Cubic Stiffness on the Subunits of a Periodic Structure for Suppression of Axial Vibration

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Abstract: Periodic structures, also known as metastructures, have vibration suppression characteristics that are not found nor common in natural materials. These characteristics are not in the chemical form of the material, but in the arrangement of the structure, that is, they are materials that can be created by manipulating physical characteristics. The study of metamaterials began with electromagnetic metamaterials. Subsequently, the study of metamaterials is extended to the field of acoustics and vibration, and for that it is necessary to implement the concept of negative mass. It is implemented through the addition of local absorbers. There are several periodic systems that can be applied in everyday life, and can be seen as finite or cyclic periodic structures. Periodic structures can be used in several areas, as they can reduce vibration without having to add mass to the system, just by moving the arrangement of the structure. A major advantage of periodic structures is that it is possible to analyze the structure dynamics by taking data from only one cell. In this paper, we compare the response of a metastructure with different combinations in the positioning and nature of restoring force (linear or nonlinear) of absorbers along the structure. The response is verified at the first mass of the structure, while a periodic excitation is applied at the last mass. Results show that, the choice of the absorber positioning depends on the range of frequencies desired for operation. Also, the value of the optimal nonlinear stiffness coefficient can change drastically depending on which frequency range is chosen.

Keywords: Metastructure, Periodic structure, Nonlinear stiffness

INTRODUCTION

The advance in the study of periodic structures is of great importance for many engineering applications, such as civil engineering, automotive and aerospace structures (Hussain, 2018). Periodic structures, also known as metastructures, have vibration suppression characteristics that are not found nor common in natural materials. According to Cveticanin (2018) these characteristics are not in the chemical form of the material, but in the arrangement of the structure, that is, they are materials that can be created by manipulating physical characteristics. The study of metamaterials began with electromagnetic metamaterials. Veselago (1968) described materials with negative electrical permittivity, negative magnetic permeability and negative refractive index. Subsequently, the study of metamaterials is extended to the field of acoustics and vibration, and for that it is necessary to implement the concept of negative mass. That is a concept since it is not possible for a material to have negative mass, and it is implemented through the addition of local absorbers. The first to implement this concept were Milton and Willis (2007). There are several periodic systems that can be applied in everyday life, and can be seen as finite or cyclic periodic structures, such as: building blocks, the fuselage structure of an airplane, a tube having identical supports positioned equidistantly, materials composites or even a ring with identical supports positioned at angular intervals identical to each other. Reichl and Inman (2017) performed a study with the feasibility of adding distributed absorbers to a structure without a total increase of the mass of the structure, and they have shown that the dynamics of the vibration absorber added to the structure is responsible for attenuating the vibration of the system and not the addition of mass. This confirms its application in the automotive and aerospace sectors, since the vibration is attenuated by the dynamics of the vibration absorbers, and these are branches of the industry where it is extremely important to be able to keep the mass of the structures as small as possible. Vibration in periodic structures may have its amplitude increased or decreased, depending on the frequency range in which the waves that generate the vibration are. According to Chakraborty (2001) an advantage of periodic structures is that the dynamics of these structures can be studied with the analysis of only one cell. In this paper, we compare the response of a metastructure with different combinations in the positioning and nature of restoring force (linear or nonlinear) of absorbers along the structure. The response is verified at the first mass of the structure, while a periodic excitation is applied at the last mass. Similarly to Reichl and Inman (2017), the absorbers are added to the structure without addition of mass, ie., the total mass of the system is the same with or without absorbers.

MATHEMATICAL MODELING

The model used in this paper is shown in Fig. 1, representing a metastructure used for axial vibration suppression. The host structure is composed of mass, stiffness and damping elements, named m_h, k_h, c_h, respectively. The absorbers are attached to the host mass, and have mass, stiffness and damping elements, named m_a, k_a, c_a, respectively. The stiffness of

the absorbers can be of linear or cubic form. The coordinate $q(t)$ represents the mass displacement of the host structure in the horizontal direction, and the coordinate $\hat{q}(t)$ represents the displacement of the absorber in the horizontal direction, in which the coordinates of each are in reference to the static equilibrium of each. For the analysis, the responses are compared to the response of the host structure without absorbers, called the base structure. For this study, a host structure with 5 cells in conjunction with 4 absorbers.
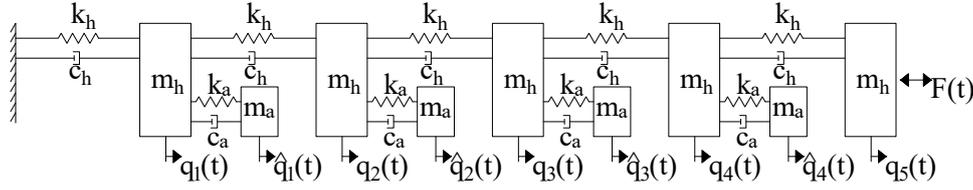

