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Asymptotic solutions for a Carreau-Yasuda film flow driven by gravity over an inclined plane.

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Abstract. *The aim of this work is to study the base flow and the temporal stability of a liquid layer driven by gravity where the fluid rheology respects the Carreau-Yasuda model. We present a review of some important works about the stability of fluid flows driven by gravity, especially those which deals with non-Newtonian fluids. In addition, we present a brief explanation of the Carreau-Yasuda model, which gives a better approximation about the rheological behavior of a generalized Newtonian fluid in the presence of a free surface. Also, we present the most important equations that are necessary in order to study base state and the temporal stability of the physical problem. In order to solve the stability problem, an asymptotic expansion with long-wave approximation was developed focusing on the zero-order solution as a preliminary result. Finally, we show the solutions obtained with the asymptotic approach for the reference flow, film thickness, and the stability problem.*

Keywords: *Gravity flow, generalized Newtonian fluid, Carreau-Yasuda model, temporal stability, asymptotic method*

1. Introduction

The study of fluid flows driven by gravity has been the object of interest of many authors through the last decades, not only because of practical engineering applications, such as friction reducing effects, reactors colling process or coating, but also because it is present in many geophysical phenomena as mud, glaciers, and lava flow. For Newtonian fluids, this class of problems has been extensively studied in the literature over the years. However, the rheological behavior of a wide class of fluids cannot be properly described by the Newtonian constitutive equation. Some authors focused on the study the stability of flows driven by gravity for fluids that are described by non-Newtonian models.

In the study of Ng and Mei (1994) it was considered the physical problem of roll waves on a shallow layer of mud, using the constitutive equation of a power-law fluid in order to describe approximately the behavior of a real non-Newtonian fluid. The authors used an integral boundary layer formulation (Chang and Demekhin, 2002). A scale separation is assumed in the cross-stream and streamwise directions, the deformation of the free surface being on a length scale that is considerably larger than the film thickness, which makes possible the introduction of a small parameter for the film. The momentum and continuity equations are integrated in depth, in this way it is possible to form a closed system of equations for the film thickness h and the local flow rate q once a closure hypothesis is made. The linear stability analysis showed that for a uniform flow the growth rate of the unstable perturbations increases monotonically with the wavenumber. This result leads the authors to the conclusion that there is no evidence in this case for a preferred wavelength for the roll waves. Using a long-wave approximation in order to find a solution the authors demonstrated that the critical Reynolds number is a function of the slope angle θ and the exponent n present in the constitutive equation, which represents the "behavior" of the fluid. If $n < 1$ the fluid will present a pseudoplastic behavior, for $n > 1$ a dilatant behavior and if $n = 1$ we have a Newtonian constitutive equation. As the last result, the authors found that, according to the linear theory, for a fluid flow where $n > \frac{1}{\sqrt{2}}$ there is no roll wave solution if the corresponding flow is stable and for $n < \frac{1}{\sqrt{2}}$ the existence of long waves with large amplitude is possible.

The work of Ruyer-Quil and Dandapat (2012) presents an attempt to modeling a power-law falling film flow over an inclined plane including second-order viscous diffusion terms. To account for streamwise viscous diffusion the authors computed at the free surface the effective viscosity, present in the power-law constitutive equation, and its derivative where, for an unperturbed interface, the strain rate goes to zero. To deal with the divergence of the viscosity the authors introduced a Newtonian plateau at small strain rate and a bound to the effective viscosity dividing the flow into a Newtonian layer at the free surface and a non-Newtonian bulk separated by a fake interface. They used a two-equation model within the framework of lubrication theory in the sense of the averaged momentum equation and the mass balance. It was showed good agreement between the Orr-Sommerfeld stability analysis and the direct numerical simulation (DNS) for both linear and nonlinear regimes. They compared the Orr-Sommerfeld analysis with and without surface tension for the spatial growth rate and the marginal stability curves for different values of the exponent n present in the power-law constitutive equation. It was noticed that the difference between the results are hardly noticeable for $n = 2$ or $n = 3$, but extremely important for $n = 1$. For the first two cases, the cut-off wavenumber is determined by the balance of inertia and viscous damping.

The study of Noble and Vila (2013) deals with the physical problem of a thin liquid layer over an inclined plane. Considering an incompressible fluid that respects the power-law constitutive equation. The authors considered the two-dimensional case of the problem under the consideration that most of the free-surface instabilities like roll waves are primarily two-dimensional. It was deduced lubrication models in order to study the stability of the constant states of the full Cauchy momentum equation and then, derive a second order shallow-water model for the fluid. To deal with the divergence of the apparent viscosity at the free surface, which introduces a singularity in the Orr-Sommerfeld formulation and in the derivation of the shallow-water models, the authors introduced a weaker formulation of the Cauchy momentum equations. This procedure made it possible to develop a linear problem that generalizes the Orr-Sommerfeld equations for Newtonian fluids, showing that the nonlinear expansions of the velocity field, in the shallow-water scaling, gives a natural expansion of the eigenfunctions and eigenvalues of the generalized Orr-Sommerfeld equations.

A multilayered flow down an inclined plane with the presence of a free surface, modeled by a Carreau constitutive equation, was studied by Weinstein (1990). The author focused on the effect of the shear-thinning on the spatial growth rate of the waves in the system. An asymptotic solution, for small shear thinning effects, was used in order to check the numerical calculations, which were performed through an iterative method. A more general solution will be presented further in this work and compared with his case for validation purposes. In his results, he found that surface mode waves propagate as in a Newtonian system in the presence of shear-thinning effects. On the other hand, the waves that appear at the interfaces between two adjacent fluids are strongly affected by the local viscosity, and the propagation is not governed by an average Newtonian viscosity of the layer, unlike the surface mode waves. It was showed that, for a two-layer system, the shear-thinning can either decrease or increase the interface growth. This effect strongly depends on whether the jump in the rate of strain in the surface is decreased or increased due to the shear-thinning.

In the work of Pinarbasi and Liakopoulos (1995) it was presented the linear stability analysis of a flow, driven by a pressure gradient in a straight channel, constituted by two fluids with different rheological models. One is a Bingham-like fluid and the other respect the Carreau-Yasuda model. Using a temporal stability analysis they reach the governing equations for the stability problem. In order to solve this new system it was implemented a pseudospectral method based on Chebyshev polynomials expansions, and the resulting generalized eigenvalue problem it was solved using the QZ algorithm. The authors showed that, for a two-layer Newtonian fluids flow, replacing the bottom fluid with a viscoplastic fluid has a stabilizing effect on the interface between both fluids for intermediate and large wavenumbers. For two viscoplastic fluids, increasing the yield stress or the stress growth exponent it is also obtained a stabilizing effect at the interface between both. For the case of two shear thinning fluids, the authors showed that increasing the zero-shear-rate viscosity ratio or the shear thinning has a destabilizing effect in the flow. Variations in the Yasuda constant a and in the onset of the shear thinning can stabilize or destabilize the flow depending upon the layer thickness ratio and the wavelength.

