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Some insights into the mode shapes' structure of MIKOTA's vibration chain

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Abstract: This contribution deals with the mode shapes of MIKOTA*'s vibration chain, a special linear multi-body oscillator showing interesting properties. Based on a proof which allows for determining all mode shapes of this linear oscillator in a successive manner, some exemplary mode shapes are discussed. The results are insights into the evolution of the i-th mode shape for an increasing number of degrees of freedom and the evolution of the mode shape for increasing index i but constant number of degrees of freedom.*

Keywords:* MIKOTA*'s vibration chain, mode shape, multi-body dynamics

INTRODUCTION

Analytical solutions of engineering problems help in understanding the underlying mechanisms of the respective mechanical behaviour. In this contribution, a rather special linear vibration chain is looked at. In detail, the mode shapes of MIKOTA's vibration chain are analyzed and discussed. MIKOTA's vibration chain is a multi-body oscillator as sketched in Fig. 1 possessing arbitrary n degrees of freedom (dof) with stiffnesses and masses

$$m_i = \frac{1}{i} m \ , \ k_i = (n-i+1)\,k \quad , \quad i = 1, 2, \ldots n \quad , \tag{1}$$

respectively. This special configuration goes back to MIKOTA (2001), who also conjectured that the resulting eigenfrequencies are

$$\Omega_l = l\Omega = l\sqrt{k/m} \quad , \quad l = 1, 2, \ldots n \quad . \tag{2}$$

However, MIKOTA did not publish a proof for his conjecture. For the difficulties arising in such a proof it is referred to

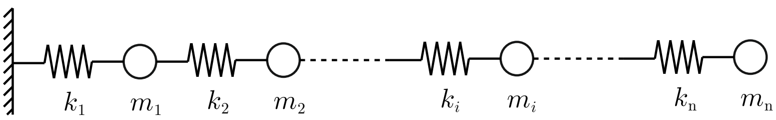


Figure 1 – Undamped vibration chain.

(Müller and Gürgöze, 2006). Meanwhile, several contributions appeared proving the equidistant nature of the eigenfrequencies, see e. g. (Müller and Hou, 2007), (Weber, Anders, and Müller, 2015). The mode shapes of MIKOTA's vibration chain show interesting properties, too. Due to the fact that the first formulæ for determining the coordinates of the mode shapes led to quite complicated expressions, see e. g. (Müller and Hou, 2007) or (Weber, Anders, and Zastrau, 2013), a deeper investigation of the mode shapes just started recently after analytical closed form solutions for the mode shapes could be provided in (Weber, Müller, and Anders, 2018). In this contribution, more insights into the structure of MIKOTA's vibration chain are presented.

STRUCTURE OF THE MODE SHAPES

The mathematical model of the considered vibration chain is given by

$$m\mathbf{M}\ddot{\mathbf{x}}(t) + k\mathbf{K}\mathbf{x}(t) = \mathbf{0} \quad , \tag{3}$$

where

$$m\mathbf{M} = \operatorname{diag}(m_i) = m\operatorname{diag}\left(\frac{1}{i}\right) \quad , \tag{4}$$

$$\begin{aligned} k\mathbf{K} &= \operatorname{diag}\{(k_1+k_2, -k_2), \dots, (-k_i, k_i+k_{i+1}, -k_{i+1}), \dots, (-k_n, k_n)\} \\ &= k\operatorname{diag}\{(2n-1, -n+1), \dots, (-n+i-1, 2n-2i+1, -n+i), \dots, (-1, 1)\} \end{aligned} \tag{5}$$

is the diagonal mass matrix and the tri-diagonal stiffness matrix, respectively.