Figure 1 – Model representing a metastructure used for axial vibration suppression with host structure and absorbers

The mass (m_b), stiffness (k_b) and damping (c_b) of the base structure are given by Eq. (1), in which n is the number of absorbers, m_{total} is the total mass of the system, E is the Young's modulus, A is the cross sectional area, L is the system length , α is the proportional damping constant. Also, the ratio between the mass of the absorbers and the host structure is given by μ.

This is due to the fact that in this work, it is desired to use a periodic structure with absorbers having the same total mass as the base structure. For this, following the methodology proposed by Reichl and Inman (2017) when adding an absorber in a region of the structure, the mass of the base structure in this same region must be smaller so that the total mass is equal.

$$m_b = \frac{m_{total}}{n+1}, \qquad k_b = \frac{EAn}{L}, \qquad c_b = \alpha k_b, \qquad \mu = \frac{\Sigma m_a}{(n+1)m_h}. \tag{1}$$

Similarly, the mass, stiffness and damping equations of the host structure (m_h, k_h, c_h) and of the absorbers (m_a, k_a, c_a) are given by Eqs. (2) and (3), in which f is the absorber tuned frequency.

$$m_h = (1-\mu)\frac{m_{total}}{n+1}, \qquad k_h = k_b, \qquad c_h = \alpha k_h. \tag{2}$$

$$m_a = \mu\frac{m_{total}}{n}, \qquad k_a = m_a f^2, \qquad c_a = \alpha k_a. \tag{3}$$

The dynamical equations of the system were obtained using Newton's second law, and are given by Eq. 4, which are shown in general form with external harmonic force, in which F_0 is the amplitude and ω is the excitation frequency.

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) + D = F(t), \qquad F(t) = F_0 sin(\omega t). \tag{4}$$

The matrices defining the equations of the base structure are given by

$$M = diag[m_b \ m_b \ m_b \ m_b \ m_b], \quad K = K_b = \begin{bmatrix} 2k_b & -k_b & 0 & 0 & 0 \\ -k_b & 2k_b & -k_b & 0 & 0 \\ 0 & -k_b & 2k_b & -k_b & 0 \\ 0 & 0 & -k_b & 2k_b & -k_b \\ 0 & 0 & 0 & -k_b & 0 \end{bmatrix}, \quad C = \alpha K, \quad D = 0. \tag{5}$$

While the matrices for the metastructure are given by

$$M = diag[m_h \ m_h \ m_h \ m_h \ m_h \ m_a \ m_a \ m_a \ m_a], \quad K_a = diag[k_a \ k_a \ k_a \ k_a], \quad O = [0 \ 0 \ 0 \ 0] \tag{6}$$

$$K = \begin{bmatrix} K_b & & -K_a \\ & & O \\ -K_a & O^T & K_a \end{bmatrix}, \quad C = \alpha K, \quad D = \begin{bmatrix} k_n\hat{q}_1^3 \\ k_n\hat{q}_2^3 \\ k_n\hat{q}_3^3 \\ k_n\hat{q}_4^3 \end{bmatrix}. \tag{7}$$

For the linear system, the stiffness coefficient k_n equals zero, meaning that the system will be linear because it will zero out the nonlinear part of the nonlinear restoring force equation Fk_n, shown in Eq. 8, and only one absorber is added to the system. Four responses are obtained, each for a different position of the single absorber in the system.

$$Fk_n = k_a\hat{q}(t) + k_n\hat{q}^3(t) \tag{8}$$

The coefficient of nonlinear stiffness k_n is obtained by Eq.9. This equation is obtained by making the ratio of how many times the value of the nonlinear force is smaller or greater than the linear force. In this paper it is considered only the case where the nonlinear force is greater than the linear force.

$$k_n = \frac{k_a(a-1)}{x^2} \tag{9}$$

NUMERICAL ANALYSIS

Table 1 shows the values used to calculate the mass, stiffness and damping which are used in this section, and are based on Reichl and Inman (2017).

