DINAME 2019 - Proceedings of the XVIII International Symposium on Dynamic Problems of Mechanics,
M.A. Savi, T.G. Ritto and W.M. Bessa (editors), ABCM, Buzios, RJ, Brazil, March 10th to 15th, 2019

Intrusive Stochastic Galerkin Method Applied to Rotor Systems

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Abstract: This abstracts describes the use of the polynomial chaos expansion approach in the intrusive Stochastic Galerkin method in a rotor supported by hydrodynamic bearings. The bearings dynamic coefficients are used as uncertain parameters. The stochastic result of the proposed method has been tested through comparison with the results obtained by Monte Carlo simulations.

Keywords: Uncertainty, Stochastic Galerkin, Monte Carlo, Rotor System, Frequency Response

INTRODUCTION

Concerned about the design of rotating system, many physical parameters affect the dynamic behavior of such equipment. In particular, machines supported by hydrodynamic bearings are known to be highly influenced by these components. Nevertheless is expected that the characterization of the bearing forces and dynamics coefficients receives particular attention (Hamrock, 2004). In an attempt to investigate the effect of uncertain bearing parameters (lubricant viscosity and radial clearance) in the dynamic response of a rotor this abstract proposes a methodology to solve the system using the intrusive stochastic Garlerkin method. The stochastic Galerkin method is a non-sampling method that uses the internal product of the space of orthogonal polynomials that approximate generalized chaos. Ghanem and Spanos (1991) presented the first relevant works that used the method, nonlinear vibrations were analyzed and the results were validated through the comparison with the solution obtained by Monte Carlo simulations. The main difficulty of such approach is that it requires a new solver and previous deterministic solvers cannot be used as in more traditional sampling methodologies as Monte Carlo and Latin Hypercub. The present abstract is organized as follows: first the rotor model is present then the bearing parameters characterization. In order to use the intrusive stochastic Galerkin method it was required to model the dynamic coefficients of the bearing as stochastic variables instead of the lubricant viscosity and radial clearance because of the nonlinear relationship of these parameter with the system dynamics. Later the polynomial chaos approximation of the response is shown and the results presented and compared to a Monte Carlo simulation.

ROTOR SYSTEM

Kramer (1993) presents the equation of movement of the disk and the bearings in a rotating system constituted of flexible rotor, supported on identical flexible bearings, containing a hard disk in the center, excited by the disk unbalanced mass. This system is represented by Fig.1. The equations of motion can be written as Eq. (1). Where m is the mass of the rotor, me is the unbalanced momentum, K is the stiffness coefficient and c a proportional damping of the flexible shaft. $[K_{ij}]$ and $[C_{ij}]$ are the dynamic coefficients of the bearing. The geometric data of the rotor is shown in Tab. 1.

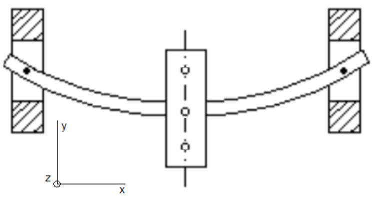

Figure 1: Rotor System

$$
\begin{aligned}
m\ddot{u}_y + c\dot{u}_y + K(u_y - y_m) &= me\omega^2 cos\omega t \\
m\ddot{u}_z + c\dot{u}_z + K(u_z - z_m) &= me\omega^2 sin\omega t \\
K(y_m - u_y) + 2(K_{xx}y_m + K_{xy}z_m + C_{xx}\dot{y}_m + C_{xy}\dot{z}_m) &= 0 \\
K(z_m - u_z) + 2(K_{xy}y_m + K_{yy}z_m + C_{xy}\dot{y}_m + C_{yy}\dot{z}_m) &= 0
\end{aligned} \tag{1}
$$

Table 1: Data of the Rotor system

	BEARING	SHAFT	DISK
Length (mm)	20	730	20
Diameter (mm)	30	10	90
Radial Clearance (mm)	$9 \cdot 10^{-2}$		
Lubricant Viscosity (Pa.s)	$7 \cdot 10^{-2}$		

Bearing Characterization

Reynolds Equation

The differential equation governing the pressure distribution in fluid film lubrication is known as Reynolds equation. Once its solution is determined one can establish all the forces acting on the bearing, its load capacity W, and its dynamic coefficients, $[K_{ij}]$ and $[C_{ij}]$.