The corresponding eigenvalue/eigenvector problem is described by

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right)\mathbf{u}_i = \mathbf{0} \quad , \quad \omega_i^2 = \frac{m}{k}\Omega_i^2 \quad , \quad i = 1, \dots, n \quad . \tag{6}$$

According to (Weber, Müller, and Anders, 2018), the solution of (6) is calculated by an iterative algorithm:

$$\omega_i = i \quad , \quad \mathbf{u}_i = \left[\mathbf{I}_n - \mathbf{U}_{i-1}\boldsymbol{\mu}_{i-1}^{-1}\mathbf{U}_{i-1}^T\mathbf{M}\right]\mathbf{y}_i \tag{7}$$

where

$$\mathbf{U}_{i-1} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{i-1}\end{bmatrix} \quad , \quad i = 2, \dots, n+1$$

is a shortened modal matrix of dimension $n \times (i-1)$,

$$\boldsymbol{\mu}_{i-1} = \mathbf{U}_{i-1}^T\mathbf{M}\mathbf{U}_{i-1} = \operatorname{diag}(\mu_1, \dots \mu_{i-1}) \tag{8}$$

represents the generalized orthogonality matrix of order $(i-1)$ with respect to the mass matrix $\mathbf{M}$. Finally, $\mathbf{y}_i$ is a "power" vector

$$\mathbf{y}_i = \left[1^i, 2^i, 3^i, \dots, n^i\right]^T \quad . \tag{9}$$

The proof is sketched shortly. Assuming

$$\omega_{i+1} = i+1 \quad , \quad \mathbf{u}_{i+1} = \mathbf{y}_{i+1} + \mathbf{U}_i\mathbf{s}_i \tag{10}$$

with some unknowns $\mathbf{s}_i$, then the eigenvalue problem (6) reads for the index $i+1$

$$\left(\mathbf{K} - (i+1)^2\mathbf{M}\right)(\mathbf{y}_{i+1} + \mathbf{U}_i\mathbf{s}_i) = \mathbf{0} \quad . \tag{11}$$

Multiplying eq. (11) from the left with $\mathbf{U}_i^T$ and taking $\mathbf{U}_i^T\mathbf{K} = \boldsymbol{\Omega}_i^2\mathbf{U}_i^T\mathbf{M}$ into account where

$$\boldsymbol{\Omega}_i^2 = \operatorname{diag}\left(\omega_1^2, \omega_2^2, \dots, \omega_i^2\right) = \operatorname{diag}\left(1, 4, \dots, i^2\right) \tag{12}$$

then eq. (11) leads to

$$\left(\boldsymbol{\Omega}_i^2 - (i+1)^2\mathbf{I}_i\right)\left(\mathbf{U}_i^T\mathbf{M}\mathbf{y}_{i+1} + \boldsymbol{\mu}_i\mathbf{s}_i\right) = \mathbf{0} \quad . \tag{13}$$

The matrix expression within the left brackets is non-singular such that

$$\mathbf{s}_i = -\boldsymbol{\mu}_i^{-1}\mathbf{U}_i^T\mathbf{M}\mathbf{y}_{i+1} \tag{14}$$

holds. Inserting this result into eq. (10) confirms eq. (7) for the index $i+1$. This completes the proof.

By this, we have with $\mathbf{u}_i = [u_i(1), u_i(2), \dots, u_i(n)]^T$ and $k = 1, \dots, n$

$$i=1: \quad \omega_1 = 1, \quad u_1(k) = k \quad , \tag{15}$$

$$i=2: \quad \omega_2 = 2, \quad u_2(k) = k^2 - \frac{2n+1}{3}k \quad , \tag{16}$$

$$i=3: \quad \omega_3 = 3, \quad u_3(k) = k^3 - \frac{3}{5}(2n+1)k^2 + \frac{1}{5}\left[\frac{3}{2}n(n+1)+1\right]k \quad , \tag{17}$$

and

$$i=4: \quad \omega_4 = 4, \quad u_4(k) = k^4 - \frac{6}{7}(2n+1)k^3 + \frac{1}{7}[6n(n+1)+5]k^2 - \frac{2}{35}(2n+1)[n(n+1)+3]k \tag{18}$$

etc. Herein, use was made of the finding of Kochendörffer (1963) that the eigenvectors of a matrix – and thus the mode shapes of a vibration chain – can be expressed in terms of polynomials.

