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Dynamics of a parametric *N*-pendulum with a view on energy harvesting from the motion of a reciprocating oil pumping unit

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Abstract: The nonlinear dynamics of a parametric pendulum system excited by the motion of a reciprocating oil pumping unit is explored. The final goal of this investigation is to recover energy from such machines by means of pendulum harvesters located on a walking beam. Being the parametric pendulum a potential well system, the study is concentrated in rotational responses which are the most energetic state. A pendulum with multiple concentric masses, i.e. a N-pendulum, is considered in the design. This provides versatility, since it allows the tuning of the natural frequency of the system. The nodding motion provided by the pumping unit is a planar motion, thus containing both vertical and horizontal components. Such feature has a strong influence in the dynamics of the system, as introduces an asymmetry in the governing equation. It is shown that, by virtue of this asymmetry, the pendulum system experiences a complex rotational behavior, very different to the widely studied classic parametric pendulum with a sinusoidal forcing.

Keywords: pendulum, pumpjack, four-bar linkage, energy harvesting, nonlinear dynamics

INTRODUCTION

Among the large number of devices aimed to scavenge ambient energy, pendulum harvesters are attractive due to simplicity in the mechanisms involved, in combination with the high kinetic energy available in its rotational motion. The concept is very simple and intuitive: it consists of a pendulum, with a supporting structure fixed to a source of motion. If a stable rotation of the pendulum can be achieved, part of the kinetic energy can be converted into electrical energy by a generator attached to the pendulum axis (Wiercigroch, 2010). Two sources of ambient vibrations are mainly considered by the researchers: vibrating machines (Dotti et al., 2017) and the motion of the sea waves (Nandakumar et al., 2012; Yurchenko and Alevras, 2013; Reguera et al., 2016). In both cases, rotations can be achieved only for some forcing scenarios. But in the case of vibrating machines, the motion imposed to the pendulum is generally harmonic and thus more predictable than the stochastic motion of the sea waves. This predictability is a very important feature in the design of a suitable pendulum harvester, since it implies that the forcing parameters are known *a priori* in the design process.

Oil pumping units, also known as *pumpjacks*, are generally located at isolated areas where there is a lack of electrical infrastructure. In such conditions, energy harvesting is appreciated to feed systems with low power demands, such as structural health monitoring sensors or warning lights. A pumpjack converts the rotatory motion of a motor into a reciprocating motion, which drives the pump. This resulting reciprocating motion, a *nodding motion*, is contained within the plane of the pumpjack main mechanism, which is the well-known planar four-bar linkage (McCarthy and Soh, 2011). If a pendulum harvester is placed over the walking beam of the pumpjack, the nodding motion represents the input forcing of the system.

We study the dynamics of a parametric *N*-pendulum system excited by the nodding motion of a pumpjack. Aiming to design a suitable pendulum harvester, the dynamics of rotating attractors is quantified. This is performed by means of bifurcation analyses and explorations in the phase plane and the space of forcing parameters.

MATHEMATICAL MODEL

Figure 1a shows a schematic *N*-pendulum, which is contained in the xy plane. Each of the N masses are separated by a fixed angular distance $\psi = 2\pi/N$, and arranged concentrically around a pivot axis. Length l_1 is assumed to be greater than or equal to any of the other lengths (l_2, l_3,..., l_N), and the same assumption is made for mass m_1 with respect to m_2, m_3,..., m_N. Both l_1 and $m_1$constitute the main arm, as pointed. The angle θ is measured with respect to the main arm and denotes the angular position relative to the vertical direction. A time-dependent motion with components $X(t)$ and $Y(t)$ is imposed to the pivot axis. The center of mass of the system is located at the point C. From classic mechanics of systems of particles (Chow, 1995), kinetic, potential and dissipative energy terms can be derived respectively as

$$T = \tfrac{1}{2} M \left\{ \left[Y' + l_C \theta' \sin(\theta+\varphi) \right]^2 + \left[X' + l_C \theta' \cos(\theta+\varphi) \right]^2 \right\} + \tfrac{1}{2} K_0 (\theta')^2 ,$$
$$V = gM \left[l_1 - l_C \cos(\theta+\varphi) \right], \quad D = \tfrac{1}{2} h (\theta')^2 \sum_{i=1}^{N} l_i^2 , \tag{1}$$

where g is gravity, $\varphi = \arctan[(x_C/y_C)|_{\theta=0}]$ is the phase shift of the center of mass C with respect to the main arm and $l_C = (x_C^2 + y_C^2)^{1/2}|_{\theta=0}$ is the distance from C to the origin; x_C and y_C are the coordinates of C and $M = \Sigma\, m_i$.