Table 1 – System parameters

Parameter	Symbol	Value	Unity
Young's Modulus	E	1970	MPa
Length	L	0.45	m
Cross Sectional Area	A	0.0009	m
Total Mass	m_{total}	0.597	kg
Absorber Tuned Frequency	f	3430	Hz
Mass Ratio	μ	0.3	-
Proportional Damping Constant	α	2.5^{-5}	-

In this section we show the results obtained with different positions of the absorbers along the structure, as well as the comparison of the linear and nonlinear systems. The analyzed values are obtained from the vibration response of the first mass of the host system, while the excitation is in the last mass.

For the case where the system is considered nonlinear, to find the stiffness coefficient k_n it is assumed that the nonlinear force at the vibration amplitude 2^{-4} [m] is five times greater than the linear force. After finding the coefficient of nonlinear stiffness, results are obtained considering that there is an absorber in all positions of the system, and that one of them is nonlinear. The position of the nonlinear absorber is changed to observe the response of the system to these different situations.

Figure 2 (a) shows the frequency response of the linear system, with the four position absorber changes and (b) shows the structure with all the absorbers, one of which is nonlinear. The dashed lines show the frequency values that were used in Figs. 3 and Figs. 4, in which several values of initial conditions were tested, and only for the excitation frequency of 8000 Hz and nonlinear absorber in the first position that appeared more than one limit cycle.

Figure 3 shows the phase plane of the different absorber positions with an excitation frequency $\omega = 4700$ Hz where in Fig.(a) is the linear system and in Fig. (b) is the nonlinear system, in which this frequency was chosen to show the influence of the position of the absorber on the response, where only by changing the position of the absorber it is possible to have a better vibration attenuation response.

Figure 4 shows the phase plane of the different absorber positions with an excitation frequency $\omega = 8000$[Hz], this frequency has been chosen to show that, depending on the position of the nonlinear absorber, more than one limit cycle may appear. Where in Fig.(a) is the linear system and in Fig.(b) is the nonlinear system, where there are absorbers in all positions and in only one will have the nonlinear absorber. In the nonlinear case it's possible to observe that when the absorber is in the first position two limit cycles emerge, that is, depending on the initial condition the system will go to one or the other cycle. The initial conditions used were: condition 1 = displacement and velocity values equal to zero for all masses, condition 2 = initial displacement value of the first mass of the host structure equal to 0.005 and all other values of displacement and initial velocity equal to zero.

In general, for both the linear system and the nonlinear system, the variable that most impacts the choice of the absorber positioning if a lower amplitude of vibration is desired will be the excitation frequency at which the system will be subjected.
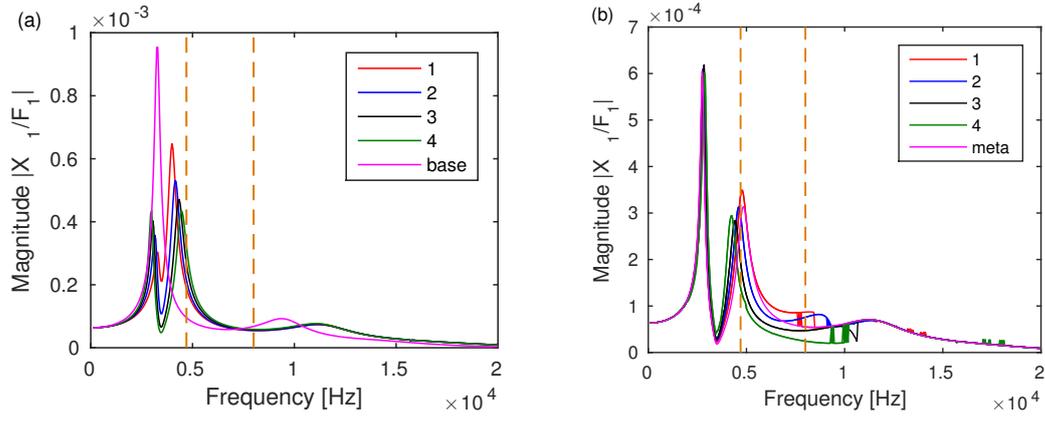
Figure 2 – (a) FRF of the linear system with change of a position absorber. (b) FRF of the nonlinear system, with all absorbers and change of position of a nonlinear absorber. In which the numbers in the caption mean in which position the absorber is inserted. Base in the linear system is the structure without absorbers. Meta in the nonlinear system is the structure considering linear absorbers in all positions.

