The Use of Poincaré Maps for Diagnosis of Heart Dynamics

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Abstract: Biological rhythms are fundamental for the understanding of the physiological functioning of organisms, being useful in diseases prevention and treatments. This work deals with the cardiac system analysis using a mathematical model composed by three nonlinear oscillators coupled by time-delayed connections. Numerical simulations show a good variety of responses, including normal and pathological rhythms as atrial flutter, atrial fibrillation, ventricular flutter and two different ventricular fibrillation cases. Poincaré maps are employed as a tool to analyze and diagnose cardiac rhythms. Two different approaches are proposed in order to build Poincaré maps. Results show that the model is capable to capture the main behaviors of the cardiac system and that Poincaré maps can be used as a diagnosis tool.

Keywords: Nonlinear dynamics, Chaos, Cardiac rhythms, DDEs, Poincaré maps

INTRODUCTION

Natural phenomena have essential nonlinear characteristics responsible for the variety and richness of behaviors. Rhythms constitute one of the most relevant manifestations of natural systems being possible to be regular or irregular in time and space. In this regard, periodic and non-periodic dynamics can be related to either normal or pathological physiological functioning. This idea motivates the natural system analysis through a dynamical perspective that usually can be performed based on either mathematical models or time series analysis (Savi, 2005).

The cardiac system is one of the possibilities where dynamical perspective has showing to be useful, being applicable either for clinical or control perspectives. In brief, heart is a muscular organ activated by electrical stimuli with the function of pumping blood through all the organs and tissues of the body. In mammals, the heart is divided into 4 cavities: 2 atria and 2 ventricles, as shown in Figure 1. The conduction of the electrical impulse in the cardiac system can be understood as a network of self-excitatory elements formed by sinoatrial node (SA), atrioventricular node (AV) and His-Purkinje complex (HP) (Gois & Savi, 2009; Glass, 2009). The initial excitation occurs in the SA node, natural pacemaker, and propagates as a wave, stimulating atria. Upon reaching the AV node, it initiates a pulse that excites the bundle of His and, afterward, the Purkinje fibers. The fibers distribute the stimulus to the myocardial cells, causing the ventricles contraction (Dubbin, 1996).

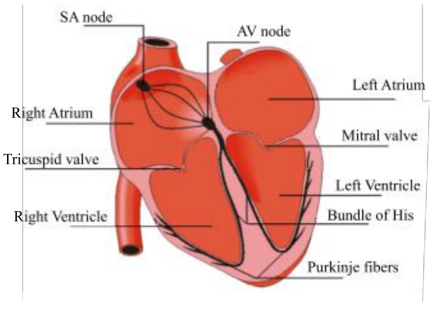

Figure 1 – Human heart schematic picture (Gois & Savi, 2009).

The analysis of the cardiac dynamics from mathematical models has been motivating several researches. Van der Pol & Van der Mark (1928) performed the first study establishing an analogy between heartbeats and electronic circuits represented by nonlinear oscillators. Grudzinski & Zebrowski (2004) proposed alterations on the original Van der Pol oscillator in order to present a more suitable description of the natural pacemaker. Santos *et al*. (2004) modeled cardiac dynamics considering two asymmetrically coupled modified Van der Pol oscillators, representing the behavior of the two cardiac pacemakers, SA and AV nodules. Gois & Savi (2009) proposed a three-coupled oscillator model in order to represent ECG signals. Besides, SA and AV nodules, His-Purkinje complex (HP) is also considered on system modeling. Each oscillator is based on the model due to Grudzinski & Zebrowski (2004) and the system has bidirectional and asymmetric time-delayed couplings to represent the time spent on impulse transmissions.

This work proposes a mathematical model to represent the cardiac functioning representing ECG signals. The model is based on the one proposed by Gois & Savi (2009), being altered in order to increase the capability to describe pathological behaviors. Basically, the model has three-coupled nonlinear oscillators with delayed couplings. It is represented by delayed differential equations being able to capture the main aspects of heart dynamics, representing normal and pathological rhythms. Poincaré section is used as an ECG analysis tool, allowing diagnosis and a proper comprehension of heart dynamics. Different procedures to build Poincaré map are discussed.