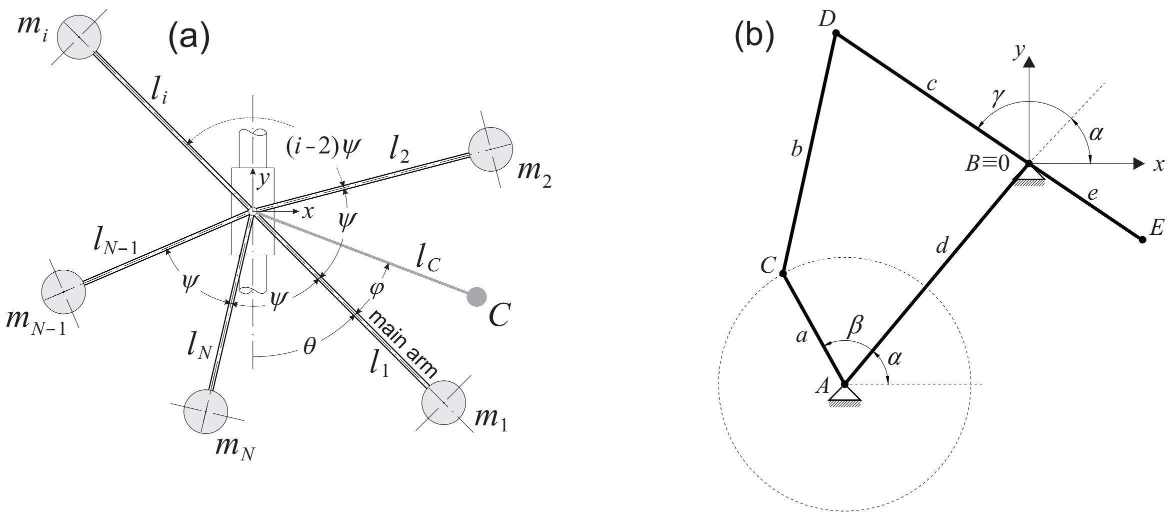

Figure 1 – (a) Scheme of the vertical parametric *N*-pendulum. (b) Planar four-bar linkage, the main mechanism of a pumpjack. The *N*-pendulum is assumed to be located at the point *E*.

The factor K_0 in (1) is associated to the kinetic energy relative to the center of mass, and it is given by

$$K_0 = \sum_{i=1}^{N} \left\{ m_i \left\{ l_i^2 + l_C^2 - 2 l_i l_C \cos\left[(i-1)\psi - \varphi \right] \right\} \right\}. \tag{2}$$

The viscous friction coefficient h in (1) is defined as independent of mass shapes, considering bobs which are very similar one to each other. By introducing (1) into the Lagrange equation for one degree of freedom non-conservative systems, the equation of motion for arbitrary $Y(t)$ and $X(t)$ is obtained as

$$I\theta'' + b\theta' \sum_{i=1}^{N} l_i^2 + M\, l_C X'' \cos(\theta+\varphi) + M\, l_C (Y'' + g) \sin(\theta+\varphi) = 0, \tag{3}$$

where the inertia of the system is defined as $I = K_0 + M\, l_C^2$.

On the other hand, let's consider the four-bar mechanism of Fig. 1b, representing a pumpjack. A motor imposes an angular motion to the linkage around the point A, at a constant angular frequency Ω. Then, the angle $\beta = \Omega\, t$ denotes the angular position of the crank a. Hence, the instantaneous position of the point E can be expressed (McCarthy and Soh, 2011) as

$$(X_E, Y_E) = -e \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot (\cos\gamma, \sin\gamma), \tag{4}$$

where α is a fixed angle representing the tilt of the mechanism with respect to the horizontal and γ is given by

$$\gamma = 2\arctan\left(\frac{2\sin\beta - \sqrt{\left[K_1 - K_3 - (1-K_2)\cos\beta \right]\left[-K_1 - K_3 + (1+K_2)\cos\beta \right] + \sin^2\beta}}{-K_1 + K_3 + (1-K_2)\cos\beta} \right). \tag{5}$$

Since $\beta = \beta(t)$, then $\gamma = \gamma(t)$. Additionally, the following constants of the mechanism are defined: $K_1 = d/a$, $K_2 = d/c$ and $K_3 = (a^2 - b^2 + c^2 + d^2)/(2\, a\, c)$.