The temporal stability of a Carreau fluid flowing down an inclined plane was considered by Rousset *et al.* (2007). The Carreau model is suitable for pure viscous fluids whose viscosity changes under an increment of the rate of deformation. With this model, it is possible to predict a power-law region and a linear relation between shear rate and viscosity. However, one of the advantages of this model compared to the power-law or Ostwald-De Waele model is that it predicts a viscosity that still remains finite while the shear rate approaches to zero, which makes it suitable for free surface flows. The authors used a standard linear stability approach for the physical problem. Two methods were used in order to solve the Orr-Sommerfeld equation and the boundary conditions. The first one was an asymptotic method, studying the stability analytically expanding the eigenvalue and the eigenvector with respect to the wavenumber, considered much larger than the depth of the liquid film. They carried the solutions until the first-order. For shear-thinning ($n < 1$) the critical Reynolds number is less than for the Newtonian case and still remains proportional to $cot\beta$, but the phase speed is larger. The second method was a numerical one, specifically a spectral collocation method based on Chebyshev polynomials for the discretization, using Gauss-Lobatto collocation points. The authors compared both solutions asymptotic and numerical for shear-thinning and the Newtonian case, and found excellent accuracy for both cases. The study showed that for a shear-thinning fluid is influenced by the viscous distribution depending on the slope angle.

The study performed by Millet *et al.* (2013) presents a linear stability temporal approach. The physical system is constituted by two shear-thinning fluids, following the four-parameter Carreau model, flowing down an inclined plane with the presence of a free surface. A spectral collocation method based on Chebyshev polynomials was used in order to solve the stability problem. For the discretization of the equations, the authors used the Gauss-Lobatto collocation points. They found that the viscosity stratification has great influence on the stability of the system. When the fluid at the bottom is less viscous, the interface instability is amplified, since it can grow without the presence of inertia. The long-wave instability at the interface is generally dominant, unless the value of the density ratio of the upper fluid is close to 1, then the short-wave instability at the interface can have larger growth rates and become dominant. Also, they found that the rheology of the fluids has great influence on the stability of the system. When the bottom layer has a less viscous fluid, the rheology of this fluid completely determines the stability of the flow, and the upper layer has almost no influence on the stability. However, when the upper fluid is less viscous, changing the rheology in this upper layer has an influence on the stability, but it is not the only influence in this case. This kind of change will affect more the surface instability, and the interface instability, for short wavelength range, whereas a change in the rheology of the bottom layer will destabilize the system in the long wavelength range. Also, the authors mentioned that they chose the focus on the Carreau model in their analysis because it captures with a good precision the features of a large number of fluids.

In this work, it was considered the two-dimensional case of the stability of a film flow driven by gravity. For Newtonian fluids it is possible to apply the result of the Squire's theorem (Squire, 1933), even for free surface flows (Yih, 1955), or stratified fluid flows (Hesla and Preziosi, 1986) where the fluids involved respect the Newtonian constitutive equation. In the work of Gupta and Rai (1968), it was used a standard linear stability analysis for the three-dimensional case of a visco-elastic fluid respecting the Rivlin-Ericksen. Once that the boundary conditions of the problem do not degenerate as the wave number α tends to zero, it is possible to solve the system using the regular perturbation procedure as done by Yih (1963). His analysis leads to the result that for under certain circumstances a Reiner-Rivlin fluid will not respect Squire's Theorem. In the study of Sahu and Matar (2010), it is analyzed the three-dimensional linear stability of a pressure-driven immiscible two-fluid channel flow. The top layer contains a Newtonian fluid and the bottom layer viscoplastic fluid respecting a Herschel-Bulkley model. They focus their analysis on the parameter range that the Squire's theorem does not exist. They solved the eigenvalue problem resulted by the combination of the Orr-Sommerfeld equation and the boundary conditions using the LAPACK open source code with a spectral collocation method (Canuto *et al.*, 2012). For the cases where the square root of the viscosity ratio is larger than the thickness ratio, the result showed the presence of three-dimensional instabilities. And for the shear mode, at high Reynolds numbers, the two-dimensional modes are more unstable than the three-dimensional ones. On the other hand, in the work of Nouar and Brancher (2007), it was considered the temporal stability of a shear-thinning fluid modeled by the Carreau law. The motivation of the authors for this study came from the possibility to delay the transition to turbulence by creating a viscosity contrast in the channel. To solve the problem it was used a Chebyshev collocation method. The main result of the authors is that the two-dimensional stabilities seem dominant compared to the three-dimensional ones. But they point out that they cannot make use of Squire theorem which is only valid for a reduced problem when terms connected with the perturbation of the viscosity are neglected.

However, there is no complete agreement in the literature about the validation of these results when the object of study is a non-Newtonian fluid, and some contradictions about the veracity of the Squire's theorem for non-Newtonian still remain in the literature.

2. Formulation of the physical problem

2.1 Governing equations

In this work, it was considered an incompressible fluid flow driven by gravity over an inclined plane. A sketch of the physical problem is represented by Fig.(1).

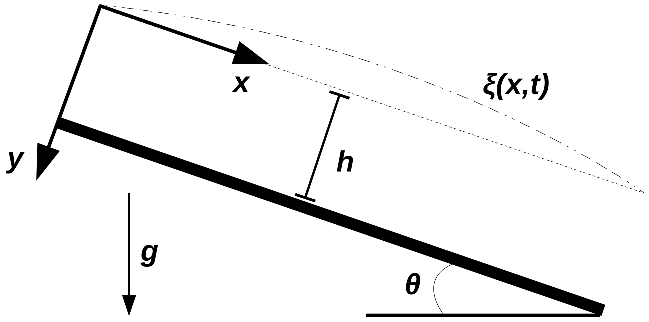

Figure 1. Thin film falling down on a inclined with planar interface and perturbed interfacial position $\xi(x,t)$.