MATHEMATICAL MODELING

The mathematical modeling of the natural pacemaker is the starting point for cardiac modeling. Van der Pol oscillator is often used in the modeling of cardiac functions because its dynamic response presents typical characteristics of biological systems such as: limit cycle, synchronization and chaos (Grudzinski & Zebrowski, 2004; Gois & Savi, 2009). Besides that, Van der Pol equation has oscillation amplitude that does not depend of the oscillation rate. The model proposed by Grudzinski & Zebrowski (2004) is a modification of the original Van der Pol oscillator replacing the restitution force by a cubic function.

Cardiac system modeling can be made from the coupling of three nonlinear oscillators. Gois & Savi (2009) proposed the use sinoatrial node (SA), atrioventricular node (AV) and His-Purkinje complex (HP) with asymmetrical and bidirectional connections in order to build a general model that is capable of reproducing the cardiac behavior. Figure 2 shows the conceptual model of this approach. In this work, the same idea of the three-coupled oscillators is considered, each one of them described by the model due to Grudzinski & Zebrowski (2004), proposing some modifications.

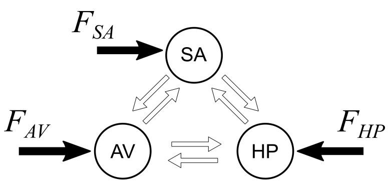

Figure 2 – Conceptual model of the general cardiac functioning.

Therefore, the cardiac system can be modeled by three oscillators (SA, AV and HP) that are coupled by time-delayed terms that represent the transmitting time spent among each one of the oscillators. Under these assumptions, the cardiac system dynamics is governed by the following equations:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= F_{SA}(t) - \alpha_{SA}\, x_2 \left(x_1 - v_{SA1}\right)\left(x_1 - v_{SA2}\right) - \frac{x_1(x_1 + d_{SA})(x_1 + e_{SA})}{d_{SA} e_{SA}} \\
&\quad - k_{AV-SA} x_1 - k^{\tau}_{AV-SA} x_3^{\tau AV-SA} - k_{HP-SA} x_1 - k^{\tau}_{HP-SA} x_5^{\tau HP-SA} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= F_{AV}(t) - \alpha_{AV}\, x_4 \left(x_3 - v_{AV1}\right)\left(x_3 - v_{AV2}\right) \\
&\quad - \frac{x_3(x_3 + d_{AV})(x_3 + e_{AV})}{d_{AV} e_{AV}} - k_{SA-AV} x_3 - k^{\tau}_{SA-AV} x_1^{\tau SA-AV} - k_{HP-AV} x_3 - k^{\tau}_{HP-AV} x_5^{\tau HP-AV} \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= F_{HP}(t) - \alpha_{HP}\, x_6 \left(x_5 - v_{HP1}\right)\left(x_5 - v_{HP2}\right) \\
&\quad - \frac{x_5(x_5 + d_{HP})(x_5 + e_{HP})}{d_{HP} e_{HP}} - k_{SA-HP} x_5 - k^{\tau}_{SA-HP} x_1^{\tau SA-HP} - k_{AV-HP} x_5 - k^{\tau}_{AV-HP} x_3^{\tau AV-HP}
\end{aligned}
\tag{1}
$$

By considering indexes m and n that can represent SA, AV or HP, and $m \neq n$, equation terms are now explained. k_{m-n} and k^{τ}_{m-n} are coupling coefficients between m and n nodes; and $x_i^{\tau_{m-n}} = x_i(t - \tau_{m-n})$ are delayed terms where τ_{m-n} is the time delay. Since the couplings have temporal lags, the system is governed by delayed differential equations (DDEs). Besides, $F_m(t) = \rho_m \sin(\omega_m t)$ is an external excitation that represents spatiotemporal stimulus and therefore, it is considered as a reduced order representation of spatiotemporal aspects. Note that this external stimulus increases the system dimension based on spatiotemporal information.