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(h^3\frac{\partial p}{\partial z}\right)=6u_b\mu\frac{\partial h}{\partial x} \quad (2)$$

The present study employs the short bearing solution, throughly discussed by Dubois (1953), Lund (1973) and Kramer (1993) it is based on the idea that in a short bearing the pressure change in the circumferential direction is small in relation to the change in the axial direction, so it's reasonable to disregard the first term in Eq. (2). This simplification allows one to find a closed-form analytical solution for the fluid film pressure distribution, load capacity and dynamic coefficient written as nonlinear functions of the eccentricity ratio ε:

$$K_{xx}=\frac{4[\pi^2(1-\varepsilon^2)(1+2\varepsilon^2)+32\varepsilon^2(1+\varepsilon^2)]}{(1-\varepsilon^2)}f^3(\varepsilon) \quad (3)$$

$$K_{yy}=4[\pi^2(2-\varepsilon^2)+16\varepsilon^2]f^3(\varepsilon) \quad (4)$$

$$K_{xy}=\frac{\pi[\pi^2(1-\varepsilon^2)(1+2\varepsilon^2)+32\varepsilon^2(1+\varepsilon^2)]}{\varepsilon\sqrt{(1-\varepsilon^2)}}f^3(\varepsilon) \quad (5)$$

$$K_{yx}=-\frac{\pi[\pi^2(1-\varepsilon^2)(1+2\varepsilon^2)+32\varepsilon^2(1+\varepsilon^2)]}{\varepsilon\sqrt{(1-\varepsilon^2)}}f^3(\varepsilon) \quad (6)$$

$$C_{xx}=\frac{2\pi[\pi^2(1-\varepsilon^2)^2+48\varepsilon^2]}{\varepsilon\sqrt{(1-\varepsilon^2)}}f^3(\varepsilon) \quad (7)$$

$$C_{yy}=2\pi\sqrt{\frac{(1-\varepsilon^2)}{\varepsilon}}[\pi^2(1+2\varepsilon^2)-16\varepsilon^2]f^3(\varepsilon) \quad (8)$$

$$C_{xy}=C_{yx}=8[\pi^2(1+2\varepsilon^2)-16\varepsilon^2]f^3(\varepsilon) \quad (9)$$

Where $f(\varepsilon)$ is defined by:

$$f(\varepsilon)=\frac{1}{\sqrt{16\varepsilon^2+\pi^2(1-\varepsilon^2)}} \quad (10)$$

The eccentricity ratio which can be determined as a function of the load capacity of the bearing, which in turn is a function of the lubricant viscosity, μ, the radial clearance C, width of the bearing, L, and the speed, $U=\Omega R$. It can be written as:

$$W=\frac{\mu UL^3}{4C^2}\frac{\varepsilon}{(1-\varepsilon^2)^2}\sqrt{\pi^2(1-\varepsilon^2)^2+16\varepsilon^2} \quad (11)$$

Uncertainty Specification

Among the variables in the bearing system, the radial clearance and the lubricant viscosity were chosen to be described as stochastic variables defined by Gamma distributions whose mean are the deterministic value, presented in Tab. 1, with a 15% of the mean standard deviation. Monte Carlo simulation were used for the pair (μ,C) to determine the stochastic behavior of all C_{ij} and K_{ij} coefficients with relation to the uncertain description of (μ,C). This enables us to generalize the procedure to any types of bearing designs and solutions of the Reynolds Equation. Other than that the highly nonlinear relationship between (μ,C) and bearing dynamic coefficients should be described in polynomial chaos expansion, which may not be possible.

The procedure is illustrated in Fig. 2, and it's summarized:

- Probability density function for μ and C, the shape and scale parameters of the Gamma distribution are defined by the mean and standard deviation values.
- Monte Carlo simulation sampling from the pdfs of μ and C, using the short bearing solution of the Reynolds Equation
- Assuming that the coefficients C_{ij} and K_{ij} can be described by Gamma distributions in each rotational speed (same as the source stochastic variables), the maximum likelihood principle is used to define the parameters of the pdf for each dynamic coefficient.