It was used a generalized Newtonian fluid model where the shear stress $\underline{\underline{\tau}}$ is a function of the shear rate $\dot{\gamma}$. This class of fluids satisfy the following constitutive equation as explained in Morrison (2001) and Macosko (1994):

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}} \qquad (1)$$

where $\eta(\dot{\gamma})$ is a scalar function and $\dot{\gamma} = |\underline{\underline{\dot{\gamma}}}|$. Once that the problem considered is a liquid layer over an inclined plane the best choice for modeling the physical problem is a Carreau model. As mentioned before the Carreau model is suitable for problems where the shear rate tends to zero, which is exactly what happens at the free surface. The Carreau or Carreau-Yasuda model is given by,

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}} \tag{2}$$

where a predicts the shape of the transition region between the zero-shear-rate plateau and the rapidly decreasing (power-law-like) portion of the viscosity versus shear-rate curvature. λ determines the shear rate at which the transition from the zero-shear-rate plateau to the power-law portion occurs. It also describes the transition between the power-law region to $\eta = \eta_\infty$. The exponent n governs the inclination of the rapidly decreasing portion of the η curve. In this model, the viscosity tends to η_∞ as the shear rate $\dot{\gamma}$ becomes larger. When the shear rate gets smaller the viscosity tends to η_0. Apart from the discussion in the literature about the validity of the Squire's theorem for non-Newtonian fluids, we considered the two-dimensional case of the physical problem. The governing equations are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + \rho g \;\; sin(\theta) \tag{4}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) + \rho g \;\; cos(\theta) \tag{5}$$

where the stress tensor components τ_{ij} and the shear rate $\dot{\gamma}$ are,

$$(\tau_{xx}, \tau_{yy}, \tau_{xy}) = \left(2\eta(\dot{\gamma})\frac{\partial u}{\partial x} \;\;,\;\; 2\eta(\dot{\gamma})\frac{\partial v}{\partial y} \;\;,\;\; \eta(\dot{\gamma})\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]\right) \tag{6}$$

$$\dot{\gamma} = \left\{2\left(\frac{\partial u}{\partial x}\right)^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2\left(\frac{\partial v}{\partial y}\right)^2\right\}^{\frac{1}{2}} \tag{7}$$

In order to fully describe the dimensionless form of the governing equations we must consider the following dimensionless quantities:

$$(\overline{x}, \overline{y}) = \left(\frac{x}{h_s}, \frac{y}{h_s}\right) \tag{8}$$

$$(\overline{u}, \overline{v}) = \left(\frac{u}{U_s}, \frac{v}{U_s}\right) \tag{9}$$

$$(\overline{t}, \overline{p}, \overline{\eta}) = \left(\frac{tU_s}{h_s}, \frac{p}{\rho U_s^2}, \frac{\eta}{\eta_0}\right) \tag{10}$$

where ρ is the density of the material, h_s is the film thickness scale and U_s is the characteristic velocity scale given by,

$$(h_s, U_s) = \left(\left[\frac{\eta_0 U_s h_s}{\rho g \;\sin(\theta)}\right]^{\frac{1}{3}}, \frac{\rho g h^s sin(\theta)}{\eta_0}\right) \tag{11}$$

Therefore, the dimensionless governing equations are written as,

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \tag{12}$$

$$\frac{\partial \overline{u}}{\partial \overline{t}} + \overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}} = -\frac{\partial \overline{p}}{\partial \overline{x}} + \frac{1}{Re}\left(\frac{\partial \overline{\tau}_{xx}}{\partial \overline{x}} + \frac{\partial \overline{\tau}_{xy}}{\partial \overline{y}}\right) + \frac{1}{Fr_x^2} \tag{13}$$

$$\frac{\partial \overline{v}}{\partial \overline{t}} + \overline{u}\frac{\partial \overline{v}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{y}} = -\frac{\partial \overline{p}}{\partial \overline{y}} + \frac{1}{Re}\left(\frac{\partial \overline{\tau}_{xy}}{\partial \overline{x}} + \frac{\partial \overline{\tau}_{yy}}{\partial \overline{y}}\right) + \frac{1}{Fr_y^2} \tag{14}$$

where the stress tensors τ_{ij}, viscosity η, shear rate $\dot{\gamma}$ become,

$$(\overline{\tau}_{xx}, \overline{\tau}_{yy}, \overline{\tau}_{xy}) = \left(2\overline{\eta}(\dot{\overline{\gamma}})\frac{\partial \overline{u}}{\partial \overline{x}} \;\;,\;\; 2\overline{\eta}(\dot{\overline{\gamma}})\frac{\partial \overline{v}}{\partial \overline{y}} \;\;,\;\; \overline{\eta}(\dot{\overline{\gamma}})\left[\frac{\partial \overline{u}}{\partial \overline{y}} + \frac{\partial \overline{v}}{\partial \overline{x}}\right]\right) \tag{15}$$

$$\overline{\eta}(\bar{\dot{\gamma}}) = I + (1-I)[1+(L\dot{\gamma})^a]^{\frac{n-1}{a}} \tag{16}$$

$$\bar{\dot{\gamma}} = \left\{ 2\left(\frac{\partial \overline{u}}{\partial \overline{x}}\right)^2 + \left[\left(\frac{\partial \overline{u}}{\partial \overline{y}}\right)^2 + 2\frac{\partial \overline{u}}{\partial \overline{y}}\frac{\partial \overline{v}}{\partial \overline{x}} + \left(\frac{\partial \overline{v}}{\partial \overline{x}}\right)^2\right] + 2\left(\frac{\partial \overline{v}}{\partial \overline{y}}\right)^2 \right\}^{\frac{1}{2}} \tag{17}$$

The definition for the dimensionless groups of Reynolds and Froude, together with I and L, are:

$$(Re, Fr_x, Fr_y, I, L) = \left(\frac{\rho U_s h_s}{\eta_0}, \sqrt{\frac{U_s^2}{gh_s \ \ sin(\theta)}}, \sqrt{\frac{U_s^2}{gh_s \ \ cos(\theta)}}, \frac{\eta_\infty}{\eta_0}, \frac{\lambda U_s}{h_s}\right) \tag{18}$$

The authors assumed that the reference flow is parallel and steady. The base velocity profile $\overline{u} = \overline{U}$ is a function of $\overline{y}$ only, the component v of the velocity is equal to zero and the pressure gradient in the x-direction is also zero. Under these considerations, the continuity equation Eq.(12) is identically satisfied. The momentum equations Eq.(13), Eq.(14) and the viscosity Eq.(16) lead to,

$$\overline{\eta}\frac{d\overline{U}}{d\overline{y}} = -\overline{y} \tag{19}$$

$$\overline{P} = \frac{\overline{y} \ \ cot(\theta)}{Re} \tag{20}$$

$$\overline{\eta}(\overline{y}) = I + (1-I)\left[1 + \left(L\frac{d\overline{U}}{d\overline{y}}\right)^a\right]^{\frac{n-1}{a}} \tag{21}$$

The boundary conditions of no-slip at the wall and the zero shear at the free surface are respectively given by,

$$\overline{U}(\overline{y}) = 0 \ \ at \ \ \overline{y} = \overline{h} \tag{22}$$

$$\overline{\eta}(\overline{y})\frac{d\overline{U}(\overline{y})}{d\overline{y}} = 0 \ \ at \ \ \overline{y} = 0 \tag{23}$$

where $\overline{h} = h/h_s$ Replacing Eq.(21) into Eq.(19) leads to a differential equation written as,

$$\left\{I + (1-I)\left[1 + \left(L\frac{d\overline{U}}{d\overline{y}}\right)^a\right]^{\frac{n-1}{a}}\right\}\frac{d\overline{U}}{d\overline{y}} = -\overline{y} \ \ for \ \ \overline{y} \in [0;\overline{h}] \tag{24}$$

A final relation can be considered by the definition of the dimensionless flow rate $\overline{Q} = Q/U_s h_s$ in the integral form. Applying a normalization with $\overline{Q} = 1$ it is possible to write the following integral,

$$\int_0^{\overline{h}} \overline{U} d\overline{y} = 1 \tag{25}$$

Equations 22, 24 and 25 establish a nonlinear problem for the velocity profile and the film thickness. There is no analytical solution in the general case for this system.