The ECG is formed by the signal of each one of the oscillators, being formed by a linear combination of the state variables given by (Gois & Savi, 2009),

$$X = ECG = \beta_0 + \beta_1 x_1 + \beta_2 x_3 + \beta_3 x_5 \tag{2}$$

where β_0, β_1, β_2 and β_3 are constants. Therefore,

$$\dot{X} = \frac{d}{dt}(ECG) = \beta_1 x_2 + \beta_2 x_4 + \beta_3 x_6 \tag{3}$$

The fourth order Runge-Kutta method with linear interpolation of time-delayed variables is used to integrate governing equations (1) (Mensour & Longtin, 1998). In order to treat the DDEs system, it is necessary to approximate their solutions for time instants before τ_{m-n}, defining an initial function. A Taylor series expansion is proposed to evaluate this function (Cunningham, 1954; Gois & Savi, 2009).

Lyapunov exponents are important to be estimated in order to identify chaotic response of cardiac systems. The time-delayed states dependence requires an appropriate approach for calculating the Lyapunov exponents. One approach is to admit that, on the interval ($t-\tau$, t), the system has infinite dimensions and therefore can be approximated by a system of ODEs of infinite dimensions (Sprott, 2007). The use of the fourth order Runge-Kutta method allows one to define $N = \frac{\tau}{h} + 1$, where N represents the number of discretizations on ($t-\tau$, t) and h is the integration time step. Hence, the ODE approach allows one to use the algorithm due to Wolf *et al*. (1985) to estimate Lyapunov exponents.

Numerical simulations of the cardiac system model are performed with the objective of presenting different system behaviors. The idea is to compare normal and pathological responses represented by ECGs. Six cases are treated: normal rhythm, atrial flutter, atrial fibrillation, ventricular flutter and two different ventricular fibrillation cases, with and without external stimulus.

DIAGNOSIS USING POINCARÉ MAP

Poincaré map is a stroboscopic representation of the dynamical system response. It reduces the time continuous dynamics to a discrete set of states, a map, allowing a better understanding of the global system dynamics. There are many ways to build a Poincaré map and two approaches are employed in this work: secant section and reference period.

Secant section map

Secant section map uses a geometrical inspiration to build Poincaré map. Basically, map points are related to the vector field that transversally crosses a specific section in an arbitrary direction. Heart dynamics analysis considers a subspace {x_1, x_3, x_5} where it is positioned a secant section at $x_1 = 0$, allowing the observation of states that crosses the section in the positive direction of x_1. Figure 3 presents an overlap of all the Poincaré maps, allowing a comparative analysis among different rhythms. In general, it is possible to identify the main differences from the normal rhythm that should be considered as the system signature reference.

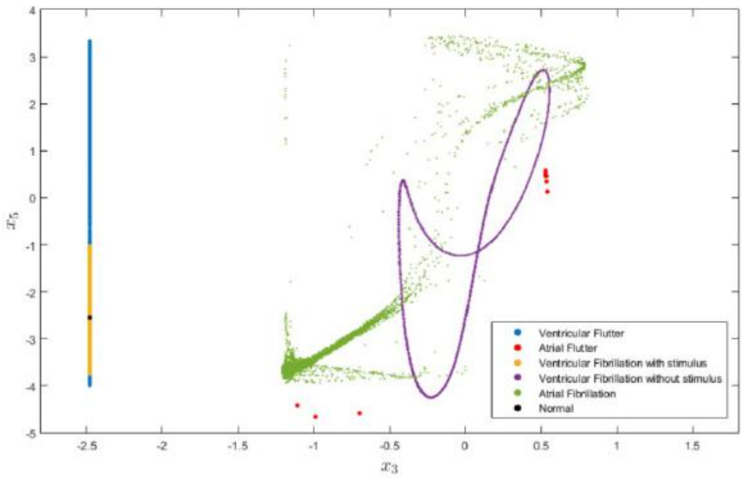

Figure 3 – Comparison of Poincaré maps built with secant section for different rhythms.