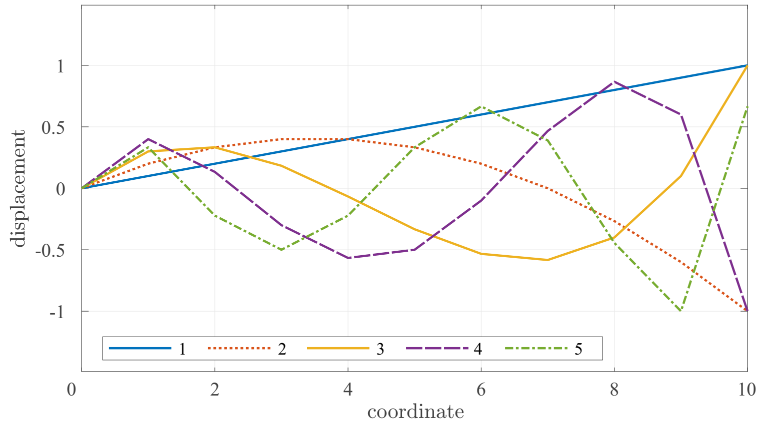


Figure 2 – Eigensolutions of MIKOTA's vibration chain for $n = 10$, where only the mode shapes $\mathbf{u}_1, \ldots, \mathbf{u}_5$ are shown. The displacements between the coordinates i are interpolated linearly.

The mode shapes of system (3) may be represented graphically in different ways. One possibility is a linear interpolation of the displacements $u_i(k-1)$, $u_i(k)$ in the coordinate interval $(k-1, k)$ with $k = 1, \ldots, n$. Herein, $u_i(0) = 0$ is defined which means that MIKOTA's vibration chain is fixed at $k = 0$, see also Fig. 1.

In general, for a conservative system (3) the mode shapes $\mathbf{u}_i$ fulfill the relations

$$\mathbf{u}_l^T \mathbf{M} \mathbf{u}_j \begin{cases} = 0 & \text{for } l \neq j \\ > 0 & \text{for } l = j \end{cases} , \quad \mathbf{u}_l^T \mathbf{K} \mathbf{u}_j \begin{cases} = 0 & \text{for } l \neq j \\ > 0 & \text{for } l = j \end{cases} . \tag{19}$$

This means that the mode shapes form an orthogonal vector system with respect to $\mathbf{M}$ and $\mathbf{K}$, respectively. However, the mode shapes of MIKOTA's vibration chain additionally show classical orthogonality properties

$$\mathbf{u}_k^T \mathbf{u}_{l-i} = 0 \quad \text{for} \quad k > l > i \tag{20}$$

such that $\mathbf{U}^T \mathbf{U}$ is tri-diagonal, cf. (Weber, Müller, and Anders, 2018).

Grammel (1943) proved that the mode shapes $\mathbf{u}_i$ represented by line segments show i nodes and i antinodes (loops) including the behaviour at the boundaries. That is, a node at the fixed beginning of the vibration chain and an antinode at the last coordinate $u_i(n)$. The same result is found by Gantmacher and Krein (1960) by using a different approach of so-called oscillation matrices. In addition, it has been shown there that the nodes of two successive mode shapes separate mutually.

In addition to the representation of the mode shapes by line segments a second representation is possible based on polynomials $P_i(x)$ with $0 \leq x \leq n$, where

$$P_i(x = k) = u_i(k) \tag{21}$$

holds and x is a continuous variable interpolating the discrete values k. Substituting k by x in eqs. (15)-(18) the polynomials

$$P_1(x) = x \quad , \tag{22}$$

$$P_2(x) = x^2 - \frac{2n+1}{3} x \quad , \tag{23}$$

$$P_3(x) = x^3 - \frac{3}{5}(2n+1)x^2 + \frac{1}{5}\left[\frac{3}{2}n(n+1) + 1\right] x \quad , \tag{24}$$