2.2 Reference flow and film thickness

In this section it is proposed an asymptotic solution similar to Weinstein (1990) where the main difference lies in the exponent of the second term in the expansion. We are also considering a limit case for a small non-Newtonian behavior when L tends to zero. In order to proceed with the asymptotic solution for the velocity profile and the film thickness, it is necessary to assume the solution for $\overline{U}$ and $\overline{h}$ as,

$$\overline{U}(\overline{y}) = \overline{U}_0 + \overline{U}_1 L^a + \mathcal{O}(L^m) \ \ with \ \ m > a \tag{26}$$

$$\overline{h} = \overline{h}_0 + \overline{h}_1 L^a + \mathcal{O}(L^m) \ \ with \ \ m > a \tag{27}$$

Considering the first approximation $\mathcal{O}(L^0)$ it is necessary to assume $\overline{U}(\overline{y}) = \overline{U}_0$ and $\overline{h} = \overline{h}_0$. Replacing these approximations in the system constituted by the equations (22), (24) and (25), we obtain the new system written as,

$$\overline{U}_0 = 0 \ \ at \ \ \overline{y} = \overline{h}_0 \tag{28}$$

$$\left\{I + (1-I)\left[1 + \left(L\frac{d\overline{U}_0}{d\overline{y}}\right)^a\right]^{\frac{n-1}{a}}\right\}\frac{d\overline{U}_0}{d\overline{y}} = -\overline{y} \tag{29}$$

$$\int_0^{\overline{h}_0} \overline{U}_0 d\overline{y} = 1 \tag{30}$$

In order to solve the system above it is necessary to discard any term equal or above of order $\mathcal{O}(L^a)$, then integrate Eq.(29) and apply Eq.(28). The result is replaced in Eq.(30) and this procedure leads to the solution at the order $\mathcal{O}(L^0)$ given by,

$$\overline{U}_0(y) = \frac{(\overline{h}_0^2 - \overline{y}^2)}{2} \tag{31}$$

$$\overline{h}_0 = \sqrt[3]{3} \tag{32}$$

Applying the second approximation $\mathcal{O}(L^a)$ it is necessary to assume $\overline{U}(\overline{y}) = \overline{U}_0 + \overline{U}_1 L^a$ and $\overline{h} = \overline{h}_0 + \overline{h}_1 L^a$. The new system for the order $\mathcal{O}(L^a)$ is written as,

$$\overline{U}_0 + \overline{U}_1 L^a = 0 \;\; at \;\; \overline{y} = \overline{h}_0 + \overline{h}_1 L^a \tag{33}$$

$$\left\{I + (1-I)\left[1 + \left(L\frac{d\overline{U}_0}{d\overline{y}} + L^{a+1}\frac{d\overline{U}_1}{d\overline{y}}\right)^a\right]^{\frac{n-1}{a}}\right\}\left(\frac{d\overline{U}_0}{d\overline{y}} + \frac{d\overline{U}_1}{d\overline{y}}L^a\right) = -\overline{y} \tag{34}$$

$$\int_0^{\overline{h}_0 + \overline{h}_1 L^a} (\overline{U}_0 + \overline{U}_1 L^a d)\overline{y} = 1 \tag{35}$$

Following the same procedure used for the $\mathcal{O}(L^0)$ it is possible to reach the solution at the order $\mathcal{O}(L^a)$ given by,

$$\overline{U}_1(\overline{y}) = \frac{(-1)^a(1-I)(1-n)(\overline{h}_0^{a+2} - \overline{y}^{a+2})}{a(a+2)} + \overline{h}_0\overline{h}_1 \tag{36}$$

$$\overline{h}_1 = \frac{(-1)^{a+1}(1-I)(1-n)\overline{h}_0^{a+1}}{a(a+3)} \tag{37}$$

Replacing Eqs.(31) and (36) into Eq.(26), and Eqs.(32) and (37) into Eq.(27) yields to the asymptotic solution for the velocity profile and the film thickness considering a small non-Newtonian behavior. The equations leads to the solutions,

$$\overline{U}(\overline{y}) = \frac{(3^{\frac{2}{3}} - \overline{y}^2)}{2} + \frac{(-1)^a(1-I)(1-n)[3^{\frac{a+2}{3}} - (a+3)\overline{y}^{a+2}]}{a(a+2)(a+3)}L^a \tag{38}$$

$$\overline{h} = \sqrt[3]{3} + \frac{(-1)^{a+1}(1-I)(1-n)3^{\frac{a+1}{3}}}{a(a+3)}L^a \tag{39}$$

which is a general case for the solution presented by Weinstein (1990). It is possible to obtain the same result by setting $a = 2$ in Eq.(38) and Eq.(39).

3. Temporal stability analysis

3.1 Perturbed equations

In order to study the stability of the flow, it was implemented a small perturbation technique. For the perturbed flow the velocity and pressure components, together with the interface fluctuation, are given by,

$$\overline{u}(\overline{x}, \overline{y}, \overline{t}) = \overline{U}(y) + \hat{u}(\overline{x}, \overline{y}, \overline{t}) \tag{40}$$

$$\overline{v}(\overline{x}, \overline{y}, \overline{t}) = \hat{v}(\overline{x}, \overline{y}, \overline{t}) \tag{41}$$

$$\overline{p}(\overline{x}, \overline{y}, \overline{t}) = \overline{P}(y) + \hat{P}(\overline{x}, \overline{y}, \overline{t}) \tag{42}$$

$$\overline{\xi}(\overline{x}, \overline{t}) = \hat{\xi}(\overline{x}, \overline{t}) \tag{43}$$