Reference period map

Poincaré map can be built by considering a reference period that defines the stroboscopic sample time. This procedure establishes the section position spaced by a period T through time, observing system dynamics states through the section. A straightforward approach to define reference period is when the system is subjected to a harmonic excitation, and the excitation frequency defines the reference period. Otherwise, it is necessary to define a proper reference period. In this regard, two different approaches are considered: external stimulus period, when the system is subjected to an external excitation; self-excitation period, otherwise. The self-excitation period is analyzed from the R-R interval measurement. In this regard, a histogram of the R-R interval is built, establishing the mean value, μ, that defines the reference period for the Poincaré map construction. Figure 4 presents a comparative analysis of all rhythms using Poincaré maps built with reference period. Once again, it is possible to identify variations of pathological responses from the normal rhythm.

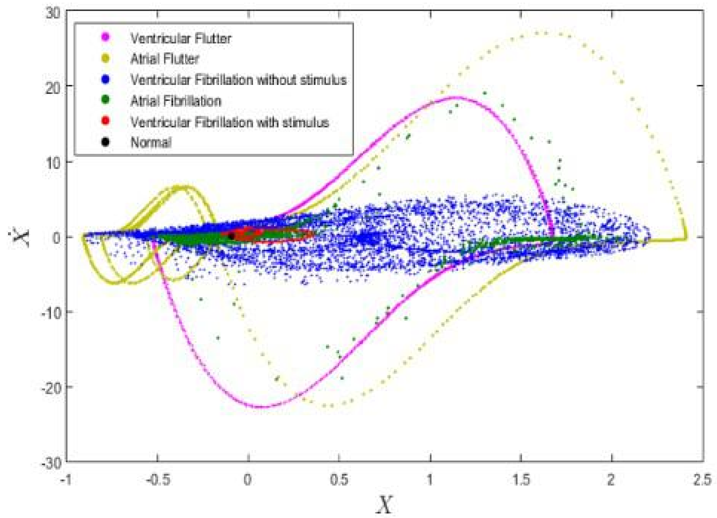

Figure 4 – Comparison of Poincaré maps built with reference period for different rhythms.

REFERENCES

Cunningham, W. J., 1954, “A nonlinear differential-difference equation of growth”, Proceedings of the National Academy of Sciences, v. 40, n. 8, p. 708-713.

Dos Santos, A. M., Lopes, S. R., Viana, R. R. L., 2004, “Rhythm synchronization and chaotic modulation of coupled Van der Pol oscillators in a model for the heartbeat”, Physica A: Statistical Mechanics and its Applications, v. 338, n. 3, pp. 335-355.

Dubin, D., 1996, “Interpretação rápida do ECG”, Editora de Publicações Biomédicas – EPUB, Rio de Janeiro.

Glass, L., 2009, “Introduction to controversial topics in nonlinear science: Is the normal heart rate chaotic?”, Chaos: An Interdisciplinary Journal of Nonlinear Science, v. 19, n. 2, pp. 028501.

Gois, S. R. S. M., Savi, M. A., 2009, “An analysis of heart rhythm dynamics using a three-coupled oscillator model”, Chaos, Solitons & Fractals, v. 41, n. 5, pp. 2553-2565.

Grudzinski, K., Zebrowski, J. J., 2004, “Modeling Cardiac Pacemakers with Relaxation Oscillators”, Physica A, v. 336, pp. 153-162, 2004.

Mensour, B., Longtin, A., 1998, “Power spectra and dynamical invariants for delay-differential and difference equations”, Physica D: Nonlinear Phenomena, v. 113, n. 1, pp. 1-25.

Savi, M. A., 2005, “Chaos and order in biomedical rhythms”, Journal of the Brazilian Society of Mechanical Sciences and Engineering, v. 27, n. 2, pp. 157-169.

Sprott, J.C., “A Simple Chaotic Delay Differential Equation”, Physics Letters A, v. 366, pp. 397-402, 2007.

Van der Pol, B., Van der Mark, J., 1928, “The Heartbeat Considered as a Relaxation Oscillator and an Electrical Model of the Heart”, Philosophical Magazine, v.6, pp.763-775.

Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A.,1985, “Determining Lyapunov exponents from a time series”, *Physica D: Nonlinear Phenomena*, v. 16, n. 3, pp. 285-317.