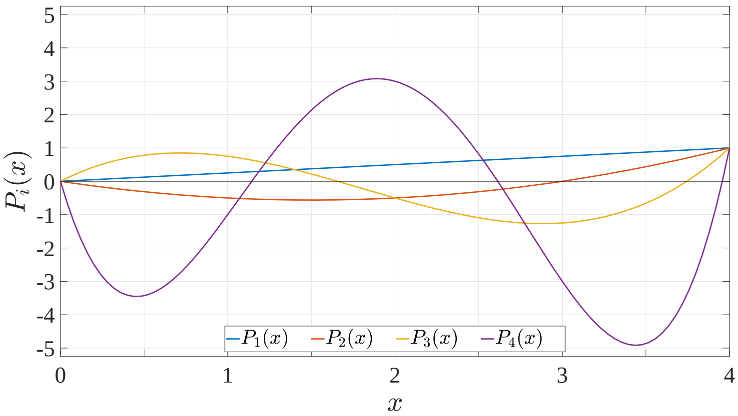
and

$$P_4(x) = x^4 - \frac{6}{7}(2n+1)x^3 + \frac{1}{7}[6n(n+1)+5]x^2 - \frac{2}{35}(2n+1)[n(n+1)+3]x \tag{25}$$

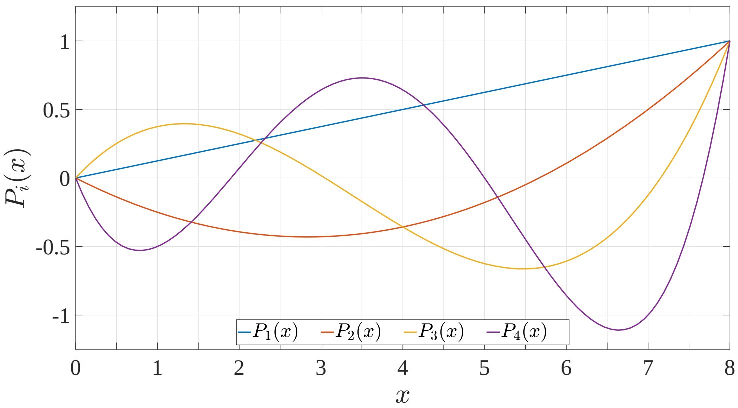
are obtained. For $n = 4$ and $n = 8$ the first four mode shapes $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ are given in Fig. 3.

While the eigenvectors $\mathbf{u}_i$ with $i = 1, \ldots, n$ show general orthogonal behaviour (19) or classical orthogonality (20), the polynomials $P_i(x)$, $i = 1, \ldots, n$ are not orthogonal in the classical sense of

$$\int_{x=0}^{n} q(x) P_l(x) P_k(x) \mathrm{d}x \quad \begin{cases} = 0 & \text{for } l \neq k \\ > 0 & \text{for } l = k \end{cases} \tag{26}$$



(a) $n = 4$



(b) $n = 8$

Figure 3 – Mode shapes no. $i = 1 \ldots 4$ represented by $P_i(x)$ for $n = 4$ and $n = 8$; the mode shapes are normalized such that $P_i(n) = 1$.

with a weighting function $q(x) > 0$, $0 \le x \le n$. This means that the $P_l(x)$ do not belong to a classical system of orthogonal polynomials, for example LEGENDRE- or LAGUERRE polynomials, see also (Müller and Weber, 2018).

Because of (21) the polynomials $P_i(x)$ also show i nodes and i antinodes (loops) such as the piecewise linear mode shape representation. Thus, the results of Grammel (1943) and Gantmacher and Krein (1960) are still valid for $P_i(x)$, $i = 1, \ldots, n$. Additionally it has been proven that if $P_i(x)$ show a node at $\bar{x}, P_i(\bar{x}) = 0$, then the neighbouring polynomials $P_{i-1}(x)$ and $P_{i+1}(x)$ have opposite signs at $\bar{x}$:

$$P_{i-1}(\bar{x})P_{i+1}(\bar{x}) < 0 \quad \text{for} \quad P_i(\bar{x}) = 0 \quad , \tag{27}$$

see exemplary Fig. 3.

A comparison of the two types of representation of the mode shapes is shown in Fig. 4 for $n = 4$ and $n = 8$. It can be seen that for increasing n the linear interpolation better approximates the polynomial interpolation.