As U and P satisfies the basic state of the flow it possible to linearize the governing equations. Applying the perturbations given by Eqs.(40), (41) and (42) into the dimensionless governing equations Eqs.(12) to (15) and neglecting all the second-order perturbations, it is possible to find the govern equations for the disturbances,

$$\frac{\partial \hat{u}}{\partial \overline{x}} + \frac{\partial \hat{v}}{\partial \overline{y}} = 0 \tag{44}$$

$$\frac{\partial \hat{u}}{\partial \overline{t}} + \overline{U}\frac{\partial \hat{u}}{\partial \overline{x}} + \hat{v}\frac{\partial \overline{U}}{\partial \overline{y}} = -\frac{\partial \hat{p}}{\partial \overline{x}} + \frac{1}{Re}\left(\frac{\partial \hat{\tau}_{xx}}{\partial \overline{x}} + \frac{\partial \hat{\tau}_{xy}}{\partial \overline{y}}\right) \tag{45}$$

$$\frac{\partial \hat{v}}{\partial \overline{t}} + \overline{U}\frac{\partial \hat{v}}{\partial \overline{x}} = -\frac{\partial \hat{p}}{\partial \overline{y}} + \frac{1}{Re}\left(\frac{\partial \hat{\tau}_{xy}}{\partial \overline{x}} + \frac{\partial \hat{\tau}_{yy}}{\partial \overline{y}}\right) \tag{46}$$

where,

$$(\hat{\tau}_{xx}, \hat{\tau}_{yy}, \hat{\tau}_{xy}) = \left(2\overline{\eta}\frac{\partial \hat{u}}{\partial \overline{x}} \ , \ 2\overline{\eta}\frac{\partial \hat{v}}{\partial \overline{y}} \ , \ \overline{\epsilon}\left[\frac{\partial \hat{u}}{\partial \overline{y}} + \frac{\partial \hat{v}}{\partial \overline{x}}\right]\right) \tag{47}$$

$$\overline{\epsilon} = I + (1-I)\left[1 + n\left(L\frac{\partial \overline{U}}{\partial \overline{y}}\right)^a\right]\left[1 + \left(L\frac{\partial \overline{U}}{\partial \overline{y}}\right)^a\right]^{\frac{n-a-1}{a}} \tag{48}$$

The no-slip condition at the solid boundary leads to,

$$\hat{u} = 0 \ \ at \ \ \overline{y} = \overline{h} \tag{49}$$

$$\hat{v} = 0 \ \ at \ \ \overline{y} = \overline{h} \tag{50}$$

and the kinematic condition at the free surface, which represents the impermeability of the interface, can be written as,

$$\frac{\partial \hat{\xi}}{\partial \overline{t}} + \hat{u}\frac{\partial \hat{\xi}}{\partial \overline{x}} - \hat{v} = 0 \ \ at \ \ \overline{y} = 0 \tag{51}$$

At the free surface, there are two dynamic conditions that are related with the continuity of the tangential and normal stresses at the interface. The first represents the viscous effect and the second one rises from the Laplace-Young relation. Both conditions can be written as,

$$\hat{\tau}_{xy} - \hat{\xi} = 0 \ \ at \ \ \overline{y} = 0 \tag{52}$$

$$\hat{p}Re + \hat{\xi}cot(\theta) - 2\overline{\eta}\frac{\partial \hat{v}}{\partial \overline{y}} + \frac{1}{We_m}\frac{\partial^2 \hat{\xi}}{\partial \overline{x}^2} = 0 \ \ at \ \ \overline{y} = 0 \tag{53}$$

where We_m is a modified Weber number defined as,

$$We_m = \frac{\eta_0 U_s h_s}{\gamma} \tag{54}$$

Once that we are considering a two-dimensional problem we can use a stream function $\hat{\Psi}$ defined as,

$$(\hat{u}, \hat{v}) = \left(\frac{\partial \hat{\Psi}}{\partial \overline{y}}, -\frac{\partial \hat{\Psi}}{\partial \overline{x}}\right) \tag{55}$$

in addition, we can introduce the dimensionless normal modes for the stream function and the interface fluctuation as,

$$\hat{\Psi}(\overline{x}, \overline{y}, \overline{t}) = \tilde{\Psi}(\overline{y})e^{i\alpha(\overline{x} - c\overline{t})} \tag{56}$$

$$\hat{\xi}(\overline{x}, \overline{t}) = \tilde{\xi}e^{i\alpha(\overline{x} - c\overline{t})} \tag{57}$$

where $\alpha = kh_s \in R$, in which k corresponds to the wave number, and $c = \frac{\omega}{k} \in C$, where ω is the complex frequency. Applying Eq.(56) and Eq.(57) into the governing equations, Eqs.(44) to (47), and also into the boundary conditions, Eqs.(49) to (53), it is possible to write the Orr-Sommerfeld equation and the boundary conditions of the problem. The Orr-Sommerfeld equation for a Carreau-Yasuda model can be written as,

$$(D^2 + \alpha^2)[D^2\overline{\epsilon} + 2D\overline{\epsilon}D + \overline{\epsilon}(D^2 + \alpha^2)]\tilde{\Psi} - 4\alpha^2 D(\overline{\eta}D\tilde{\Psi}) = i\alpha Re[(\overline{U} - c)(D^2 - \alpha^2) - D^2\overline{U}]\tilde{\Psi} \tag{58}$$

where $D^m = \frac{\partial^m}{\partial \overline{y}^m}$. The boundary conditions at the wall given by Eq.(49) and Eq.(50) yield to,

$$D\tilde{\Psi} = 0 \;\; at \;\; \overline{y} = \overline{h} \tag{59}$$

$$\tilde{\Psi} = 0 \;\; at \;\; \overline{y} = \overline{h} \tag{60}$$

and the kinematic condition Eq.(51) becomes,

$$\tilde{\Psi} - (c - \overline{U})\tilde{\xi} = 0 \;\; at \;\; \overline{y} = 0 \tag{61}$$

and the last two equations are given by the dynamic boundary conditions Eq.(52) and Eq.(53) which leads to,

$$\overline{\epsilon}(D^2 + \alpha^2)\tilde{\Psi} - \tilde{\xi} = 0 \;\; at \;\; \overline{y} = 0 \tag{62}$$

$$i\alpha Re[(c - \overline{U})D + D\overline{U}]\tilde{\Psi} - 4\alpha^2\overline{\eta}D\tilde{\Psi} + (D^2 + \alpha^2)(D\overline{\epsilon} + \overline{\epsilon}D)\tilde{\Psi} + i\alpha\left[cot(\theta) + \frac{\alpha^2}{We_m}\right]\tilde{\xi} = 0 \;\; at \;\; \overline{y} = 0 \tag{63}$$

also its possible to write the surface conditions without the fluctuation $\tilde{\xi}$ by using a combination between of Eq.(61) and Eq.(62), together with Eq.(61) and Eq.(63). In this way it is possible to find,

$$[1 + \overline{\epsilon}(\overline{U} - c)(D^2 + \alpha^2)]\tilde{\Psi} = 0 \;\; at \;\; \overline{y} = 0 \tag{64}$$

$$i\alpha Re[(c - \overline{U})D + D\overline{U}]\tilde{\Psi} - 4\alpha^2\overline{\eta}D\tilde{\Psi} + (D^2 + \alpha^2)\left[D\overline{\epsilon} + \overline{\epsilon}D + i\alpha\overline{\epsilon}\left(cot(\theta) + \frac{\alpha^2}{We_m}\right)\right]\tilde{\Psi} = 0 \;\; at \;\; \overline{y} = 0 \tag{65}$$

where the system of Eqs.(58) to (63) forms a generalized eigenvalue problem, which makes possible to find the complex velocity c and the wave number α.