Now, let's assume that the pendulum system of Fig. 1a is placed at the point E of the four-bar linkage of Fig. 1b. Then, the input motion of the parametric pendulum is given by $(X, Y) = (X_E, Y_E)$. Introducing (4) in (3), and after some algebraic manipulation, the dimensionless form of (3) can be obtained as

$$\ddot{\theta}+\eta\dot{\theta}+R\,f\left(\omega\tau\right)\cos\left(\theta+\varphi\right)+\left[R\,h\left(\omega\tau\right)+1\right]\sin\left(\theta+\varphi\right)=0, \tag{6}$$

where the dimensionless forcing parameters are defined as $\omega = \Omega/\omega_0$ and $R = e\,\omega^2\,M\,l_C/I$, the natural frequency of the system corresponds to $\omega_0 = (g\,M\,l_C/I)^{1/2}$ and the dimensionless viscous damping coefficient is $\eta = (b\,\Sigma\,l_i^2)/(I\,\omega_0)$. The superimposed dots over some symbols in (6) indicate derivation with respect to the dimensionless time $\tau = \omega_0\,t$. In (6), the input forcing is represented by the functions $f(\omega t)$ and $h(\omega t)$, which are given by

$$f(\omega t)=\cos\left(\alpha+\gamma\right)\left(\frac{d\gamma}{d\beta}\right)^2+\sin\left(\alpha+\gamma\right)\frac{d^2\gamma}{d\beta^2},\qquad h(\omega t)=-\cos\left(\alpha+\gamma\right)\frac{d^2\gamma}{d\beta^2}+\sin\left(\alpha+\gamma\right)\left(\frac{d\gamma}{d\beta}\right)^2. \tag{7}$$

Equation (6) can be simplified to obtain the governing equation of the classic parametric pendulum (i.e. with a simple sinusoidal forcing) (Clifford and Bishop, 1995, Dotti et al., 2015), by considering $f(\omega t) = 0$ and $h(\omega t) = \cos(\omega t)$.

NUMERICAL SIMULATIONS

The dynamics of *N*-pendulum systems governed by (6) is explored. Such equation is solved numerically by means of a classic Runge-Kutta method implemented in *Mathematica*. Since a stable period-1 rotation (clockwise or anticlockwise) is the desired response for energy harvesting purposes, the study is mainly focused on obtaining such steady state of the system.

Figure 2 shows the bifurcational behavior of a pendulum parametrically forced by a nodding motion. The forcing amplitude R is varied within $0 \le R \le 3$, which is assumed to be a feasible operative range of a real harvester. The forcing frequency ω is fixed at the state of main parametric resonance ($\omega = 2$). The other parameters also remain fixed, being their values chosen in order to represent a realistic pumpjack mechanism. The most striking observation is the asymmetry of the attractors with respect to $\theta = 0$ (or $\dot{\theta} = 0$). This somewhat counter-intuitive behavior differs substantially to what happens for the classic parametric pendulum (Fig. 3), for which the attractors appear in a perfectly symmetric fashion. The symmetry is broken in the diagram of Fig. 2 due to the presence of a horizontal component of the excitation, related to the cosine term in equation (6) (Mann and Koplow, 2006; Horton et al., 2011). For the rotating attractors, the asymmetry implies that for some sets of the parameters a clockwise rotation is a feasible response of the system while an anticlockwise rotation is not, and *vice versa*. This is clearly observed in Fig. 2, within the ranges $0.75 \le R \le 0.955$ and $2.057 \le R \le 2.655$.