Figure 3 – Phase plane for excitation frequency ω = 4700 [Hz]. (a) is the linear system and (b) is the nonlinear system. In which the numbers in the caption mean in which position the absorber is inserted. Base in the linear system is the structure without absorbers. Meta in the nonlinear system is the structure considering linear absorbers in all positions.
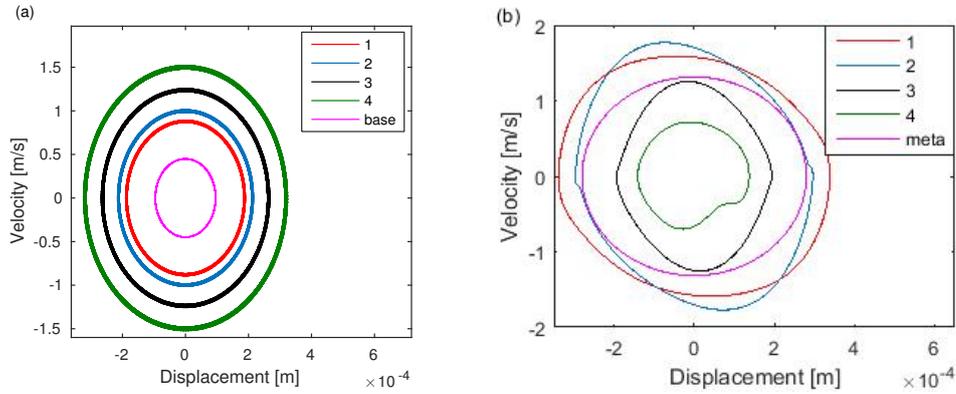
OPTIMIZATION

In this section an optimization procedure is shown to minimize the $\mathcal{H}_2$ norm of the frequency response of the system. This problem of optimization can be written by:

$$\text{Find } X = \{x_1 \ x_2\}^T \text{ which minimizes } f(X)$$

subject to constraints

$$\begin{aligned}
1 - g_1(x_1) &\leq 0 \\
-4 + g_2(x_1) &\leq 0 \\
1 - g_3(x_2) &\leq 0 \\
-10 + g_4(x_2) &\leq 0
\end{aligned} \tag{10}$$

In which x_1 is decision variable of position p and x_2 is decision variable of a, $f(X)$ is the objective function $\mathcal{H}_2$ norm, and $g_{1,2,3,4}$ are inequality constraints. The formulation of $\mathcal{H}_2$ norm is obtained considering the representation of the system in state space, in which I is the identity matrix and G is the transfer function, shown in Eq. 11.

$$G(\omega) = C(j\omega I - A)^{-1} B \tag{11}$$

The definition of the $\mathcal{H}_2$ norm is shown in Eq. 12, in which tr is the trace of the matrix.
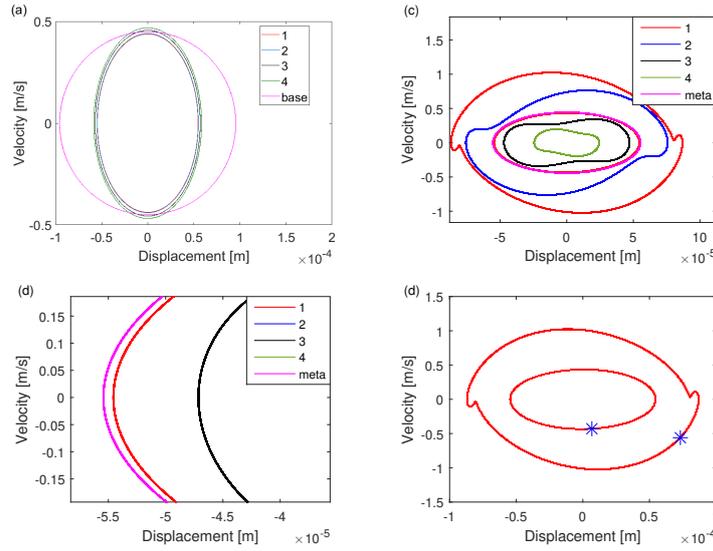
Figure 4 – Phase plane for excitation frequency ω = 8000 [Hz].(a) is the linear system,(b) is the nonlinear system, (c) is a zoom of the nonlinear figure. In which the numbers in the caption mean in which position the absorber is inserted and (d) is the Poincare section for when the non-linear absorber is in the first position. Base in the linear system is the structure without absorbers. Meta in the nonlinear system is the structure considering linear absorbers in all positions.