3.2 Asymptotic approach at zero-order

In order to study the stability of the system analytically, it was considered the velocity profile and film thickness given by Eq.(38) and Eq.(39). In this study, we present only the zero-order solution. To represent $\tilde{\Psi}$ and c as a power series of the wave number α, we used the following expansions:

$$\tilde{\Psi} = \tilde{\Psi}_0 + \alpha\tilde{\Psi}_1 + \mathcal{O}(\alpha^2) \tag{66}$$

$$c = c_0 + \alpha c_1 + \mathcal{O}(\alpha^2) \tag{67}$$

also applying Eq.(66) and Eq.(67) into the system constitute by Eqs.(58), (59), (60), (64), (65) and considering an additional normalization condition $\tilde{\Psi} = 1$ at $\overline{y} = 0$, the zero-order system becomes,

$$\overline{\epsilon}D^4\tilde{\Psi}_0 + 2D\overline{\epsilon}D^3\tilde{\Psi}_0 + D^2\overline{\epsilon}D^2\tilde{\Psi}_0 = 0 \tag{68}$$

$$D\tilde{\Psi}_0 = 0 \;\; at \;\; \overline{y} = \overline{h} \tag{69}$$

$$\tilde{\Psi}_0 = 0 \;\; at \;\; \overline{y} = \overline{h} \tag{70}$$

$$\tilde{\Psi}_0 + (\overline{U} - c_0)D^2\tilde{\Psi}_0 = 0 \;\; at \;\; \overline{y} = 0 \tag{71}$$

$$D^3\tilde{\Psi}_0 = 0 \;\; at \;\; \overline{y} = 0 \tag{72}$$

$$\tilde{\Psi}_0 = 1 \;\; at \;\; \overline{y} = 0 \tag{73}$$

In order to solve the system above, we used the software Wolfram Mathematica to solve Eq.(68) with the adequate boundary conditions which yields to the solution of the zero-order,

$$\tilde{\Psi}_0(\overline{y}) = \frac{[\pi a(a+2)\mathcal{M} + 2(a+1)\mathcal{M}sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})]\overline{y}^{a+2} - [\pi a(a+1)(a+2) + (a+1)(a+2)sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})]\overline{y}^2}{\pi a(a+1)(a+2)(\mathcal{M}\overline{h}_0^a - \overline{h}) + sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})[2(a+1)^2\mathcal{M}\overline{h}_0^{a+1} - \overline{h}(a+1)(a+2)]}$$

$$+\frac{\pi a(a+2)[2(a+1) - (a+2)\mathcal{M}\overline{h}_0^a] + 2(a+1)(a+2)sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})(1 - \mathcal{M}\overline{h}_0^a)}{\pi a(a+1)(a+2)(\mathcal{M}\overline{h}_0^a - 1) + sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})[2(a+1)\mathcal{M}\overline{h}_0^a - (a+2)]}\overline{y} + 1 \tag{74}$$

where,

$$\mathcal{M} = \frac{(-1)^a(1-I)(n-1)(a+1)L^a}{a} \tag{75}$$

also $\overline{h}_0$ and $\overline{h}$ are given by Eq.(32) and Eq.(39) respectively. Applying the solution Eq.(74) into the equation Eq.(71) we can determine the celerity at zero order as,

$$c_0 = \frac{\overline{h}_0^2[(a+1)(a+2)(a+3) - 2\mathcal{M}\overline{h}_0^a]}{2(a+1)(a+2)(a+3)}$$

$$-\frac{\pi a(a+2)(\mathcal{M}\overline{h}_0^a - \overline{h}) + sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})[2(a+1)\mathcal{M}\overline{h}_0^{a+1} - \overline{h}(a+2)]}{2\pi a(a+2) + 2(a+2)sin(\frac{\pi}{a})\Gamma(-\frac{1}{a})\Gamma(\frac{2}{a})} \tag{76}$$

4. Discussion and results

In this section, we present some of the preliminary results about the problem. It was assumed in the asymptotic analysis for the reference flow and the film thickness a limit case for a small non-Newtonian behavior (L tend to zero). So for all the calculations proposed it was used $L = 0,5$ which is related with the material relaxation time λ in Eq.(18) (λ equal zero set a Newtonian behavior), and also it was fixed $I = 0,5$. The choice for I sets that $\eta_0 = 2\eta_\infty$ and this choice is slightly arbitrary; some authors usually choose to fix $\eta_\infty = I = 0$, but in this work all parameters were fixed as non-zero. In order to compare the behavior of different fluids it was used $n = 0,5$, $n = 1$ and $n = 2$ which represent shear-thinning, Newtonian and shear-thickening behaviors, respectively. Also, the main difference from previous works is the presence of the parameter a. In order to study the effect of this parameters in the base state it was considered $a = 1$, 2, and 4 were $a = 2$ represents the four parameters Carreau model. For the reference flow, it was obtained Tab. (1) which contains the relation between the dimensionless maximum velocity at the interface $\overline{U}_{max}$ and the parameters a and n. The trivial solution $n = 1$ remains the same regardless of the variation of a as expected by the model. For $a = 2$ and $a = 4$, the velocity at the interface is slightly faster for the shear-thinning case and slower for the shear-thickening, it is possible to see that the increment in the parameter a approaches the behavior of both cases to the Newtonian one. If we consider $a = 1$ the behavior is completely the opposite and they deviate from the Newtonian case more strongly. Similar aspects were found in the asymptotic solution results for the dimensionless film thickness. As it is possible to see in Tab. (2) for the cases where $a = 2$ and 4, the film thickness h for the shear-thinning is slightly thinner compared to the Newtonian case, in contrast with the shear-thickening which is thicker than the Newtonian one. Both results are closer to the Newtonian case for $a = 4$ which is consistent with the results of $\overline{U}_{max}$ in Tab.(1). But the values for $a = 1$ are also complete the opposite expected by $a = 2$ and 4 but still consistent with the results presented by $\overline{U}_{max}$ for the same value of a.

n	$a = 1$	$a = 2$	$a = 4$
$0,5$	1,0088	1,0468	1,0409
$1,0$	1,0400	1,0400	1,0400
$2,0$	1,1025	1,0265	1,0384

Table 1. Asymptotic results for the dimensionless maximum velocity at the interface $\overline{U}_{max}$.

n	$a = 1$	$a = 2$	$a = 4$
$0,5$	1,5073	1,4235	1,4388
$1,0$	1,4422	1,4422	1,4422
$2,0$	1,3122	1,4797	1,4492

Table 2. Asymptotic results for the dimensionless film thickness $\overline{h}$.