For a parametric pendulum with an elliptical excitation, Horton et al. (2011) showed that a preferred direction of rotation (clockwise or anticlockwise) is obtained according to the direction of motion of the pivot point of the pendulum around the ellipse. A similar behavior is observed in Fig. 2, as the anticlockwise rotation can be obtained for smaller values of R than clockwise rotations. Now, the pivot point has not a predominant direction of rotation. The preference must be tracked to the motion of the crank AC of Fig. 1b, which is assumed to rotate counterclockwise. Thereby, anticlockwise rotations pick up more energy from the excitation.

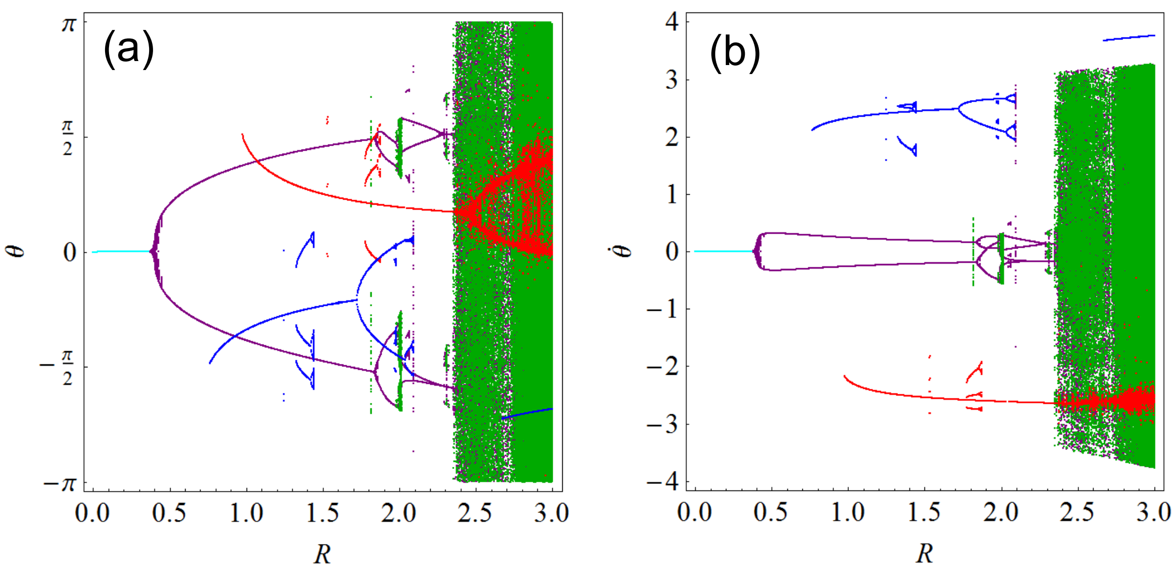

Figure 3 – Bifurcation diagrams for the main parametric resonance (ω = 2), with η = 0.1, φ = 0, K_1 = 2.94, K_2 = 1.45, K_3 = 2.06, α = 0.93. (a) Angular position. (b) Angular velocity. Reference: (•): clockwise rotations, (•): anticlockwise rotations, (•): rest, (•): oscillations, (•): tumbling chaos.