ADDENDUM

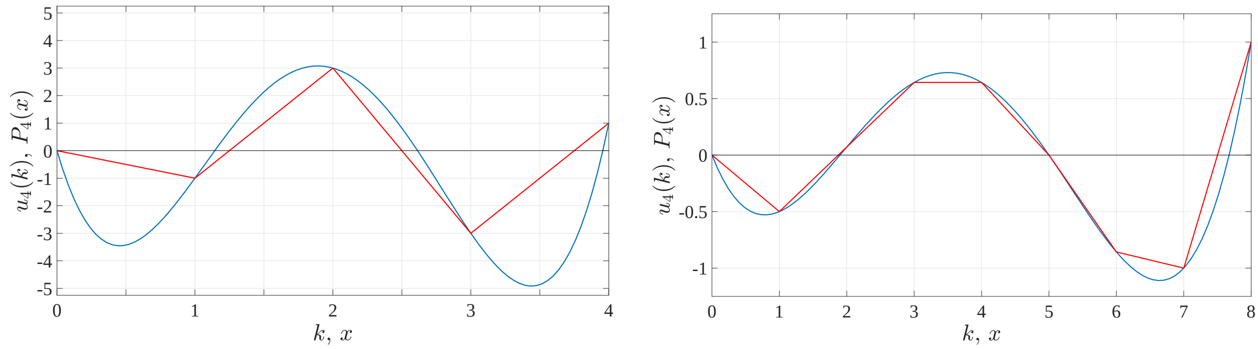
In (Müller and Hou, 2007) a relation has been shown between MIKOTA's vibration chain and an unusual problem of matrix calculus:

$$\mathbf{M}^{-1}\mathbf{K} = \left(\mathbf{M}^{-1}\mathbf{P}\mathbf{D}\right)^2 = \mathbf{A}^2 \tag{28}$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} \quad , \quad \mathbf{P} = \begin{bmatrix} & & & & 1 \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \end{bmatrix} \quad , \tag{29}$$

$$\mathbf{D}^T\mathbf{P} = \mathbf{P}\mathbf{D} \quad , \tag{30}$$



(a) $n = 4$ (b) $n = 8$

Figure 4 – Mode shape no. 4 represented by $\mathbf{u}_4(i)$ and $P_4(x)$ for $n = 4$ and $n = 8$; the mode shapes are normalized such that $u_4(n) = P_4(n) = 1$. Taken from (Müller and Weber, 2018).

$$\mathbf{A} = \begin{bmatrix} & & & -1 & 1 \\ & & -2 & 2 & \\ & \iddots & \iddots & & \\ -(n-1) & n-1 & & & \\ n & & & & \end{bmatrix} \quad . \tag{31}$$

Therefore, the eigenvalue problem of the cross-bidiagonal matrix $\mathbf{A}$ is solved by the solution of MIKOTA's problem:

$$(\lambda_i \mathbf{I}_n - \mathbf{A})\,\mathbf{u}_i = \mathbf{0} \quad \text{with} \quad \lambda_i = (-1)^{i-1} i \quad , \quad i = 1, \ldots, n \quad . \tag{32}$$

Instead of the standard eigenvalue problem (32) a general symmetric eigenvalue problem can be considered having $\mathbf{A} = \mathbf{M}^{-1}\mathbf{PD}$ in mind:

$$(\lambda_i \mathbf{M} - \mathbf{PD})\,\mathbf{u}_i = \mathbf{0} \tag{33}$$

where

$$\mathbf{PD} = \begin{bmatrix} & & & -1 & 1 \\ & & -1 & 1 & \\ & \iddots & \iddots & & \\ -1 & 1 & & & \\ 1 & & & & \end{bmatrix} \quad . \tag{34}$$

From eq. (33) it follows

$$\mathbf{u}_k^T \mathbf{PD} \mathbf{u}_i = \lambda_i \mathbf{u}_k^T \mathbf{M} \mathbf{u}_i = \begin{cases} 0 & \text{for } k \neq i \\ \lambda_i \mu_i & \text{for } k = i \end{cases} \quad , \tag{35}$$

i. e.