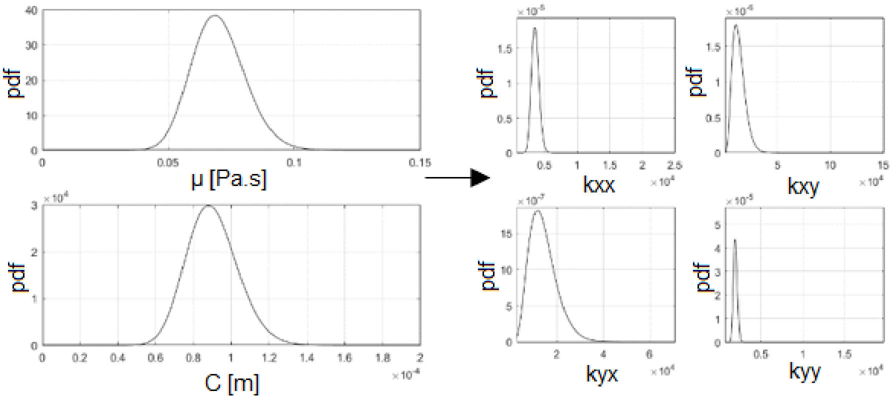

Figure 2: Monte Carlo: (C, μ) to $[K_{ij}]$

The results from the Monte Carlo simulation are then used to defined a confidence interval of one standard deviation around the mean value at each rotational speed, seen in Fig. 3. At a close analysis of Fig. 3, one can note that the direct damping (C_{ii}) and the crossed stiffness (K_{ij}) coefficients have a big dispersion which increases with the rotational speed while the direct stiffness (K_{ii}) and crossed damping (C_{ij}) coefficients present a much smaller dispersion. This is used as the basis to decide which coefficients need to be considered as a stochastic variable in the Intrusive Stochastic Galerkin Method solution of the equations of motion.

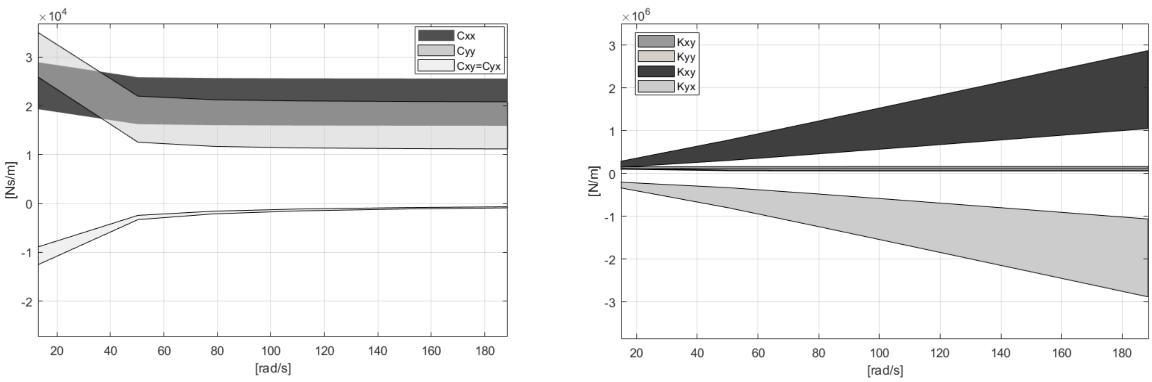

Figure 3: Interval confidence of the bearing dynamic coefficients

INTRUSIVE STOCHASTIC GALERKIN METHOD

The Wiener-Askey Polynomial Chaos

The homogeneous chaos expansion was first proposed by Wiener (1938) and it achieved optimal performance in the representation of Gaussian variables. In order to represent Gamma distributed parameters, the generalization introduced by Xiu (2002) was followed allowing the approximation of the equation of motion's solution and stochastic parameters of the rotor system:

$$y_m = \sum_{k=0}^{P} A_k(t)\Psi_k(\xi) \quad z_m = \sum_{k=0}^{P} B_k(t)\Psi_k(\xi) \quad u_y = \sum_{k=0}^{P} D_k(t)\Psi_k(\xi) \quad u_z = \sum_{k=0}^{P} E_k(t)\Psi_k(\xi) \tag{12}$$

$$K_{xy} = \sum_{k=0}^{P} M_k\Psi_k(\xi) \quad K_{yx} = \sum_{k=0}^{P} N_k\Psi_k(\xi) \quad C_{xx} = \sum_{k=0}^{P} P_k\Psi_k(\xi) \quad C_{yy} = \sum_{k=0}^{P} R_k\Psi_k(\xi) \tag{13}$$

As a result of the rapid increase in dimension (and computational cost) required by the representation of a large number of stochastic variable in this scheme (Eq. (14)), only the crossed stiffness and direct damping coefficients were chosen to be so. The other dynamic coefficients were considered as deterministic parameters. The use of the second order polynomials (d=2) with four stochastic parameters (p=4) results in a polynomial basis of order (P+1) 15.