$$\mathcal{H}_2 = ||G||_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} tr(G^*(j\omega)G(j\omega))d\omega \tag{12}$$

The $\mathcal{H}_2$ norm can be interpreted by the area below the frequency response. In this paper, the area is obtained by an the trapezoidal method. The $\mathcal{H}_2$ norm values are obtained for four regions, being them, region 1 (1500 to 3500 Hz), region 2 (3500 to 6500 Hz), region 3 (6500 to 12550 Hz) and region 4 (10 to 20000 Hz). These values were chosen as regions of optimization, because regions 1 and 2 are where high amplitude values are found, region 3 is where characteristic of multistabilities exists and region 4 is the entire frequency range used in the simulations. These regions are exemplified in Fig. 5 where in (a) is the frequency response for $a = 5$ and (b) $a = 10$. In Tab. 2 the values of the $\mathcal{H}_2$ norm for these two situations are shown, in which p is the position of the nonlinear absorber. The lowest values analyzed in each region are in bold.

Table 2 – $\mathcal{H}_2$ norm

	region 1	region 2	region 3	region 4
Without absorbers	0.2888	0.7487	0.3890	1.7153
Linear absorbers	0.3600	0.4177	0.3732	1.4411
a = 5 e p = 1	**0.3597**	0.4753	0.4301	1.5564
a = 5 e p = 2	0.3666	0.4091	0.4071	1.4738
a = 5 e p = 3	0.3788	0.3535	0.3274	1.3500
a = 5 e p = 4	0.3870	**0.3339**	0.2614	1.2749
a = 10 e p = 1	0.3601	0.4724	0.4408	1.5676
a = 10 e p = 2	0.3684	0.4228	0.3870	1.4704
a = 10 e p = 3	0.3824	0.3669	0.3142	1.3564
a = 10 e p = 4	0.3891	0.3446	**0.1876**	**1.1946**

To perform the optimization, some situations were considered. First the two decision variables are a p. The second situation is that only the value of a is used as decision variable, while the position of the nonlinear absorber is fixed. The value of the variable a influences in the coefficient of nonlinear stiffness k_n through Eq. 9.

To solve the optimization problem a constrained nonlinear optimization method was used. The method used was sequential quadratic optimization, which is a method resulting from the application of the Newton method to the minimization of the Lagrangian function of the problem. This method is based on the idea of approaching, in each iteration, the nonlinear problem by quadratic programming subproblems (RAO, 2009). The solution of these subproblems generates a search direction for the linear method, which is where the value of the desired function, in this case, the $\mathcal{H}_2$ norm is evaluated.
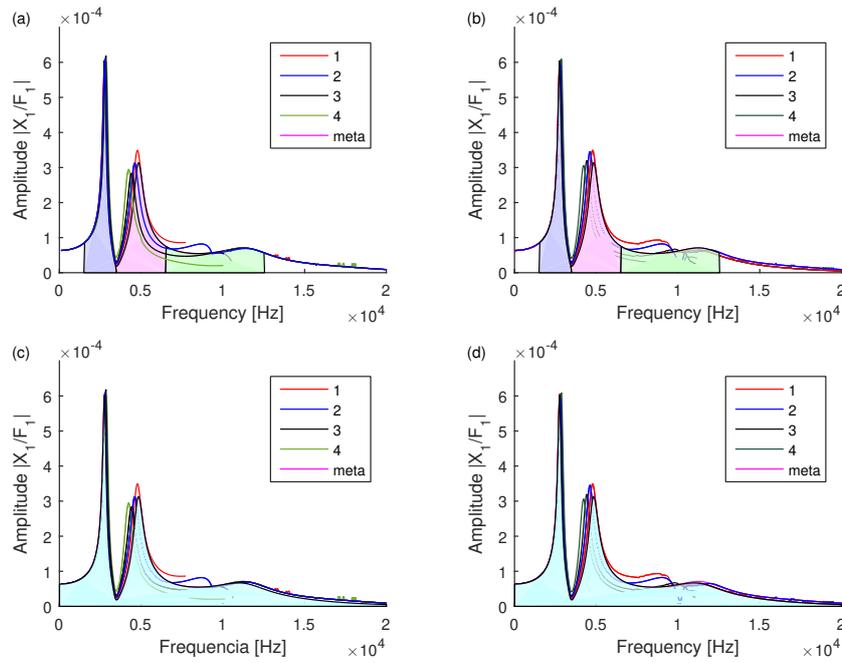
Figure 5 – (a) FRF of the system for $a = 5$, (b) FRF of the system to the value of a equal to 10, (c) FRF of the system to the value of a equal to 10 where the shaded region is the region 4 and (d) FRF of the system to the value of a equal to 10 where the shaded region is the region 4. Where the shaded regions show which regions were the $\mathcal{H}_2$ norm was optimized

