac{d\theta}{dx} = \dfrac{M}{EI} \\ \\ \dfrac{dV}{dx} = -EIk^4\,w \\ \\ \dfrac{dM}{dx} = -V \end{cases} \Rightarrow \frac{d}{dx}\begin{Bmatrix} u \\ \theta \\ V \\ M \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/EI \\ -EIK^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u \\ \theta \\ V \\ M \end{Bmatrix}. \tag{8}$$

And the matrix $\mathbf{G}$ is given as:

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/EI \\ -EIK^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{9}$$

MODELING OF THE ONE-DIMENSIONAL PHONONIC CRYSTAL

The phononic crystal unit cell contains two materials, which are polymer and steel. Figure 1 shows a schematic view of the phononic crystal and the unit cell composed by a sequence of polymer, steel and polymer.

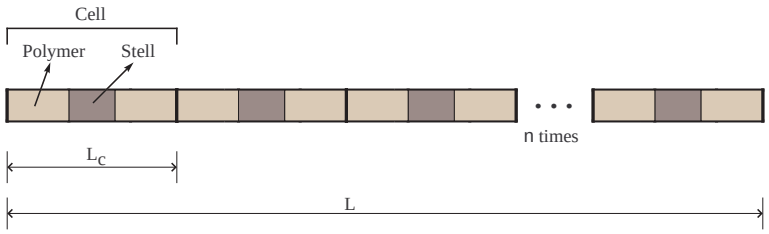

Figure 1 – One-dimensional phononic crystal with the unit cells composed by polymer-steel-polymer.

For a 1D phononic crystal, the Eq.(3) can be partitioned in terms of left and right side of unit cell as:

$$\mathbf{y}_r^{(m)} = \mathbf{T}_c \mathbf{y}_l^{(m)}, \tag{10}$$

with the subscripts r and l corresponding to the right and left nodes of the $m-th$ cell. While $\mathbf{T}_c$ represents the transfer matrix of the cell, calculated by:

$$\mathbf{T}_c = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_1, \tag{11}$$

where the subscript 1 and 2 indicates the proprieties of the polymer and the steel, respectively.

Using the coupling relation at the cells interface ($\mathbf{y}_l^{(m+1)} = \mathbf{y}_r^{(m)}$) and applying Floquet-Bloch theorem (Mead, 1973) it has:

$$\mathbf{T}_c \mathbf{y}_l^{(m)} = e^{-i\cdot kL} \mathbf{y}_l^{(m)} \quad \text{or} \quad \mathbf{T}_c \mathbf{y}_l = e^{-i\cdot kL} \mathbf{y}_l. \tag{12}$$

The eigenproblem of Eq.(12) is solved and the eigenvalues produces dispersion diagrams and the eigenvectors the wave modes. It is important to note that the eigenvectors and eigenvalues are independent of the position of the cell in the structure. To each n degree of freedom has $2n$ eigenvalues that can be ordered appropriately into two groups: $|e^{\mu_j}| \leq 1$, $j = 1,2,\ldots,n$ indicate waves propagating to the right and $|e^{-\mu}| \geq 1$, $j = 1,2,\ldots,n$ correspond to waves traveling to the left. Therefore, wave amplitudes can be obtained as follows (Silva et al, 2014):

$$\mathbf{y}^{(m)} = \sum_j^n e^{-i\cdot k_j L} \Phi_j \mathbf{y}_j^{(m)} \qquad \text{with } m = 1,2,3\ldots N_c, N_{c+1}, \tag{13}$$

where $j = 1,2,\ldots,n$ with n as the number of eigenvalues and N_c the total number of unit-cells.

NUMERICAL RESULTS

The results were obtained considering a unidimensional structure with 20 unit cells of length $L_c = 0.025\,\text{m}$. Unit cell length is formed by 45% polymer + 10% steel + 45% polymer. A clamped-free boundary condition is applied to the crystal. A squared cross-sectional area is used with $A = 4 \times 10^{-4}\text{m}^2$. Table 1 shows polymer and steel properties. The structure was excited as a rod with a longitudinal force (case 01) and later as a beam with a transversal force (case 02).

Table 1 – Properties used for numerical simulation.