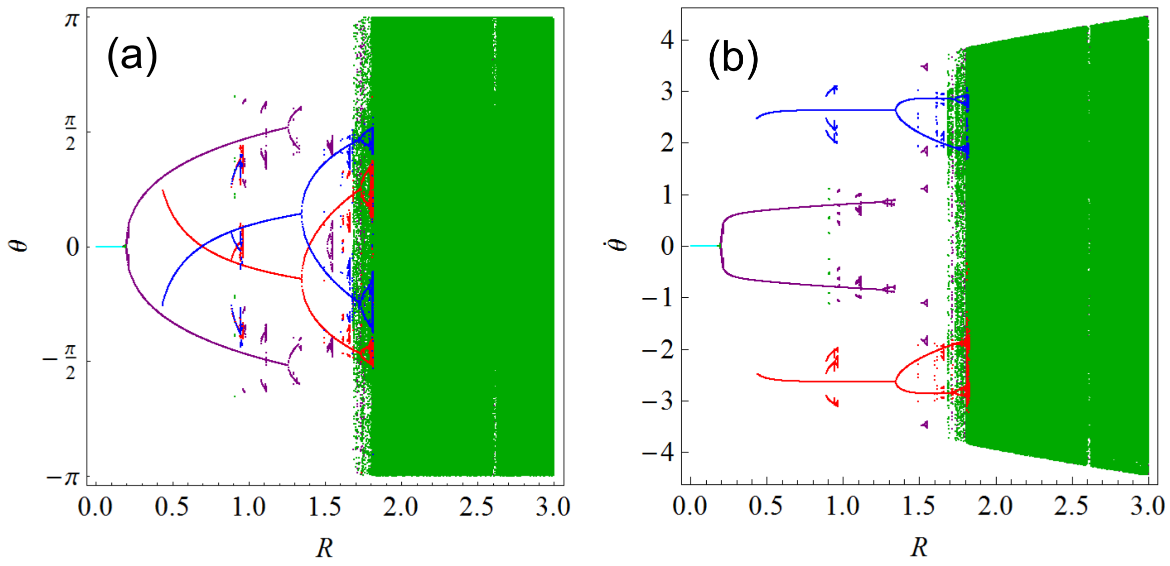

Figure 3 – Bifurcation diagrams for the main parametric resonance (ω = 2) of the classic parametric pendulum, with η = 0.1. (a) Angular position. (b) Angular velocity. Reference: (•): clockwise rotations, (•): anticlockwise rotations, (•): rest, (•): oscillations, (•): tumbling chaos.

By comparing Fig. 2 and Fig. 3, one can verify that the excitation by nodding motion favors the development of period-1 rotations, as they can be reached within $0.75 \leq R \leq 2.35$ and $2.7 \leq R \leq 3$. This represents 63 % of the assumed operative range. On the other hand, with a sinusoidal excitation, period-1 rotations are possible only for 30 % of such range.

Basins of attraction, i.e. the sets of initial conditions leading to an attractor, also show evidence of such asymmetry. Fig. 4a presents a situation where anticlockwise rotations are possible but no basin of clockwise rotations is observed. In Fig. 4b, both rotating attractors are available but their basins are different in term of robustness: 17.6% of the initial conditions produce clockwise rotations while 18.21% produce anticlockwise rotations. The attractors are also at different stages of fractalization in Fig. 4b, as can be seen. The anticlockwise rotation appears first and also vanishes first, in a crisis at $R \approx 2.1$. At that stage, period-1 clockwise rotation is still a feasible response. Fig. 4c illustrates such situation, with a fairly robust basin of negative rotations.

Figure 2 evidences a coexistence of a rotations and tumbling chaos, for $2.65 \leq R \leq 3$. At the main resonance, such feature is not possible with a simple sinusoidal excitation. The associated basin is shown in Fig. 4d. Anticlockwise rotations can be reached with high initial velocities, as a robust basin is observed. Now, most of the initial conditions (97.98 %) allow obtaining a period-2 clockwise rotatory response, after long chaotic transients (Szemplinska-Stupnicka et al., 2000). Tumbling chaos, on the other hand, is very difficult to obtain.

In order to obtain a global picture of the dynamics of period-1 rotations, we constructed a parameter space in the R-ω plane for a nodding excitation (Fig. 5). This space is also based on extensively numerical simulations. The bifurcational behavior of the classic parametric pendulum is also included in Fig. 5, by means of the black curves J, G and U. For increasing R, period-1 rotations born at a fold bifurcation (indicated by J, J^P and J^N in Fig. 5) and loose stability at a flip (period-doubling) bifurcation (curves G, G^P and G^N). By further increasing R, period-1 rotations regain stability at the intermediate rotation zones delimited by the curves H^N-K^N and H^P-K^P. , which present fractal boundaries. Rotations coexist with tumbling-chaos at this point, and the system exhibits final state sensitivity: minute variations of the initial state may change the attractor ultimately chosen (Thompson and Stewart, 2003). Clockwise rotations are reached after very long periods of transient chaos for $R > 2.1$, next to the curve H^N. Thus, such curve could not be clearly identified in these calculations, and a deeper investigation of this area is required. In addition, for high values of forcing, a small area of clockwise rotations is obtained, delimited by curves E^N and F^N in Fig. 5.