a and p as decision variables

For the first optimization situation, a is constrained between 1 and 10, and p is constrained between 1 and 4, using integer values, since they are the only possible positions for the nonlinear absorber in the studied structure.

In Tab. 3 the $\mathcal{H}_2$ norm values are shown for the optimal values found, as well as the optimal values of a and p, in which the optimal value of each region was obtained generating the frequency response only for the region under analysis. The smallest values of each analyzed region are in bold.

Table 3 – Optimized $\mathcal{H}_2$ norm having two decision variables

	region 1	region 2	region 3	region 4	p	a
Optimization of region 1	**0.3659**	0.4060	0.4051	1.4676	2	3.5033
Optimization of region 2	0.3744	0.3343	0.3262	1.3252	3	1.9900
Optimization of region 3	0.3752	**0.3298**	**0.3255**	**1.3208**	3	1.8361
Optimization of region 4	0.3744	0.3343	0.3262	1.3252	3	1.9899

a as decision variable

For the second optimization situation, a is constrained between 1 and 10. The position p is fixed in different values. For example, when the position is fixed 1, the simulation was performed to find the optimal value of a for this position and the study regions. This is repeated for the four possible positions of the non-linear absorber. Table 4 shows the values of $\mathcal{H}_2$ norm for the optimal values found, as well as the optimal values of the variable a. This table was divided into four smaller tables, where each one refers to the fixed position of the non-linear absorber.

Comparing the two situations where it is desired to minimize the values of the $\mathcal{H}_2$ norm, it is possible to observe that the second situation, where the positions were fixed and the variation only made in a, smaller values of $\mathcal{H}_2$ norm were obtained.

Now comparing the values of the second situation with the nominal values (non-optimized), where the variable a was used as equal to 5 and equal to 10. In region 1, the value of $\mathcal{H}_2$ non-optimized with $p = 1$ and $a = 5$ are similar, and are the lowest ones obtained, so the optimization was successful for region 1. In region 2, the lowest value of $\mathcal{H}_2$ norm obtained was the optimized with fixed $p = 4$, resulting in $a = 1.9900$. In region 3, the optimization was not able to find a value smaller than for $p = 4$ and $a = 10$, where in this case the $\mathcal{H}_2$ norm value obtained was 0.1876 and the lowest value obtained by the optimization was when the position was set to 4, with a value of $a = 6.1354$ which generated a $\mathcal{H}2$

Table 4 – Optimized $\mathcal{H}_2$ norm having a decision variable

p = 1	a	$\mathcal{H}_2$ norm	p = 2	a	$\mathcal{H}_2$ norm
region 1	1.9900	**0.3607**	region 1	5.4348	0.3676
region 2	5.8789	0.4727	region 2	1.9900	0.3972
region 3	5.5723	0.4201	region 3	1.9900	0.3848
region 4	4.8229	1.5523	region 4	5.6962	1.4751
p = 3	a	$\mathcal{H}_2$ norm	p = 4	a	$\mathcal{H}_2$ norm
region 1	1.9900	0.3750	region 1	5.0182	0,3874
region 2	1.9900	0.3334	region 2	1.9900	**0.3038**
region 3	5.9729	0.3283	region 3	6.1354	**0.2547**
region 4	1.9900	1.3391	region 4	1.3870	**1.2059**

norm value of 0.2547. In region 4, the value without going through the optimization process with $p = 4$ and $a = 10$ are similar, and the lowest obtained, so the optimization was successful for region 4. Table 5 shows the values of $\mathcal{H}_2$ norm for each region studied for the metastructure with linear absorbers in all four positions, the lowest values of $\mathcal{H}_2$ norm with optimization, and finally the reduction of the $\mathcal{H}_2$ norm.