Property	Stell	Polymer
Elastic Modulus [MPa]	210×10^9	0.48×10^9
Density [kg m^{-3}]	7800	935
Damping factor	0.01	0.02

Figure 2 shows the dispersion diagram for the phononic crystal unit cell calculated by STM. This graph relates the unit cell wavenumber versus analyzed frequency band. The wavenumber of the cell is calculated as $k_c = \ln|\mu|/-iL_c$. At the bandgap region, the wavenumber has a non-zeros imaginary part indicating the behavior of evanescence waves in the structure. The values of the frequency that the bandgap starts occurs when the real part of $k_c L_c$ is equal to π or 0. To the first case, it can be see three frequency bandgaps indicated by the imaginary part of diagram. And to the second case shows two bandgaps regions. The Bragg limit is given as $\max|k_c| = \frac{\pi}{L_c} \approx 125.7\,\text{m}^{-1}$.

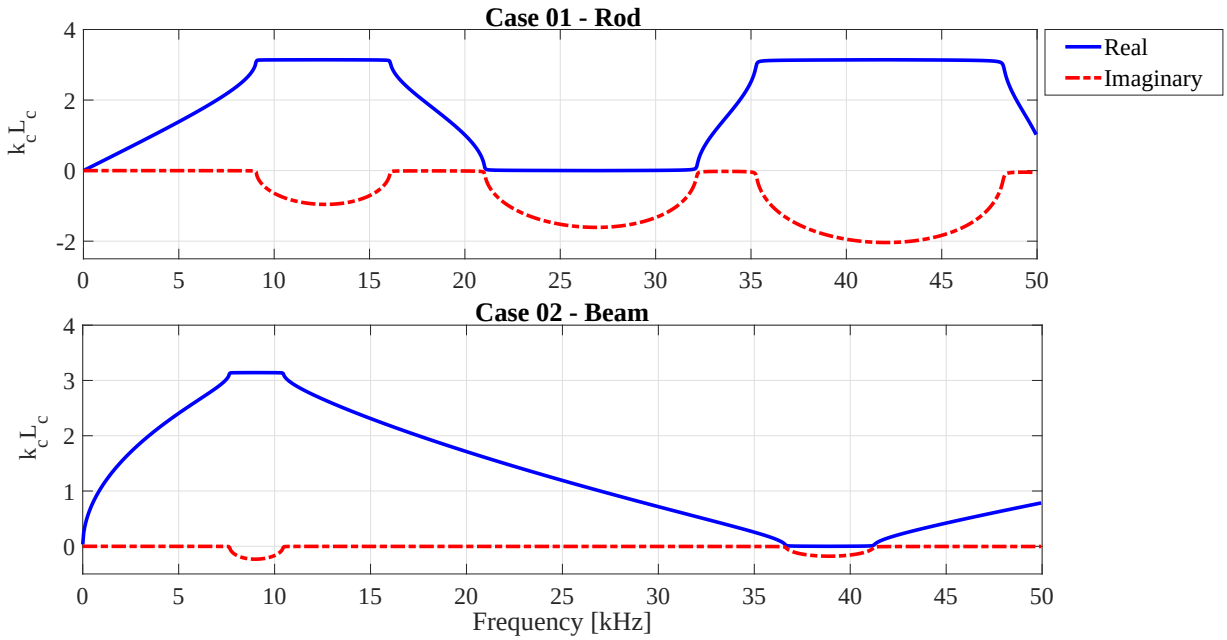

Figure 2 – Phononic Crystal Dispersion Diagram by STM.

The forced response was calculated by STM using the wave expansion method described in the Eq.(13). It was obtained the transmittance and the receptance for the structure. The transmittance relates the reaction force to the excitation force, evaluated in $x = N_c \times L_c$. The receptance is the ratio between displacement ($x = 0$) and excitation force. Figure 3 shows the transmittance and receptance curves calculated for rod. It is possible to see that the react force and the displacement reduces significantly in the bandgaps regions. For the beam case, as show in the Fig. 4, the reduction of the magnitude of the reaction force and the displacement was most noticeable in the higher frequency bandgap at 36.6-41.2 kHz.