$$\mathbf{U}_n^T \mathbf{PD} \mathbf{U}_n = \mathbf{\Lambda}_n \boldsymbol{\mu}_n, \quad , \quad \mathbf{\Lambda}_n = \operatorname{diag}(\lambda_i) \tag{36}$$

holds. Therefore, the eigenvectors $\mathbf{u}_i$ with $i = 1, \ldots, n$ of the MIKOTA problem form an orthogonal vector system with respect to $\mathbf{PD}$. Nevertheless, the $\mathbf{u}_i$ are not eigenvectors of $\mathbf{PD}$ but of (33).

If one is interested in the eigenvectors of $\mathbf{PD}$, then the procedure of eq. (28) to pass over from $\mathbf{M}^{-1}\mathbf{K}$ to $\mathbf{A}$ has to be realized in reverse order: To look for a vibration chain with $\bar{\mathbf{M}} = \mathbf{I}_n$ and $\bar{\mathbf{K}} = (\mathbf{PD})^2 = \mathbf{D}^T\mathbf{D}$. This leads to a well-known oscillator chain which has been investigated by Grammel (1943) and Klotter (1960). It is the so-called "chain B" with

$$\bar{\mathbf{K}} = (\mathbf{PD})^2 = \mathbf{D}^T\mathbf{D} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & & \\ & & -1 & 2 & 1 \\ & & & -1 & 1 \end{bmatrix} \quad . \tag{37}$$

The eigenvalue problem

$$\left(\bar{\mathbf{K}} - \omega_i^2 \mathbf{I}_n\right) \mathbf{w}_i = \mathbf{0} \tag{38}$$

has the solution

$$\mathbf{w}_i = [w_i(k)] \quad , \quad w_i(k) = \sin(k\varphi_i) \quad , \quad k = 1, \ldots, n \quad , \tag{39}$$

$$\omega_i^2 = 2(1-\cos\varphi_i) \quad , \quad \varphi_i = \frac{2i-1}{2n+1}\pi \quad , \quad i = 1, \ldots, n \quad . \tag{40}$$

Therefore, the eigenvalue problem of $\mathbf{PD}$ is solved by the same eigenvectors $\mathbf{w_i}$ and the eigenvalues

$$\lambda_i(\mathbf{PD}) = (-1)^{i-1}\, 2\sin\frac{\varphi_i}{2} \quad , \quad i = 1, \ldots, n \quad . \tag{41}$$

The orthogonality relations are as usual:

$$\mathbf{w}_k^T \mathbf{w}_i \begin{cases} = 0 & \text{for } k \neq i \\ > 0 & \text{for } k = i \end{cases} \quad , \quad \mathbf{w}_k^T \bar{\mathbf{K}} \mathbf{w}_i = \begin{cases} 0 & \text{for } k \neq i \\ \omega_i^2 \mathbf{w}_i^T \mathbf{w}_i & \text{for } k = i \end{cases} \quad . \tag{42}$$

SUMMARY AND RESULTS

MIKOTA's vibration chain is a multi-body oscillator possessing interesting properties. Due to these properties, the rather special MIKOTA's vibration chain can serve as a reference case and application example for several techniques which allow the solution of practical problems. This vibration chain has been investigated completely confirming the eigenfrequencies $\Omega_l = l\Omega$ with the first eigenfrequency $\Omega = \sqrt{k/m}$ and determining the eigenvectors $\mathbf{u}_l$ successively by a recursion formula. The mode shapes have been illustrated by means of both line segments from $(i, u_l(i))$ to $(i+1, u_l(i+1))$ and by polynomials $P_l(x)$, where the relation $u_l(i) = P_l(x = i)$ holds. Both approaches show l nodes and l antinodes (loops) for the l-th mode shape. Although the polynomials do not form a system of classical orthogonal polynomials, it seems that there exist similar properties. However, this conjecture has to be proven in future work.

In addition, a relation between mechanical vibration chains and unusual cross-bidiagonal matrices has been used to solve their eigenvalue problems.

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