$$P+1=\frac{(d+p)!}{d!p!} \tag{14}$$

The inner product in the Hilbert space of the variables is defined by:

$$\langle f,g\rangle=\int_D f(\theta)g(\theta)W(\theta)d\theta \tag{15}$$

Where W is the weighting function corresponding to the Wiener-Askey polynomial chaos basis. In the multidimensional case, W is represented by the joint pdf (Eq. (16)) and Eq. (15) becomes a multidimensional integral.

$$W=\prod w(\theta_i) \tag{16}$$

Since Gamma distributions were used to approximate the behavior of the bearings dynamic coefficients the Laguerre chaos polynomials are the best suitable to approximate both the solution and the stochastic variables. For notation convenience we rewrite the equation of motion as:

$$\mathscr{L}(t,X,\varepsilon)=f(t,\varepsilon) \tag{17}$$

Where X represents the whole solution:

$$\mathbf{X}=\begin{Bmatrix} u_y \\ u_z \\ y_m \\ z_m \end{Bmatrix}=\sum_{i=0}^{P}x_i(t)\Psi_i(\xi) \tag{18}$$

Substituting (18) in (17):

$$\mathscr{L}\left(t,\sum_{i=0}^{P}x_i(t)\Psi_i(\xi),\xi\right)=f(t,\xi) \tag{19}$$

A Galerkin projection of the Eq. (19) onto the polynomial basis Ψ_k, ensures the error is orthogonal to the space spanned by the basis Ψ_k:

$$\left\langle \mathscr{L}\left(t,\sum_{i=0}^{P}x_i(t)\Psi_i(\xi),\xi\right),\Psi_k\right\rangle=\langle f(t,\xi),\Psi_k\rangle \tag{20}$$

Equation (20) becomes a set of 60 (15x4) deterministic coupled differential equations for the Wiener-Askey Polynomial chaos coefficient of the solutions u_y, u_z, y_m and z_m. This procedure is detailed for the Eq. (1.4) in the next section.

General Procedure

- Substitution of the Wiener-Askey polynomial chaos expansion, Eq.(12), in the equation of motion (1.4):

$$\begin{split} K\left(\sum_{i=0}^{P}B_i(t)\Psi_i(\xi)-\sum_{i=0}^{P}E_i(t)\Psi_i(\xi)\right)+ \\ 2\left(\sum_{k=0}^{P}\sum_{i=0}^{P}N_kA_i(t)\Psi_k(\xi)\Psi_i(\xi)+K_{yy}\sum_{i=0}^{P}B_i(t)\Psi_i(\xi)+C_{yx}\sum_{i=0}^{P}\dot{A}_i(t)\Psi_i(\xi)+\sum_{k=0}^{P}\sum_{i=0}^{P}R_k\dot{B}_i(t)\Psi_k(\xi)\Psi_i(\xi)\right)=0 \end{split} \tag{21}$$

- Galerkin projection onto the polynomial basis $\{\Psi_j\}$:

$$\begin{split} K\left(B_i(t)\left\langle\Psi_i(\xi)\Psi_j(\xi)\right\rangle-E_i(t)\left\langle\Psi_i(\xi)\Psi_j(\xi)\right\rangle\right)+2\left(\sum_{k=0}^{P}\sum_{i=0}^{P}N_kA_i(t)\left\langle\Psi_k(\xi)\Psi_i(\xi)\Psi_j(\xi)\right\rangle+\right. \\ \left.+K_{yy}B_i(t)\left\langle\Psi_i(\xi)\Psi_j(\xi)\right\rangle)+C_{yx}\dot{A}_i(t)\left\langle\Psi_i(\xi)\Psi_j(\xi)\right\rangle+\sum_{k=0}^{P}\sum_{i=0}^{P}R_k\dot{B}_i(t)\left\langle\Psi_k(\xi)\Psi_i(\xi)\Psi_j(\xi)\right\rangle\right)=0 \end{split} \tag{22}$$

- Polynomials Orthogonal Properties:

$$\begin{split} K\left(B_i(t)\left\langle\Psi_i(\xi)^2\right\rangle-E_i(t)\left\langle\Psi_i(\xi)^2\right\rangle\right)+2\left(\sum_{k=0}^{P}\sum_{i=0}^{P}M_kA_i(t)e_{kij}+K_{yy}B_i(t)\left\langle\Psi_i(\xi)^2\right\rangle)+\right. \\ \left.+C_{xy}\dot{B}_i(t)\left\langle\Psi_i(\xi)^2\right\rangle+\sum_{k=0}^{P}\sum_{i=0}^{P}R_k\dot{B}_i(t)e_{kij}\right)=0 \end{split} \tag{23}$$