Fig. 5 indicates that stable rotations are obtained with a higher value of R, if a nodding motion is the applied external motion, in comparison with simple sinusoidal. A different result was obtained by Horton et al. (2011) for an elliptical excitation. However, probably curves J^P and J^N depend on the dimensions of the four-bar linkage, which have not been varied in this study. Thus, this observation should not be taken as a final conclusion.

CONCLUSIONS

This article aims to contribute to the development of pendulum systems for energy extraction from reciprocating oil pumpjack units. The dynamics of the parametric pendulum with an external excitation provided by the nodding motion of a four-bar linkage was addressed.

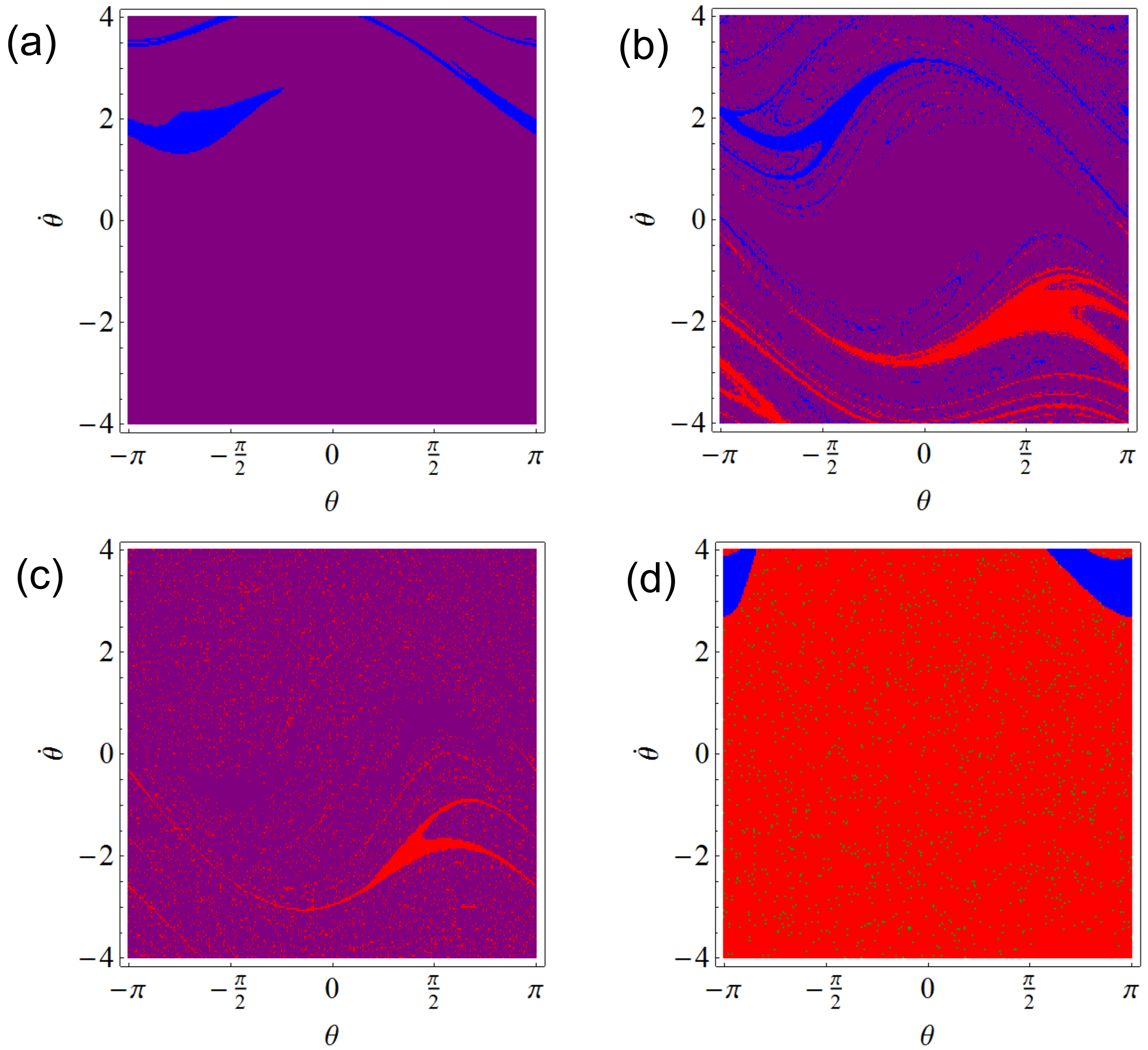

Figure 4 – Basins of attraction for the main parametric resonance (ω = 2), with η = 0.1, φ = 0, K_1 = 2.94, K_2 = 1.45, K_3 = 2.06, α = 0.93. (a) *R* = 0.9. (b) *R* = 1.6. (c) *R* = 2.3. (d) *R* = 2.8. Reference: (•): clockwise rotations, (•): anticlockwise rotations, (•): rest, (•): oscillations, (•): tumbling chaos.

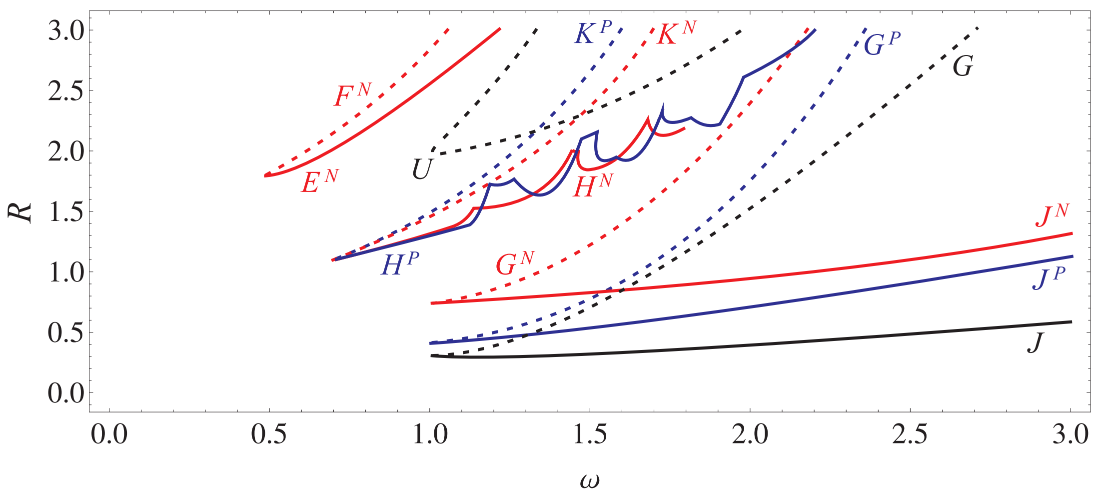

Figure 5 – Two-parameter space *R*-ω, with η = 0.1, φ = 0, K_1 = 2.94, K_2 = 1.45, K_3 = 2.06, α = 0.93. (a) Angular position. (b) Angular velocity. Bifurcation curves corresponding to: (–): clockwise (negative) rotations for nodding excitation, (–): anticlockwise (positive) rotations for nodding excitation, (–): rotations of the classic parametric pendulum, with η = 0.1.