Table 5 – Comparison between the values obtained for the norm $\mathcal{H}_2$

	Linear absorbers	Optimization	Difference	%
region 1	0.3600	0.3607 (p=1 e a=1,99)	-0.0007	+0.19
region 2	0.4177	0.3038 (p=4 e a=1,99)	0.1139	-27.27
region 3	0.3732	0.2547 (p=4 e a=6,1354)	0.1185	-31.76
region 4	1.4411	1.2059 (p=4 e a=1,3870)	0.2352	-16.33

In Tab. 5 it is possible to see that for region 1 the optimization was not efficient, obtaining similar value to the system only with linear absorbers. For the other regions there were reductions in the norm $\mathcal{H}_2$, even for region 3 that did not find the lowest possible value in the optimization process the reduction was 31 %. Figure 6 shows the frequency responses for each of the optimal values obtained, compared to the frequency responses of the system with only linear absorbers.
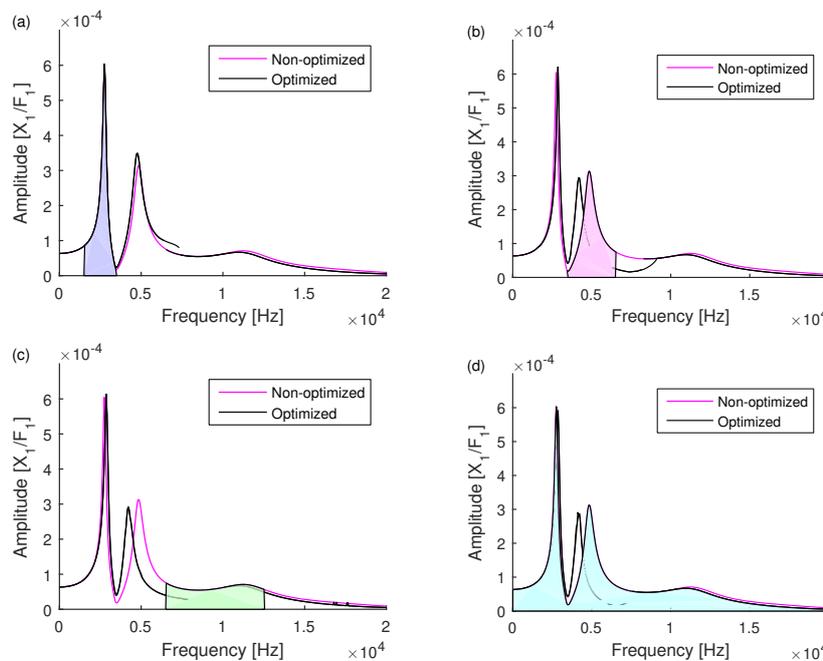

Figure 6 – The FRFs in pink are of the system only with linear absorbers, the FRF of black color in (a) is for optimal values for region 1, in (b) is for optimal values for region 2, in (c) is for optimal values for region 3 and in (d) is for optimal values for region 4

CONCLUSION

In the course of this paper it is possible to note that the choice of adding absorbers to the structure depends directly on the frequency range to be used, once this is the frequency at which the absorber is to be tuned. For when the structure already contains absorbers and even if it is still desired to decrease the vibration in a certain region by adding a non-linear cubic stiffness, a study must be made to which position this absorber will most impact in the desired way, and what will be the coefficient of nonlinear stiffness. Comparing the values presented in the tables contained in the optimization section for lower frequency values, as for region 1, the position that obtained a lower $\mathcal{H}_2$ norm was in the first position, ie near the position the response was obtained . For the other regions adding the coefficient of nonlinear stiffness in the last position showed to be more efficient, that is, closer to where the system is being excited.

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