Where:

$$e_{kij} = \left\langle \Psi_k(\xi)\Psi_i(\xi)\Psi_j(\xi) \right\rangle, \quad [\mathbf{H}] = \left\langle \Psi_i(\xi)^2 \right\rangle$$

$$\sum_{k=0}^{P}\sum_{i=0}^{P} M_k A_i(t) e_{kij} = \sum_{k=0}^{P}\sum_{i=0}^{P} M_k A_i(t) \left\langle \Psi_k(\xi)\Psi_i(\xi)\Psi_j(\xi) \right\rangle = A_i(t)[[\mathbf{M}]_{kxy}$$
$$\sum_{k=0}^{P}\sum_{i=0}^{P} R_k \dot{B}_i(t) e_{kij} = \sum_{k=0}^{P}\sum_{i=0}^{P} R_k \dot{B}_i(t) \left\langle \Psi_k(\xi)\Psi_i(\xi)\Psi_j(\xi) \right\rangle = R_i(t)[[\mathbf{M}]_{cyy}$$

$[\mathbf{M}]_{kyx}$, $[\mathbf{M}]_{cyy}$ and $[\mathbf{H}]$ are squared matrices of order (P+1).

- Finally:

$$K\,\{\mathbf{B}\}\,[\mathbf{H}] - \{\mathbf{E}\}\,[\mathbf{H}] + 2\left(\{\mathbf{A}\}\,[\mathbf{M}]_{kyx} + K_{yy}\{\mathbf{B}\}\,[\mathbf{H}] + C_{yx}\,\{\dot{\mathbf{A}}\} + \{\dot{\mathbf{B}}\}\,[\mathbf{M}]_{cyy}\right) = 0 \qquad (24)$$

Where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$ are vectors of the PCE coefficients.

Representation of Random Inputs by PCE

In order to represent the stochastic coefficients in the form presented by Eq. (13) one needs to perform the inner product in the Hilbert space spanned by the Wiener-Askey basis $\{\Psi_i\}$:

$$\langle C_{xx}\Psi_i \rangle = \sum_{k=0}^{P} P_k \langle \Psi_k(\xi)\Psi_i \rangle \qquad (25)$$

Using the polynomials orthogonal properties, we can write for each input:

$$M_k = \frac{\left\langle K_{xy}\Psi_k \right\rangle}{\left\langle \Psi_k^2 \right\rangle} \quad N_k = \frac{\left\langle K_{yx}\Psi_k \right\rangle}{\left\langle \Psi_k^2 \right\rangle} \quad P_k = \frac{\left\langle C_{xx}\Psi_k \right\rangle}{\left\langle \Psi_k^2 \right\rangle} \quad R_k = \frac{\left\langle C_{yy}\Psi_k \right\rangle}{\left\langle \Psi_k^2 \right\rangle} \qquad (26)$$

This projection can be done by analytical and numerical approaches. In order to perform it analytically both the pdf and the inverse pdf of the random input (C_{ij} or K_{ij}) and the random variable (ξ) must be known, and they have to be transformed into the same probability space before the inner space can be calculated, this methodology is thoroughly discussed by Xiu (2002).

A more straight-forward approach to find the PCE coefficients is done numerically by Monte Carlo integration, allowing the inner product to be calculated directly as long as the pdf for the input and the random variable are known. To perform the Monte Carlo integration we need to generate a large set of uniformly distributed random number and use it as seed for both the input and the random variable, this fulfills the requirement that both comes from the same probability space. Fig. (4) shows the convergence of the PCE coefficients for the C_{xx} in 20 Hz with the number of samples used in the Monte Carlo integration, also worth noting that only the first three coefficients are non-zero.