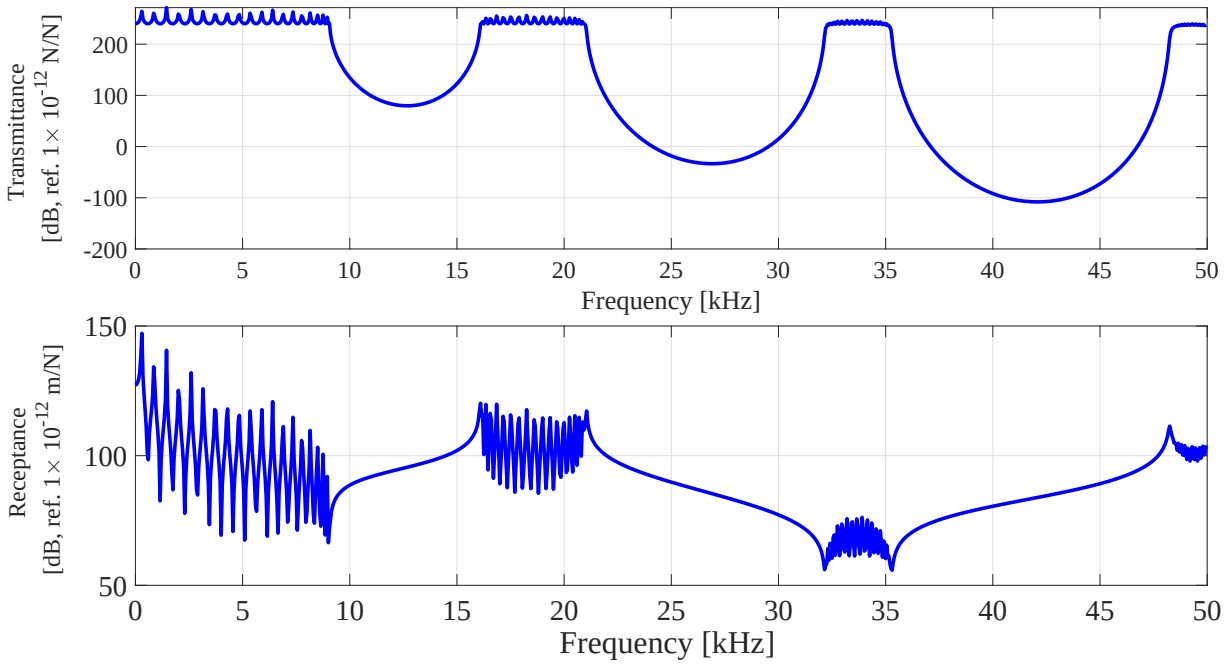

Figure 3 – Transmittance and receptance with response and excitation in $x = 0$ calculated by STM for rod.

The STM methodology was compared with the FEM and SEM. In Figure 5 the three methods are compared based on the displacement response. The curves obtained by STM and SEM presents good agreement for both cases. Particularly, the STM has an advantage to evaluated phononic crystals because avoid the ill-conditioned problem in the inversion of the term of the DSM to calculate the TM.