Nodding motion has not only vertical but also a horizontal component, which produces a symmetry-breaking scenario, thus generating substantial differences with respect to the classic parametric pendulum with a sinusoidal excitation. New period-1 rotating attractors appear which are not possible with a sinusoidal excitation. These rotations were proven to exist for ranges of damping where energy extraction is feasible, which is important in terms of energy harvesting since it means that more combinations of the excitation parameters allow rotational motion, increasing the possibilities for energy extraction.

We conclude that with a suitable design, based on a correct configuration of non-dimensional excitation and damping parameters, steady rotations can be easily reached and predicted for a pendulum harvester installed on an oil pumpjack unit. This suitable design does not represent a serious problem if the excitation is harmonic, as it happens in this situation.

Energy harvesting by means of rotating pendulum or *N*-pendulum harvesters is an attractive idea, due to their simplicity and scalability, and this work aims to contribute to the knowledge on the field. But several steps must be taken until this could be a practical and commonly implemented technology. Further research is needed –numerical and experimental–, which must include: optimization of harvester devices for a maximum energy extraction, development of a suitable generator for the conversion of kinetic to electric energy, influence of the proper generator dynamics, influence of the synchronization phenomenon in multi-pendular systems, and several others. Nevertheless, we think that all these aspects will be dominated soon given the growing demand for clean energy.

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