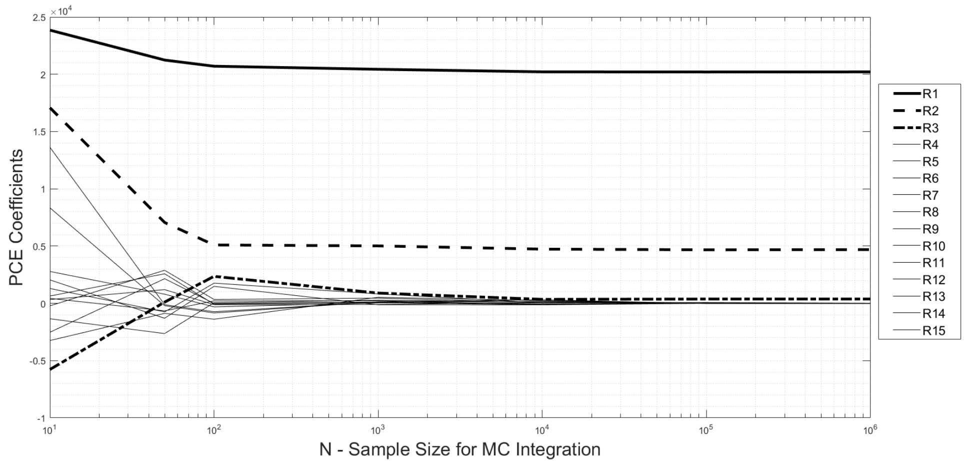

Figure 4: Convergence of PCE Coefficients

In Fig. (5) and (6) one can see how close the PCE representation is of the actual pdf of the random inputs. One can only note a small difference in the near zero region for some coefficients, which could be corrected by the use of higher order PCE, but because of the cost of adding higher orders discussed previously, this was not implemented.

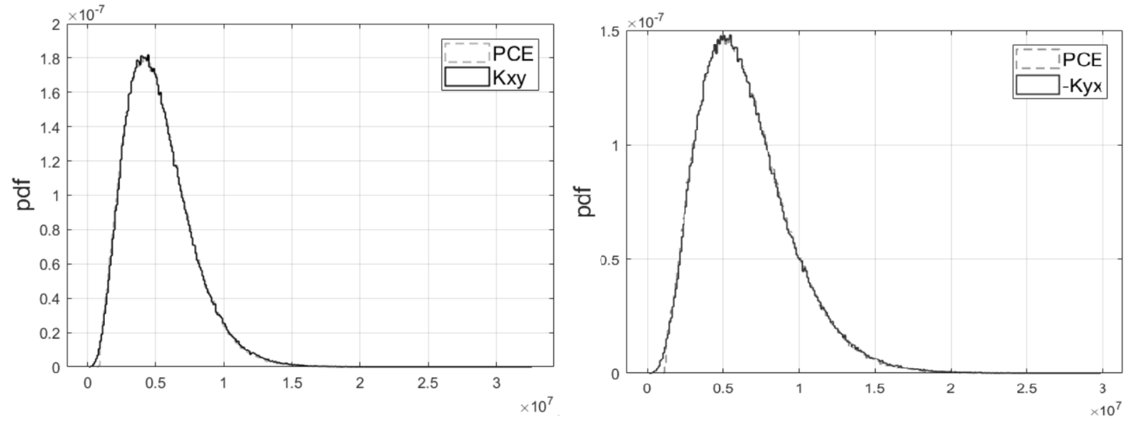

Figure 5: Pdf Representation by PCE

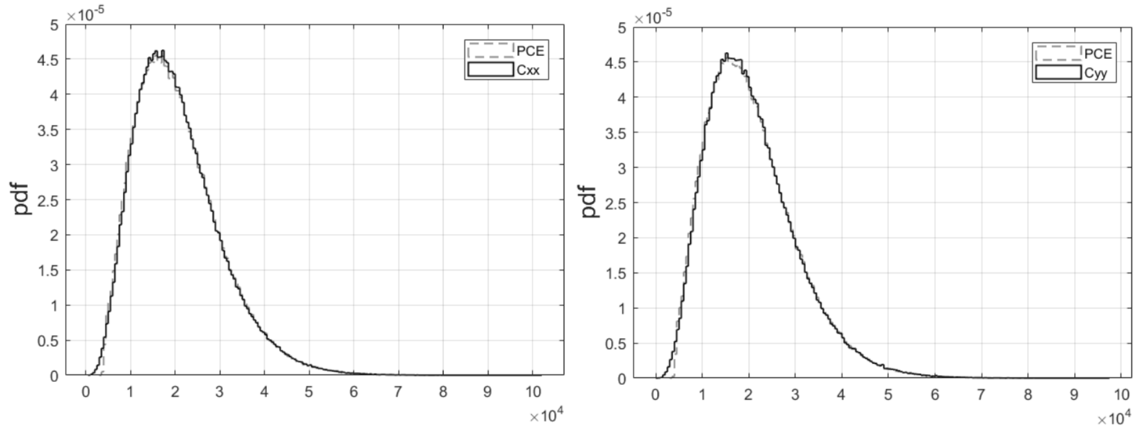

Figure 6: Pdf Representation by PCE

RESULTS

The equation of motion of the rotor system, Eq. (1), can be written:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \tag{27}$$

The Frequency Response Function (FRF) can be obtained by solving the linear system **AX=F**, where **F** is the unbalance force and **A**:

$$\mathbf{A}(\omega) = -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K} \tag{28}$$

Solving the system of each rotational velocity ω gives the FRF.