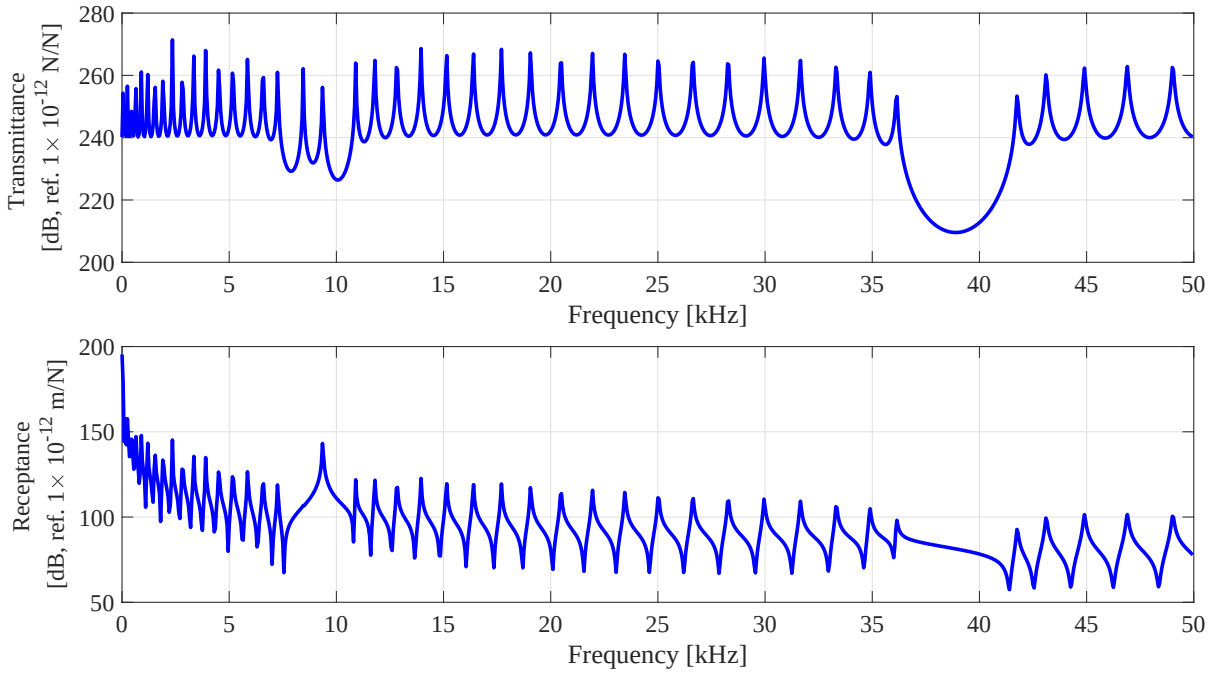

Figure 4 – Transmittance and receptance with response and excitation in $x = 0$ calculated by STM for beam.

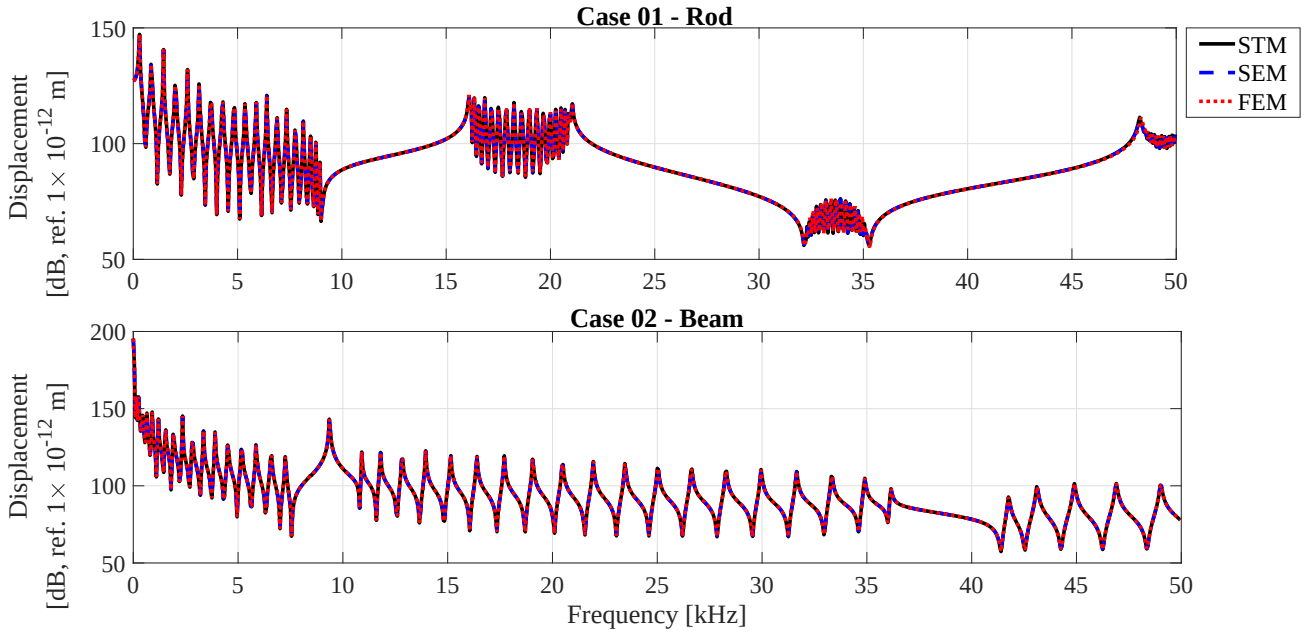

Figure 5 – Displacement response calculated by FEM, SEM and STM.

CONCLUSION

In the present work was investigated the use of the Spectral Transfer Matrix (STM) methodology associated with the Floquet-Bloch theorem for the evaluation of one-dimensional phononic crystals. The numerical results shown good agreement with the already consolidated methods (FEM and SEM), and made possible to obtain the dispersion diagrams and a clear identification of band gaps. Also, the approach allows to obtain force responses for the crystals. In addition, the use of STM presented a very good computational efficiency as compared to SEM and FEM for the evaluated structure.

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