Figure (2) shows the normalized velocity profiles for different values of n. As mentioned previously about Tab.(1), it is possible to see only a small variation for the velocity profile with the shear-thinning slightly below the Newtonian case and the shear-thickening above, but even closer to the curve $n = 1$. This small difference is expected considering the limitation of the asymptotic approach for small non-Newtonian behavior. For the case of $a = 4$, the three curves were

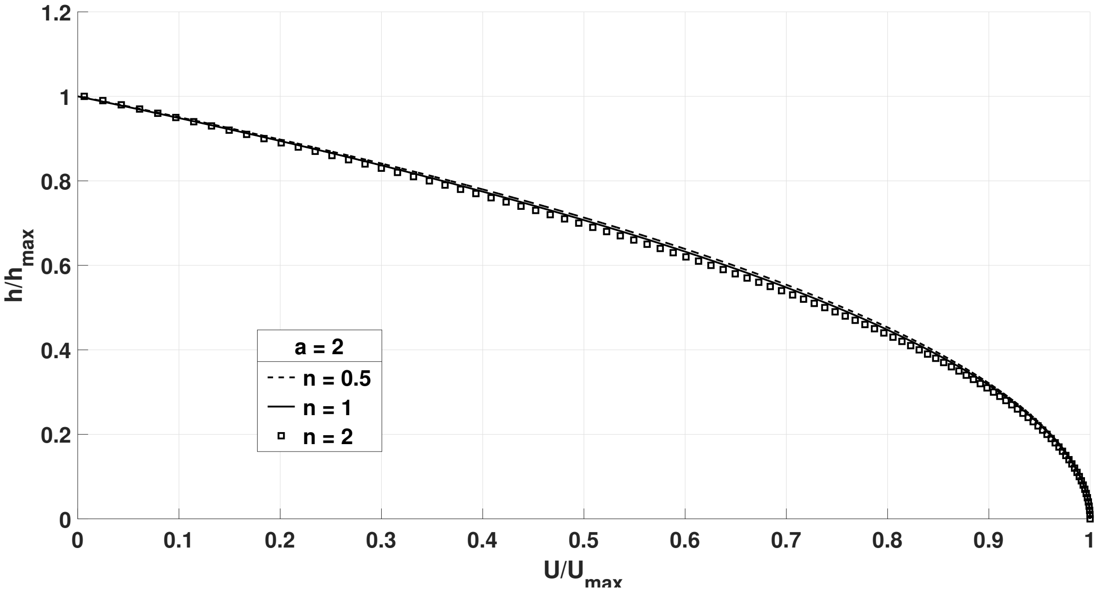

Figure 2. Velocity profiles as a function of the film thickness for $a = 2$.

basically overlapping each other. For $a = 1$ the inversion between the profiles also was detected, but a small error start to appear at the wall.

Figure (3) and Fig.(4) present the normalized shear rate profiles $\overline{\dot{\gamma}}$ for the cases $a = 2$ and $a = 4$, respectively. The results for the shear-thinning and shear-thickening are consistent with the previous results. Comparing both figures we can detect a small shift in the curvature for the shear-thinning case. On the other hand, for the shear thickening case, the difference provoked in the shear rate for a variation in the parameter a is considerably strong. The shear rate profile was shifted from around $\overline{h}/\overline{h}_{max} = 3,7$ to $\overline{h}/\overline{h}_{max} = 6,0$ and the change in the magnitude of the shear rate is even stronger, with the difference between the two cases being around the unity.

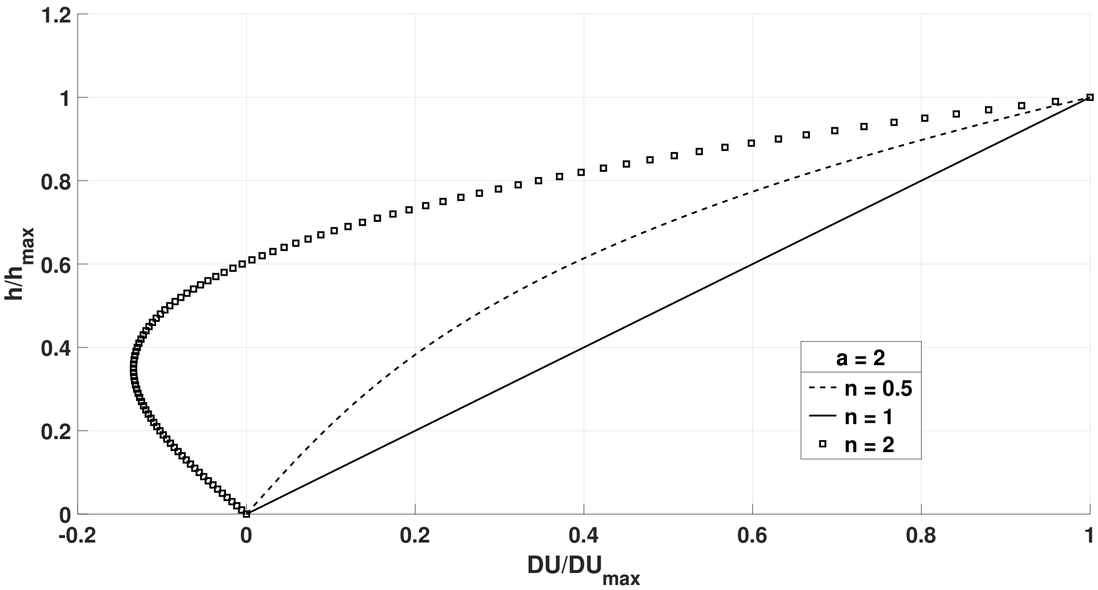

Figure 3. Shear rate $\overline{\dot{\gamma}}$ as a function of the film thickness for $a = 2$.

Figure (5) and Fig.(6) show the viscosity as a function of the film thickness. It is important to note that the previous behavior noticed in the shear rate for both cases become consistent with this results. We note that the variation for the shear-thickening viscosity around $\overline{h}/\overline{h}_{max} = 3,7$ and $\overline{h}/\overline{h}_{max} = 6,0$ in Fig.(5) and Fig.(6) are consistent with the data presented for the shear rate. For $a = 4$ the shear-thickening cases fluctuate more closely of the Newtonian case compared with the case for $a = 2$, which diverges to its maximum value close to the wall. Once that the values of the viscosity start