Intrusive Stochastic Galerkin Method

Using the same procedure detailed previously for the Eq. (1.4) to write all the equations of motion one finds a new system of equations that represents the stochastic response and the solution vector is now composed by the PCE coefficients:

$$\begin{bmatrix} m[\mathbf{H}] & z & z & z \\ z & m[\mathbf{H}] & z & z \\ z & z & z & z \\ z & z & z & z \end{bmatrix} \begin{Bmatrix} \ddot{D} \\ \ddot{E} \\ \ddot{A} \\ \ddot{B} \end{Bmatrix} + \begin{bmatrix} Ce[\mathbf{H}] & z & z & z \\ z & Ce[\mathbf{H}] & z & z \\ z & z & 2[\mathbf{M_{cxx}}] & 2C_{yx}[\mathbf{H}] \\ z & z & 2C_{yx}[\mathbf{H}] & 2[\mathbf{M_{cyy}}] \end{bmatrix} \begin{Bmatrix} \dot{D} \\ \dot{E} \\ \dot{A} \\ \dot{B} \end{Bmatrix} + $$
$$+ \begin{bmatrix} Ke[\mathbf{H}] & z & -Ke[\mathbf{H}] & z \\ z & Ke[\mathbf{H}] & z & -Ke[\mathbf{H}] \\ -Ke[\mathbf{H}] & z & [\mathbf{H}](Ke+2K_{xx}) & 2[\mathbf{M_{kxy}}] \\ z & -Ke[\mathbf{H}] & 2[\mathbf{M_{kyx}}] & [\mathbf{H}](Ke+2K_{yy}) \end{bmatrix} \begin{Bmatrix} D \\ E \\ A \\ B \end{Bmatrix} = \begin{Bmatrix} \langle me\omega^2\Psi_k \rangle \\ -j\langle me\omega^2\Psi_k \rangle \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{29}$$

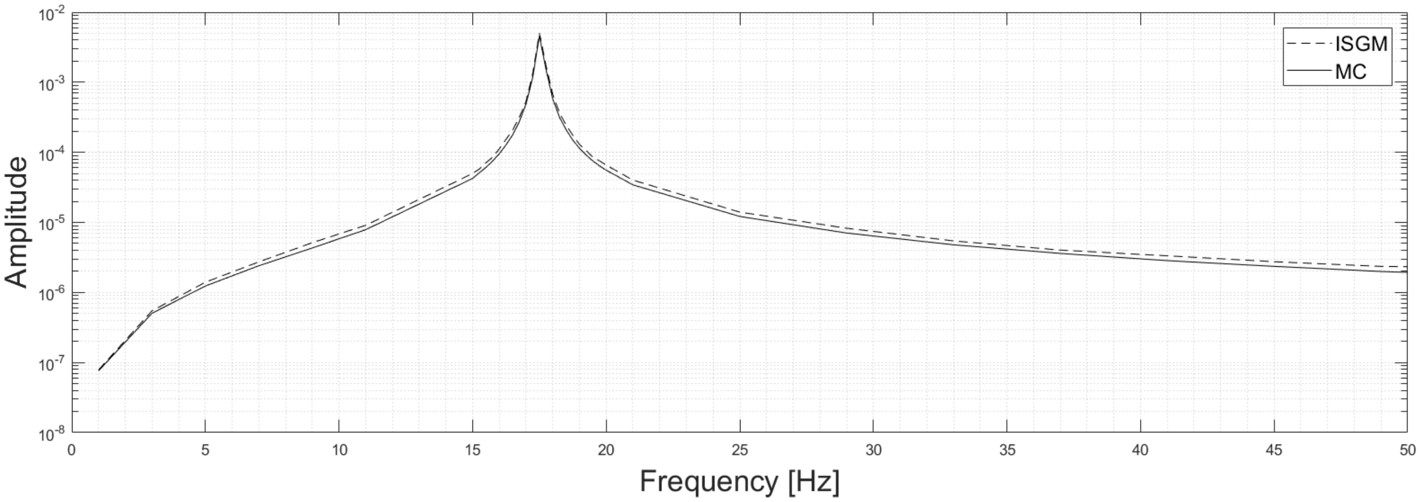

Figure 7: Monte Carlo vs Intrusive Stochastic Galerkin: Mean values FRF

Where z is a zero matrix of order (P+1) and Eq. (29) is used to calculate the FRF of the stochastic response of the system.

Monte Carlo Simulation

A Monte Carlo simulation was also implemented where a big number of realizations of the random input (the dynamic coefficients C_{xx}, C_{yy}, K_{xy} and K_{yx}) were generated based on their adjusted pdfs. For each realization the inputs are fixed and the problem becomes deterministic. From solving the system for each realization a statistical information is extracted.

Conclusion

In this study, the intrusive stochastic Galerkin method was implemented in order to evaluate the PCE coefficients of the stochastic response of a simple rotor. The results show that the Polynomial Chaos Expansion is effective to evaluate the propagation of uncertainties in this type of system. Albeit the need of changing the mathematical representation of the system, making it impossible to use directly previously implemented codes, the ISGM still offers benefits over traditional sampling methods.

The main advantage is the lower computing process required when compared to sampling methods, even though the ISGM increases the order of the system (from a set of 4 ODEs to a set of 60 ODEs in this study) it requires only one solution and (as long as the parameters correctly represents the uncertainties of the system and the correct polynomials are used) there's always convergence to the stochastic response.

Limitations of this approach includes the representation of the uncertain parameters as linear differential equation coefficients. That's the main reason to change from bearing clearance and viscosity to dynamic bearing coefficients as random variables, depending on the system this may not be possible. Another difficulty is the fact that the inner product from Eq. (15) becomes a multidimensional integral that can become difficult to compute if many stochastic parameters (p from Eq. (14)) are required.

The mean values of the FRF for the disk center can be seen in Fig. (7) in semi-log scale and its confidence interval with 99.7% ($\mu - 3\sigma < X < \mu - 3\sigma$) is shown in Fig. (8) in linear scale. The difference between both methods' is less than 5% in the mean value of the FRF and 20% in the standard deviation maximum values: $\sigma_{max} = 2.93 \cdot 10^{-4}$ for ISGM and $\sigma_{max} = 3.64 \cdot 10^{-4}$ for MC. This shows that the intrusive stochastic Galerkin method results in a similar level of accuracy of Monte Carlo simulation.

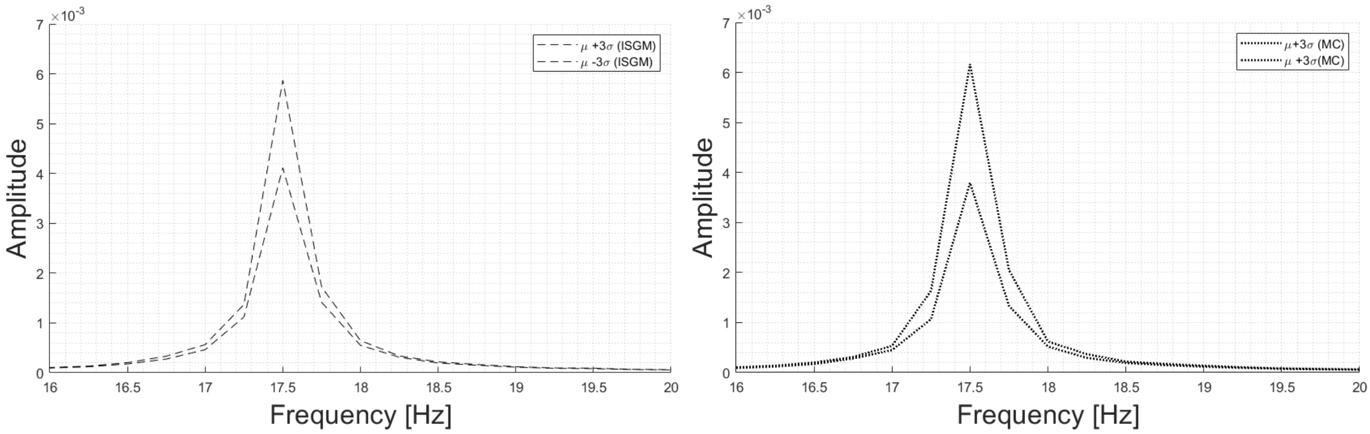

Figure 8: Monte Carlo vs Intrusive Stochastic Galerkin: Confidence Interval FRF

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