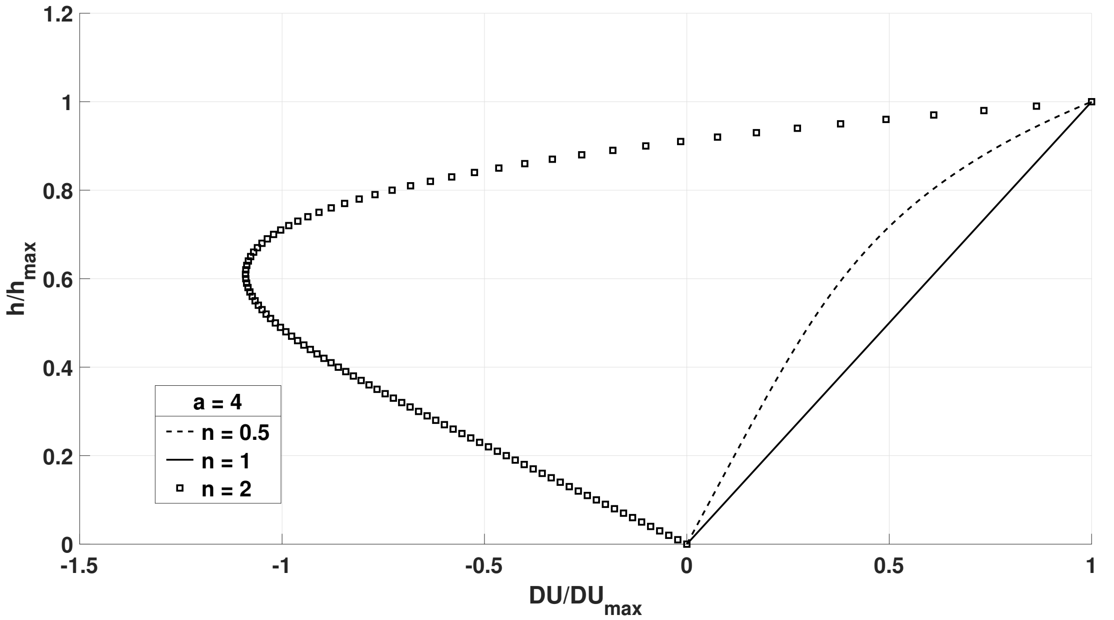

Figure 4. Shear rate $\overline{\dot{\gamma}}$ as a function of the film thickness for $a = 2$.

to approach to the free surface, both non-Newtonian cases tend to a Newtonian behavior. This convergence starts even sooner for $a = 4$.

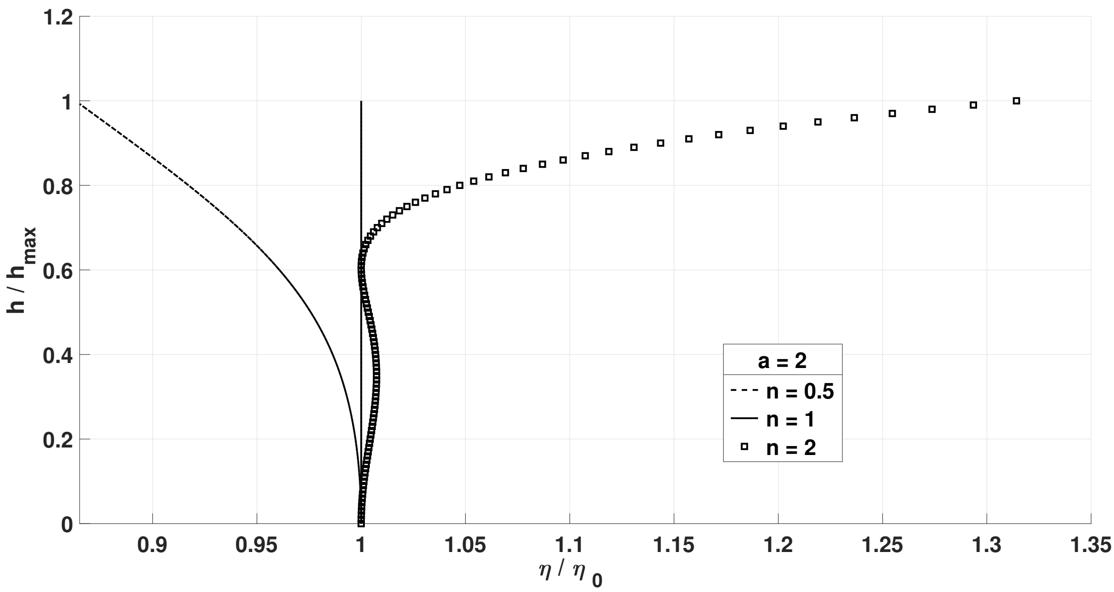

Figure 5. Dimensionless viscosity $\overline{\eta}$ as a function of the film thickness for $a = 2$.

For the temporal stability problem an asymptotic approach it was also used. As a preliminary result, it was obtained the solution at the zero order given by Eq.(74) and Eq.(76). It is important to notice that at the zero order there is no instability present. Also, the equations show a great dependence of the parameter a which appears in all gamma functions. The solution of Eq.(68) have four hypergeometric functions and even with a strong simplification, the solutions are still heavily dependent on the parameter a.

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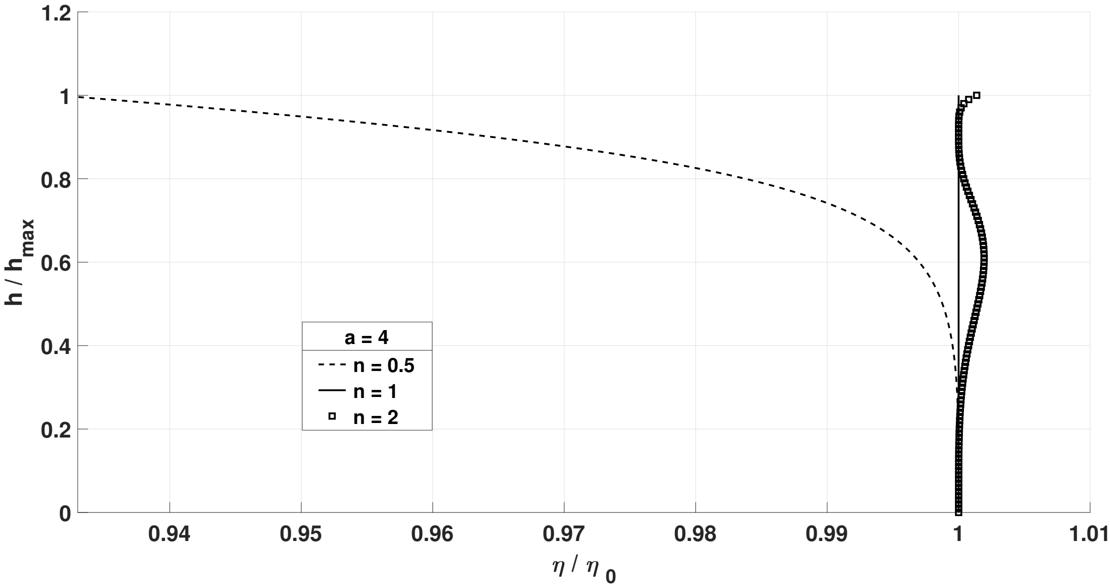

Figure 6. Dimensionless viscosity $\overline{\eta}$ as a function of the film thickness for $a = 4$.

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